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Carleson Measure Associated with the Fractional Heat Semigroup of Schrödinger Operator

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Abstract. Let $L=-\Delta+V$ be a Schrödinger operator, where Δ is the Laplacian on $\mathbb{R}^d$ and the nonnegative potential V belongs to the reverse Hölder class $B_{d/2}$. In this paper, we define a new version of Carleson measure associated with the fractional heat semigroup of Schrödinger operator L. We will characterize the Campanato spaces and the predual spaces of the Hardy spaces by the new Carleson measure.

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Key words: Schrödinger operator, reverse Hölder class, Carleson measure, fractional heat semigroup, Campanato spaces.

1 Introduction

The Schrödinger operators with potential satisfying the reverse Hölder inequality have been studied by various authors. Some basic results are established in Fefferman [8], Zhong [18] and Shen [12]. The Hardy type spaces H_L^p, $d/(d+\delta)<p\leq 1$ for some $\delta>0$, and BMO type space BMO_L associated with a Schrödinger operator L are studied by Dziubański-Zienkiewicz [5,6] and Dziubański *et al.* [4]. In this

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article, we investigate fractional heat semigroup related to the Schrödinger operator L, then we use it to define a new version of Carleson measure to characterize the dual spaces and predual spaces of the Hardy space H_L^p, $d/(d+\delta)<p<1$.

Let $L=-\Delta+V$ be a Schrödinger operator on $\mathbb{R}^d$, $d\geq 3$, where Δ is the Laplacian and $V\not\equiv 0$ is a nonnegative potential belonging to the reverse Hölder class B_q for some $q\geq d/2$, i.e.,

$$\left(\frac{1}{|B|}\int_B V^q(x)dx\right)^{1/q}\leq C\left(\frac{1}{|B|}\int_B V(x)dx\right) \quad \text{for every ball} \quad B. \tag{1.1}$$

Without loss of generalization, we assume that $V\in B_{q_0}$ for some $d/2<q_0<d$ and set $\delta_0=2-d/q_0$ and $\delta=\min(1,\delta_0)\leq 1$, and throughout the paper we keep this assumption and the meanings of q_0, δ_0 and δ.

Let $\{T_t^L\}_{t>0}=\{e^{-tL}\}_{t>0}$ be the semigroup of linear operators generated by $-L$ and $K_t^L(x,y)$ be their kernels. Since V is nonnegative, the Feynman-Kac formula implies that

$$0\leq K_t^L(x,y)\leq K_t(x-y)=(4\pi t)^{-d/2}e^{-(4t)^{-1}|x-y|^2}, \tag{1.2}$$

where $K_t(x)$ is the convolution kernels of the heat semigroup $\{T_t\}_{t>0}=\{e^{t\Delta}\}_{t>0}$. The estimate (1.2) can be improved as follows. We introduce the auxiliary function $\rho(x,V)=\rho(x)$ defined by

$$\rho(x)=\sup\left\{r>0:\ \frac{1}{r^{d-2}}\int_{B(x,r)}V(y)dy\leq 1\right\}. \tag{1.3}$$

It is well known that $0<\rho(x)<\infty$ and there exists $k_0\geq 1$ such that

$$\frac{1}{C}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0}\leq\frac{\rho(y)}{\rho(x)}\leq C\left(1+\frac{|x-y|}{\rho(x)}\right)^{k_0/(k_0+1)}. \tag{1.4}$$

In particular, $\rho(y)\sim\rho(x)$ if $|x-y|<C\rho(x)$ (cf. [12, Lemma 1.4]). Then we have the following estimates for $K_t^L(x,y)$.

Proposition 1.1 ([7, Theorem 4.10]). *For every $N>0$, there is a constant $C_N>0$ such that*

$$K_t^L(x,y)\leq C_N t^{-d/2}e^{-(5t)^{-1}|x-y|^2}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

Proposition 1.2 ([7, Proposition 4.11]). *For every $N>0$, there exist $C_N>0$ and $0<\delta'<\delta$ such that, for all $|h|\leq\sqrt{t}$,*

$$\left|K_t^L(x+h,y)-K_t^L(x,y)\right|\leq C_N\left(\frac{|h|}{\sqrt{t}}\right)^{\delta'}t^{-d/2}e^{-At^{-1}|x-y|^2}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$