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Dirichlet Eigenvalue Problem of Degenerate Elliptic Operators with Non-Smooth Coefficients

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Dedicated to Professor Chao-Jiang Xu on the occasion of his 65th birthday

Abstract. The aim of this review is to introduce some recent results in eigenvalues problems for a class of degenerate elliptic operators with non-smooth coefficients, we present the explicit estimates of the lower bound and upper bound for its Dirichlet eigenvalues.

AMS subject classifications: 35P15, 35J70

Key words: Dirichlet eigenvalues, weighted Sobolev spaces, degenerate elliptic operators, homogeneous dimension.

1 Introduction

Eigenvalue problem is an important research topic in spectral theory and has widely applications in physics. For the classical elliptic operators, the pioneer research work obtained by Weyl [52] in 1911 gave the following remarkable asymptotic formula describing the distribution of large eigenvalues of the Dirichlet

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Laplacian Δ in a bounded domain $\Omega\subset\mathbb{R}^n$:

$$\lambda_k\sim(2\pi)^2|B_1|^{-\frac{2}{n}}\left(\frac{k}{|\Omega|}\right)^{\frac{2}{n}}\quad\text{as}\quad k\to+\infty, \tag{1.1}$$

where $|B_1|$ is the volume of the unit ball in $\mathbb{R}^n$ and $|\Omega|$ is the volume of Ω. This formula was conjectured independently by Sommerfeld [50] and Lorentz [40] in 1910. Then in 1961, Pólya [47] proved that the above asymptotic relation (1.1) is in fact a one-sided inequality if Ω is a plane domain which tiles $\mathbb{R}^2$ (his proof also works in $\mathbb{R}^n$) and he conjectured that, for any domain in $\mathbb{R}^n$, we have

$$\lambda_k\geq(2\pi)^2|B_1|^{-\frac{2}{n}}\left(\frac{k}{|\Omega|}\right)^{\frac{2}{n}}\quad\text{for any}\quad k\geq 1. \tag{1.2}$$

The Polya's conjecture has attracted a lot of attention in the estimation of eigenvalues. Lieb [38] proved an inequality like (1.2) for any domain in $\mathbb{R}^n$ but with a constant $\widetilde{C}_n$ that differs from the constant $(2\pi)^2|B_1|^{-2/n}$ by a factor. Later in 1983, Li and Yau [37] gave the following lower bound of λ_k by a simple approach:

$$\sum_{j=1}^{k}\lambda_j\geq\frac{n}{n+2}(2\pi)^2|B_1|^{-\frac{2}{n}}\cdot k^{1+\frac{2}{n}}\cdot|\Omega|^{-\frac{2}{n}}\quad\text{for any}\quad k\geq 1. \tag{1.3}$$

On the other hand, for the upper bounds of Dirichlet eigenvalues of Laplacian, under some conditions Kröger [34] obtained that

$$\sum_{j=1}^{k}\lambda_j\leq\frac{n}{n+2}(2\pi)^2|B_1|^{-\frac{2}{n}}\cdot k^{1+\frac{2}{n}}\cdot|\Omega|^{-\frac{2}{n}}+C_0k^{1+\frac{1}{n}} \tag{1.4}$$

holds for sufficient large k, where C_0 is a positive constant depending on Ω. For more results on the upper bound estimates of the ratio $\frac{\lambda_{k+1}}{\lambda_1}$, one can see the papers [12, 13, 24, 35] as well as the references therein. Other related results in eigenvalue problems for the non-degenerate elliptic operators, one can refer to [26, 36, 42].

Degenerate elliptic operators have been intensively studied in the late 1960s and is still an active research field. The Hörmander type operator

$$\Delta_X:=-\sum_{j=1}^{m}X_j^*X_j$$

(also called the finitely degenerate elliptic operator) generated by a system of smooth vector fields $X=(X_1,\ldots,X_m)$ with Hörmander's condition is an important