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Kirchhoff-Type Problem with Mixed Boundary Condition in a Variable Exponent Sobolev Space

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Abstract. In this paper, we consider a mixed boundary value problem for the stationary Kirchhoff-type equation containing $p(\cdot)$-Laplacian. More precisely, we are concerned with the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show the existence of at least one, two or infinitely many nontrivial weak solutions according to hypotheses on given functions.

AMS subject classifications: 35H30, 35D05, 35J60, 35J70

Key words: Kirchhoff-type problem, mixed boundary value problem, $p(\cdot)$-Laplacian type equation, weak solutions.

1 Introduction

In this paper, we consider the following Kirchhoff-type problem:

$$\begin{cases} -M\big(\Phi(u)\big)\operatorname{div}\big[\boldsymbol{a}(x,\nabla u(x))\big]=f\big(x,u(x)\big) & \text{in } \Omega, \\ u(x)=0 & \text{on } \Gamma_1, \\ M\big(\Phi(u)\big)\boldsymbol{n}(x)\cdot\boldsymbol{a}\big(x,\nabla u(x)\big)=g\big(x,u(x)\big) & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

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Here Ω is a bounded domain of $\mathbb{R}^N$ $(N\geq 2)$ with the Lipschitz-continuous ($C^{0,1}$ for short) boundary Γ such that:

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are disjoint open subsets of } \Gamma, \quad \overline{\Gamma_1}\cup\overline{\Gamma_2}=\Gamma, \quad \Gamma_1\neq\emptyset, \tag{1.2}$$

and the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. The function $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$. Then the operator $u\mapsto \operatorname{div}[\boldsymbol{a}(x,\nabla u(x))]$ is more general than the $p(\cdot)$-Laplacian

$$\Delta_{p(x)}u(x)=\operatorname{div}\left[|\nabla u(x)|^{p(x)-2}\nabla u(x)\right]$$

and the mean curvature operator

$$\operatorname{div}\left[(1+|\nabla u(x)|^2)^{(p(x)-2)/2}\nabla u(x)\right].$$

This generality brings about difficulties and requires some conditions. The function $M=M(s)$ defined in $[0,\infty)$ satisfies the following condition (M).

(M) $M:[0,\infty)\to[0,\infty)$ is a continuous and monotone increasing (i.e., non-decreasing) function, and there exist $0<m_0\leq m_1<\infty$ and $l\geq 1$ such that

$$m_0 s^{l-1}\leq M(s)\leq m_1 s^{l-1} \quad \text{for all} \quad s\geq 0.$$

Furthermore, the function $\Phi(u)$ is defined by

$$\Phi(u)=\int_\Omega A\big(x,\nabla u(x)\big)\,dx, \tag{1.3}$$

where $A(x,\boldsymbol{\xi})$ is a function on $\Omega\times\mathbb{R}^N$ satisfying $\boldsymbol{a}(x,\boldsymbol{\xi})=\nabla_{\boldsymbol{\xi}}A(x,\boldsymbol{\xi})$. Thus we impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $g:\Gamma_2\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions. The first equation in (1.1) is non-local in the sense that the equation is not a pointwise identity according to the term $M(\Phi(u))$.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics [31], in electrorheological fluids [9,17,22,24].

For physical motivation to the problem (1.1), we consider the case where $\Gamma=\Gamma_1$ and $p(x)=2$. Then the equation

$$M\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\Delta u(x)=f\big(x,u(x)\big) \tag{1.4}$$

is the Kirchhoff equation which arises in nonlinear vibration, namely