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Symmetric Reduction and Hamilton-Jacobi Equations of the Controlled Underwater Vehicle-Rotor System

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In Memory of Professor Jerrold E. Marsden

Abstract. As an application of the theoretical results, in this paper, we study the symmetric reduction and Hamilton-Jacobi theory for the underwater vehicle with two internal rotors as a regular point reducible RCH system, in the cases of coincident and non-coincident centers of the buoyancy and the gravity. At first, we give the regular point reduction and the two types of Hamilton-Jacobi equations for a regular controlled Hamiltonian (RCH) system with symmetry and a momentum map on the generalization of a semidirect product Lie group. Next, we derive precisely the geometric constraint conditions of the reduced symplectic forms for the dynamical vector fields of the regular point reducible controlled underwater vehicle-rotor system, that is, the two types of Hamilton-Jacobi equations for the reduced controlled underwater vehicle-rotor system, by calculations in detail. These work reveal the deeply internal relationships of the geometrical structures of the phase spaces, the dynamical vector fields and the controls of the system.

AMS subject classifications: 70H33, 70H20, 70Q05

Key words: Underwater vehicle with internal rotors, regular controlled Hamiltonian system, coincident and non-coincident centers, regular point reduction, Hamilton-Jacobi equation.

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1 Introduction

It is well-known that the study of the stability and the drift of underwater vehicle dynamics is an important research work given by Leonard and Marsden [12], in which the underwater vehicle system is considered as a Hamiltonian system with rigid motion symmetry, the authors gave the semidirect product reduction for the underwater vehicle system and the careful analysis of the stability for the reduced underwater vehicle dynamics, also see Leonard [10, 11] on the study of underwater vehicle system.

It is worthy of noting that Marsden *et al.* [21] defined a regular controlled Hamiltonian (RCH) system, which is a Hamiltonian system with the external force and the control. In general, an RCH system, under the actions of the external force and the control, is not Hamiltonian, however, it is a dynamical system closely related to a Hamiltonian system, and it can be explored and studied by extending the methods for the external force and the control used in the study of Hamiltonian systems. Thus, we can emphasize explicitly the impact of the external force and the control in the study of the RCH systems. Now, a natural idea is that one may consider the underwater vehicle with the internal rotors, and a control torque acting on the internal rotors, as a model of the Hamiltonian system with the control. In particular, Marsden *et al.* [21], gave the regular point reduction and the regular orbit reduction for an RCH system with symmetry and a momentum map, by analyzing carefully the geometrical and the topological structures of the phase space and the reduced phase space of the corresponding Hamiltonian system. This work not only gave a variety of reduction methods for the RCH systems, but also showed a variety of relationships of the controlled Hamiltonian equivalence of these systems. Thus, it is a natural problem to study the underwater vehicle-rotor system with the control torque acting on the internal rotors as a regular point reducible RCH system on the generalization of a semidirect product Lie group. Moreover, as an application of the theoretical result of the RCH system with symmetry, one can derive the regular point reduced controlled underwater vehicle-rotor systems, by calculations in detail, in the cases of coincident and non-coincident centers of the buoyancy and the gravity, respectively.

On the other hand, we note that Hamilton-Jacobi theory is an important research subject in mathematics and analytical mechanics; see Abraham and Marsden [1], Arnold [3] and Marsden and Ratiu [18], and the Hamilton-Jacobi equation is also fundamental in the study of the quantum-classical relationship in quantization, and it plays an important role in the study of stochastic dynamical systems; see Woodhouse [36], Ge and Marsden [7], and Lázaro-Camí and Ortega [9]. For these reasons, the equation is described as a useful tool in the