Commun. Math. Res.
doi: 10.4208/cmr.2023-0034

Vol. **40**, No. 2, pp. 154-190
May 2024

Symplectic Conditions on Grassmannian, Flag, and Schubert Varieties

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Received 1 August 2023; Accepted 3 January 2024

Abstract. In this paper, a description of the set-theoretical defining equations of symplectic (type C) Grassmannian/flag/Schubert varieties in corresponding (type A) algebraic varieties is given as linear polynomials in Plücker coordinates, and it is proved that such equations generate the defining ideal of variety of type C in those of type A. As applications of this result, the number of local equations required to obtain the Schubert variety of type C from the Schubert variety of type A is computed, and further geometric properties of the Schubert variety of type C are given in the aspect of complete intersections. Finally, the smoothness of Schubert variety in the non-minuscule or cominuscule Grassmannian of type C is discussed, filling gaps in the study of algebraic varieties of the same type.

AMS subject classifications: 14M15, 14L30, 15A15

Key words: Grassmannian variety, generalized flag variety, Schubert variety, Plücker embedding, complete intersection.

1 Introduction

Grassmannian and flag varieties, which stem from linear algebra, are important study objects in the interplay of algebraic geometry, representation theory, and combinatorics. The symplectic Grassmannian and flag variety have also attracted considerable interest from researchers (e.g. [1,7]). As one of the best-understood examples of singular projective varieties, the Schubert variety plays an important

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role in the study of generalized Grassmannian/flag varieties. Its relation with the cohomology theory on Grassmannian was first proposed by Hermann Schubert as early as the 19th century and later featured as the 15th problem among Hilbert's famous 23 problems.

Let k be an algebraically closed field with $\mathrm{char}(k)=0$, and $e_1,e_2,\cdots,e_n$ the standard basis of the linear space k^n. For any $d\leq n$, put

$$\begin{aligned} I_{d,n} &= \left\{ \underline{i}=(i_1,i_2,\cdots,i_d) \,|\, 1\leq i_1<i_2<\cdots<i_d\leq n \right\} \\ &= \left\{ d-\text{subsets of } \{1,2,\ldots,n\} \right\}. \end{aligned}$$

Then $\{e_{\underline{i}}=e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_d} \,|\, \underline{i}\in I_{d,n}\}$ forms a basis of $\wedge^d k^n$. Denote its dual basis in $(\wedge^d k^n)^*$ by $\{p_{\underline{i}} \,|\, \underline{i}\in I_{d,n}\}$

$$p_{\underline{i}}(e_{\underline{j}})=\begin{cases} 1, & \text{if} \quad \underline{i}=\underline{j}, \\ 0, & \text{if} \quad \underline{i}\neq\underline{j}. \end{cases}$$

Then, $\{p_{\underline{i}} \,|\, \underline{i}\in I_{d,n}\}$ can be viewed as the homogeneous (projective) coordinates on $\mathbb{P}(\wedge^d k^n)$. Let $\mathrm{Gr}(d,n)$ be the Grassmannian variety formed by the d-dimensional subspace of k^n (if in the scheme-theoretical language, the closed points only). There is the famous Plücker embedding

$$\begin{aligned} \mathrm{Gr}(d,n) &\rightarrow \mathbb{P}(\wedge^d k^n)=\mathbb{P}^{\binom{n}{d}-1}, \\ \mathrm{Span}\{\boldsymbol{v_1},\boldsymbol{v_2},\cdots,\boldsymbol{v_d}\} &\mapsto [\boldsymbol{v_1}\wedge\boldsymbol{v_2}\wedge\cdots\wedge\boldsymbol{v_d}]. \end{aligned}$$

Additionally, $\{p_{\underline{i}} \,|\, \underline{i}\in I_{d,n}\}$ can be regarded as the homogeneous coordinates on $\mathrm{Gr}(d,2n)$. Everything discussed in this article is under this Plücker embedding.

Let

$$\boldsymbol{J}=\begin{bmatrix} & & 0 & & 1 \\ & & & \cdots & \\ & & 1 & & 0 \\ 0 & & -1 & & \\ & \cdots & & & \\ -1 & & 0 & & \end{bmatrix}_{2n\times 2n}.$$

Then, the symplectic Grassmannian is exactly

$$\mathrm{Gr}^C(d,2n)=\left\{ V\in\mathrm{Gr}(d,2n) \,|\, V\perp \boldsymbol{J}V \right\}, \quad 1\leq d\leq n,$$

where, for the column vectors $\boldsymbol{u},\boldsymbol{v}\in k^{2n}$, $\boldsymbol{u}\perp\boldsymbol{v}$ means $\boldsymbol{u}^T\boldsymbol{v}=0$. We employ the superscript C because of its connection to the classic linear algebraic group of type C, i.e., the symplectic group Sp_{2n}. In contrast, we sometimes use a superscript A for objects corresponding to the special linear group SL_{2n}.