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Zero Viscosity-Diffusivity Limit for the Incompressible Boussinesq Equations in Gevrey Class

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Dedicated to Professor C.-J. Xu on occasion of his 65th birthday

Abstract. In this paper, we study the zero viscosity-diffusivity limit for the incompressible Boussinesq equations in a periodic domain in the framework of Gevrey class. We first prove that there exists an interval of time, independent of the viscosity coefficient and the diffusivity coefficient, for the solutions to the viscous incompressible Boussinesq equations. Then, based on these uniform estimates, we show that the solutions of the viscous incompressible Boussinesq equations converge to that of the ideal incompressible Boussinesq equations as the viscosity and diffusivity coefficients go to zero. Moreover, the convergence rate is also given.

AMS subject classifications: 35Q35, 76D03, 76D09

Key words: Gevrey class, incompressible Boussinesq equation, analyticity, zero viscosity-diffusivity limit, convergence rate.

1 Introduction

The Cauchy problem of d-dimensional incompressible Boussinesq equations on torus $\mathbb{T}^d$ takes the following form:

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$$\begin{cases} \partial_t u^{\nu,\kappa} + u^{\nu,\kappa}\cdot\nabla u^{\nu,\kappa} - \nu\Delta u^{\nu,\kappa} + \nabla p^{\nu,\kappa} = \theta^{\nu,\kappa} e_d, \\ \operatorname{div} u^{\nu,\kappa} = 0, \\ \partial_t \theta^{\nu,\kappa} + u^{\nu,\kappa}\cdot\nabla\theta^{\nu,\kappa} - \kappa\Delta\theta^{\nu,\kappa} = 0, \\ u^{\nu,\kappa}(x,0) = u_0(x), \quad \theta^{\nu,\kappa}(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $u^{\nu,\kappa} = (u_1^{\nu,\kappa}, u_2^{\nu,\kappa}, \ldots, u_d^{\nu,\kappa})$ represents the fluid velocity field, $\theta^{\nu,\kappa}$ may be interpreted physically as a thermal variable (e.g., when $\kappa > 0$), or a density variable (e.g., when $\kappa = 0$) at point $x = (x_1, \ldots, x_d) \in \mathbb{T}^d$ at time t, and $p^{\nu,\kappa}$ represents the scalar pressure. $e_d = (0,0,\ldots,1)$ denotes the unit normal vector in x_d direction. The parameters $\nu > 0$ and $\kappa > 0$ denote the viscosity coefficient and the diffusivity coefficient, respectively.

The Boussinesq equations (1.1), which model geophysical flows such as atmospheric fronts and oceanic circulation, play an important role in the study of Raleigh-Bernard convection, see [4, 9, 13]. Due to the significance of the physical background, the incompressible Boussinesq equations have been studied by many physicists and mathematicians.

When we take $\theta^{\nu,\kappa} = 0$ in the system (1.1), it is reduced into the classical incompressible Navier-Stokes equations. One of the most important problems in fluid dynamics is the inviscid limit of the incompressible flow. So far many results are available on the inviscid limit to the incompressible Navier-Stokes equations in the whole space $\mathbb{R}^d$ or on the torus, for instance, see [2, 3, 5, 7, 10] and the reference therein. Specially, we have studied the vanishing viscosity limit of the incompressible Navier-Stokes equations in Gevrey class in a torus and obtained the convergence rate in Gevrey class [2]. However, in the presence of a physical boundary, according to the classical Prandtl boundary layers theory [12, 14], the inviscid limit of the incompressible Navier-Stokes equations in a domain with non-slip boundary condition is still an outstanding open problem up to now and only a few results for some specific cases available. Sammatino and Caflisch first investigated the Prandtl theory and the vanishing viscosity limit problem in the analytic setting in [15,16], in which they implied the method of the abstract Cauchy-Kovalevskaya theorem. By using the vorticity formulation, Maekawa [8] studied the same problems for 2D case under the assumption that the initial vorticity of outer Euler flows should vanish in a neighborhood of boundary. Wang *et al.* [17] developed an energy method for the inviscid limit problem in the analytic setting. Recently, the results in [15, 16] were generalized to Gevrey class in [6]. When considering the incompressible Boussinesq equations (1.1), similar situation happens when either the viscosity $\varepsilon \to 0$ or the diffusivity $\kappa \to 0$ or both $\nu \to 0$ and $\kappa \to 0$. It was remarked by Moffat [11] that the understanding of how the topology changes when the diffusivity $\kappa \to 0$ is a very interesting problem.