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On the Kernel of the Borel's Characteristic Map of Lie Groups

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Abstract. For compact and connected Lie group G with a maximal torus T the quotient space G/T is canonically a smooth projective manifold, known as the complete flag manifold of the group G. The cohomology ring map $c^*: H^*(B_T) \to H^*(G/T)$ induced by the inclusion $c: G/T \to B_T$ is called the Borel's characteristic map of the group G [7,8], where B_T denotes the classifying space of T. Let G be simply-connected and simple. Based on the Schubert presentation of the cohomology $H^*(G/T)$ of the flag manifold G/T obtained in [10,11], we develop a method to find a basic set of explicit generators for the kernel $\ker c^* \subset H^*(B_T)$ of the characteristic map c.

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1 Introduction

In this paper the integral cohomology ring of a space X is denoted by $H^*(X)$, unless otherwise stated. The Lie groups G under consideration are always assumed to be compact, simply-connected and simple. According to the classification of compact Lie groups of Cartan, these groups consist of three infinite families of the classical Lie groups $SU(n), Spin(n), Sp(n)$, as well as the five exceptional ones G_2, F_4, E_6, E_7, E_8.

Let G be a Lie group with a maximal torus T. The inclusion $T\subset G$ induces the fibration

$$G/T \overset{c}{\hookrightarrow} B_T \overset{\pi}{\to} B_G, \tag{1.1}$$

where B_T (resp. B_G) denotes the classifying space of the group T (resp. G), and where the fiber space G/T is canonically a complex projective manifold, called the complete flag manifold of the group G. Assume that $\dim T = n$, and let $\{\omega_1,\dots,\omega_n\} \subset H^2(B_T)$ be a set of fundamental dominant weights of G [3] and $\mathbb{Z}[\omega_1,\dots,\omega_n]$ the ring of integral polynomials in the weights $\omega_1,\dots,\omega_n$. Then the induced map of the fiber inclusion c on cohomologies

$$c^*: H^*(B_T) = \mathbb{Z}[\omega_1,\dots,\omega_n] \ \to \ H^*(G/T) \tag{1.2}$$

is called the Borel's characteristic map of the group G (e.g. [7,8]). Since c^* is a ring map, its kernel is an ideal in $\mathbb{Z}[\omega_1,\dots,\omega_n]$

$$\ker c^* = \big\{ p \in \mathbb{Z}[\omega_1,\dots,\omega_n] \,|\, c^*(p) = 0 \big\}.$$

According to the basis theorem of Hilbert, there exists a minimal system of homogeneous polynomials $p_1,\dots,p_m \in \mathbb{Z}[\omega_1,\dots,\omega_n]$ with $\deg p_1 \le \cdots \le \deg p_m$ such that $\ker c^*$ is the ideal $\langle p_1,\dots,p_m\rangle$ generated by $p_1,\dots,p_m$. For convenience, we call such a system $\{p_1,\dots,p_m\}$ a basic sequence of generators of the ideal $\ker c^*$. The problem we are about to study is:

Problem 1.1. For a Lie group G with a maximal torus T find a basic sequence of generators $\{p_1,\dots,p_m\}$ of the ideal $\ker c^*$.

For the cohomologies with rational coefficients certain information about a basic sequence of generators of the ideal $\ker c^* \otimes \mathbb{Q} \subset \mathbb{Q}[\omega_1,\dots,\omega_n]$ are known. Firstly, Borel [2] has shown that

Lemma 1.1. *For a Lie group G with rank $\dim T = n$, there exist n homogeneous polynomials $q_1,\dots,q_n \in H^*(B_T;\mathbb{Q}) = \mathbb{Q}[\omega_1,\dots,\omega_n]$ such that the map c^* induces an isomorphism of algebras*

$$H^*(G/T;\mathbb{Q}) = \mathbb{Q}[\omega_1,\dots,\omega_n] / \langle q_1,\dots,q_n\rangle. \tag{1.3}$$