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Schur Complement Based Preconditioners for Twofold and Block Tridiagonal Saddle Point Problems

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Abstract. In this paper, we consider using Schur complements to design preconditioners for twofold and block tridiagonal saddle point problems. One type of the preconditioners are based on the nested (or recursive) Schur complement, the other is based on an additive type Schur complement after permuting the original saddle point systems. We analyze different preconditioners incorporating the exact Schur complements. We show that some of them will lead to positively stable preconditioned systems if proper signs are selected in front of the Schur complements. These positive-stable preconditioners outperform other preconditioners if the Schur complements are further approximated inexactly. Numerical experiments for a 3-field formulation of the Biot model are provided to verify our predictions.

AMS subject classifications: 65F08, 65M22, 74F20

Key words: Schur complement, block tridiagonal systems, positively stable preconditioners, Routh-Hurwitz stability criterion.

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1 Introduction

Many application problems will lead to twofold and/or block tridiagonal saddle point linear systems. Important examples include mixed formulations of the Biot model [1,9,22,26,33], the coupling of fluid flow with porous media flow [10,21,29], hybrid discontinuous Galerkin approximation of Stokes problem [17], liquid crystal problem [3,36] and optimization problems [23,26,28,34,38]. Some of these problems (or after permutations) will lead to a twofold saddle point problem [3,12,21,23,24,38,40,41] (or the so-called double saddle point problem) of the following form:

$$\mathcal{A}x=\begin{bmatrix} A_1 & B_1^T & 0 \\ C_1 & -A_2 & B_2^T \\ 0 & C_2 & A_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}. \tag{1.1}$$

A negative sign in front of A_2 is just for the ease of notation. After simple permutations, the system matrix of (1.1) can be rewritten into the following form:

$$\mathcal{A}=\begin{bmatrix} A_1 & 0 & B_1^T \\ 0 & A_3 & C_2 \\ C_1 & B_2^T & -A_2 \end{bmatrix}. \tag{1.2}$$

We call the system matrix in (1.2) permutation-equivalent to that in (1.1). Without causing confusion, we continue to use the notation $\mathcal{A}$ for the permuted matrix (1.2). The linear system in (1.2) arises naturally from the domain decomposition methods [30,39]. In this work, we only assume that A_1 is invertible and the global system matrix $\mathcal{A}$ is invertible. Many special cases, e.g. if $A_2=0$, or $A_3=0$, or $A_2=A_3=0$ can be cast into the above forms of twofold saddle point systems. Our discussions will try to cover all these special cases.

The above 3-by-3 block linear problems (1.1) and (1.2) can be naturally extended to n-tuple cases. For example, when the system matrix in (1.1) is extended to the n-tuple case, it is the block tridiagonal systems discussed in [38]. When the system matrix in (1.2) is extended to the n-tuple case, it corresponds to the linear system resulting from the domain decomposition method for elliptic problems with $n-1$ subdomains. In many references, these linear systems are assumed to be symmetric. No matter whether it is symmetric or not, $\mathcal{A}$ is generally indefinite. For solving such a system in large-scale computations, Krylov subspace methods with preconditioners are usually applied. The analysis in [25,31] indicates that one should employ Schur complement based preconditioners [5,15,16,27] and