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Recent Results on Recurrent Solutions and Limit Distributions of SDEs

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Abstract. The limit distribution for homogeneous Markov processes is studied extensively and well understood, but it is not the case for inhomogeneous Markov processes. In this paper, we review some recent results on inhomogeneous Markov processes generated by non-autonomous stochastic (partial) differential equations (SDE in short). Under some suitable conditions, we show that the distribution of recurrent solutions of SDEs constitutes the limit distribution of the corresponding inhomogeneous Markov processes.

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1 Introduction

Recurrence is one of central topics in the fields of dynamical systems and probability theory and it has essentially similar meaning in both fields, which describes the asymptotic behaviors and complexity of dynamical systems and Markov processes. On one hand, it follows from Poincaré recurrence theorem and Birkhoff recurrence theorem that recurrence exists extensively in dynamical systems. On the other hand, it is known that (positive) recurrence is essentially equivalent to the existence of invariant (probability) measures for Markov processes. The

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notion of Poisson stability (also called recurrence in the literature) was first introduced by Poincaré in his famous work [50] in the late 19th century; he pointed out that the orbits of aperiodic solutions are stable in the sense of Poisson in all bounded Hamilton systems. Poisson stable motions in dynamical systems include the following different classes: stationary, periodic, quasi-periodic, almost periodic [6–8], almost automorphic [5, 58], Birkhoff recurrent [4], Levitan almost periodic [40], almost recurrent [2], pseudo-periodic [9], pseudo-recurrent [52, 54] etc. The recurrence and positive recurrence was introduced by Kolmogorov [39]. It seems that recurrence has more abundant meaning and levels in dynamical systems than that in probability theory.

Invariant measures or limit distributions are extensively studied for homogeneous Markov processes, but it is not the case for inhomogeneous Markov processes. Inhomogeneity is essentially rooted in the time dependence of the system evolution law (specifically in the field of stochastic differential equations, inhomogeneity corresponds to non-autonomy of equations), so the inhomogeneous Markov process has a deeper, wider and more natural theoretical background than the homogeneous one. Similar to the study in the homogeneous case, the primary issue to consider in the inhomogeneous case is: how to reasonably and effectively describe the limit distribution of the inhomogenous Markov process (including reasonable definitions, existence problems, and the characterization of its properties, etc.)? This is a question of fundamental importance in the study of Markov processes, and its investigation needs to develop new theories, methods and tools. Our idea is to introduce the theory and method of recurrence in dynamical systems to the study of the limit distribution of inhomogeneous Markov processes; the idea stems from the natural connection between the essentially similar meaning of recurrence in both fields. Indeed, we will show that the distribution of recurrent solutions of an SDE constitutes the limit distribution of the corresponding inhomogeneous Markov process generated by this SDE.

The paper is organized as follows. In Section 2, we discuss the Markov process and its limit distributions. In Section 3, we recall basic idea and concepts of recurrent motions in dynamical systems. In Section 4, we review recent works on recurrent solutions for SDEs, and finally we conclude in Section 5 that the distribution of recurrent motions is the limit distribution of the corresponding inhomogeneous Markov process.

2 The Markov process and its limit distribution

The Markov process is named after A.A. Markov who introduced the concept in 1907 with discrete time and finite states, which is a kink of "memoryless" stochas-