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Invariance of Conjugate Normality Under Similarity

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Abstract. An operator T on a separable, infinite dimensional, complex Hilbert space $\mathcal{H}$ is called conjugate normal if $C|T|C=|T^*|$ for some conjugate linear, isometric involution C on $\mathcal{H}$. This paper focuses on the invariance of conjugate normality under similarity. Given an operator T, we prove that every operator A similar to T is conjugate normal if and only if there exist complex numbers λ_1,λ_2 such that $(T-\lambda_1)(T-\lambda_2)=0$.

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1 Introduction

Let $\mathcal{H}$ be a separate, complex Hilbert space with $\dim\mathcal{H}=\infty$. We write $\mathcal{B}(\mathcal{H})$ for the collection of all bounded linear operators on $\mathcal{H}$.

Definition 1.1. *Let $C:\mathcal{H}\to\mathcal{H}$ be a map. We say that C is a conjugation if*

(i) $C(ax+y)=\overline{a}Cx+Cy$ for all $x,y\in\mathcal{H}$ and any $a\in\mathbb{C}$,

(ii) C is bijective with $C^{-1}=C$,

(iii) $\langle Cx,Cy\rangle=\langle y,x\rangle$ for all $x,y\in\mathcal{H}$.

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Definition 1.2. *Let $T\in\mathcal{B}(\mathcal{H})$. We say that T is conjugate normal if there exists a conjugation C on $\mathcal{H}$ so that $C|T|C=|T^*|$ (in this case, T is said to be C-normal).*

The study of C-normal operators was initiated by Ptak *et al.* [17], where basic properties of conjugate normal operators are developed. The term "conjugate normal" was first used in [15]. The class of conjugate normal operators includes complex symmetric operators. An operator $T\in\mathcal{B}(\mathcal{H})$ is called complex symmetric if there exists a conjugation C on $\mathcal{H}$ so that $CTC=T^*$. We refer the reader to [4–7, 9, 11, 16, 22, 25] for more results concerning complex symmetric operators.

The conjugate normality is quite different from the complex symmetry. On one hand, a complex symmetric operator T must be biquasitriangular (i.e., $\operatorname{ind}(T-z)=0$ whenever $T-z$ is semi-Fredholm). However, a conjugate normal operator is not necessarily biquasitriangular (see [20, Example 4.1]). Also, the class of conjugate normal operators contains skew symmetric operators [23], which differs a lot from complex symmetric operators. On the other hand, it can be seen from [6, 8, 14, 25] that conjugate normality differs a lot from the complex symmetry for weighted shifts and partial isometries.

Recently the class of conjugate normal operators has received a lot of attention. Conjugate normal weighted shifts and partial isometries are classified in [14, 15]. A refined polar decomposition of conjugate normal operators was provided in [20]. In [18], the Cartesian decomposition for conjugate normal operator was established. A representation of compact conjugate normal operators was provided in [19]. Two recent papers [12, 13] are devoted to studying operator transforms of conjugate normal operators.

The aim of this paper is to explore the invariance of conjugate normality under similarity. Recall that two operator $A,B\in\mathcal{B}(\mathcal{H})$ are said to be similar if $AX=XB$ for some invertible $X\in\mathcal{B}(\mathcal{H})$, denoted by $A\sim B$. For $A\in\mathcal{B}(\mathcal{H})$, we denote $\mathcal{S}(A)=\{X\in\mathcal{B}(\mathcal{H}):A\sim X\}$ and call $\mathcal{S}(A)$ the similarity orbit of A.

In general, the conjugate normality is not stable under similarity.

Example 1.1. Let $T,A\in\mathcal{B}(\mathbb{C}^3)$ with

$$T=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\quad A=\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}$$

relative to the canonical orthonormal basis $\{e_1,e_2,e_3\}$ of $\mathbb{C}^3$. Then $T\sim A$, and T is conjugate normal with respect to the conjugation C on $\mathbb{C}^3$ given by

$$Ce_i=e_{4-i},\quad i=1,2,3.$$