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Reducing Subspaces of Toeplitz Operators on N_φ-type Quotient Modules on the Torus*

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Abstract: In this paper, we prove that the Toeplitz operator with finite Blaschke product symbol $S_{\psi(z)}$ on N_φ has at least m non-trivial minimal reducing subspaces, where m is the dimension of $H^2(\Gamma_\omega) \ominus \varphi(\omega)H^2(\Gamma_\omega)$. Moreover, the restriction of $S_{\psi(z)}$ on any of these minimal reducing subspaces is unitary equivalent to the Bergman shift M_z.

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1 Introduction

Let D denote the open unit disk in the complex plane $\mathbb{C}$ and T^2 be cartesian product of two copies of T, where T is the unit circle. It is well known that T^2, as usually is endowed with the rotation invariant Lebesgue measure, is the distinguished boundary of D^2. Let $dm(z)$ denote the normalized Lebesgue measure on T and $\mathrm{d}m(z)\mathrm{d}m(\omega)$ be the product measure on the torus T^2. The Bergman space is denoted by $L_a^2(D)$ and Bergman shift is denoted by M_z. Let $H^2(\Gamma^2)$ be the Hardy space on the two dimensional torus T^2. We denote by z and ω the coordinate functions. Shift operators T_z and T_ω on $H^2(\Gamma^2)$ are defined by $T_z f = zf$ and $T_\omega f = \omega f$ for $f \in H^2(\Gamma^2)$. Clearly, both T_z and T_ω have infinite multiplicity. A closed subspace M of $H^2(\Gamma^2)$ is called a submodule (over the algebra $H^\infty(D^2)$), if it is invariant under multiplications by functions $H^\infty(D^2)$. Equivalently, M is a submodule if it is invariant for both T_z and T_ω. The quotient space $N : H^2(\Gamma^2) \ominus M$ is called a quotient module. Clearly, $T_z^* N \subset N$ and $T_\omega^* N \subset N$. In the study here, it is necessary to distinguish the classical Hardy space in the variable z and that in the variable ω, for which we denote

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by $H^2(\Gamma_z)$ and $H^2(\Gamma_\omega)$, respectively. In this paper, we look at submodules of the form $[z\text{-}\varphi(\omega)]$, where φ is an inner function in $H^2(\Gamma_\omega)$ and $[z\text{-}\varphi(\omega)]$ is the closure of $(z\text{-}\varphi)H^\infty(\Gamma^2)$ in $H^2(\Gamma^2)$. For simplicity we denote $[z\text{-}\varphi(\omega)]$ by M_φ. $N_\varphi = H^2(\Gamma^2) \ominus M_\varphi$ denote N_φ-type quotient modules on the torus. For a function $\psi \in H^\infty(D^2)$, we define the Toeplitz operator S_ψ on N_φ with symbol ψ by

$$S_\psi(f) = P_{N_\varphi}(\psi f), \qquad \forall f \in N_\varphi,$$

where P_{N_φ} is a projection from $H^2(\Gamma^2)$ to N_φ.

The quotient module N_φ has a very rich structure. In deed, when φ is inner, N_φ can be identified with the tensor product of two well-known classical spaces, namely the quotient space $H^2(\Gamma) \ominus \varphi H^2(\Gamma)$ and the Bergman space $L_a^2(D)$. Clearly, when $\varphi(\omega) = \omega$, N_φ is unitary equivalent to $L_a^2(D)$. In fact, it is shown in [1] that $\{T_z, T_\omega, H^2(\Gamma^2)\}$ is the minimal super-isometrical dilation of M_z. Then the reducible problem of Toeplitz operator with finite Blaschke product on the Bergman space is turned to the reducible problem of Toeplitz operator with finite Blaschke product on N_ω. It is obtained in [1] that Toeplitz operator with finite Blaschke product $S_{\psi(z)}$ on N_ω has at least a reducing subspace M, moreover, $S_\psi|_M \cong M_z$. In this paper, we prove that when φ is a non-constant inner function, the conclusion like that in [1] is also true.

2 Preliminaries

In order to prove the main theorem, we need the following lemma.

Lemma 2.1[2] *Let $\varphi(\omega)$ be a one variable non-constant inner function and $\{\lambda_k(\omega) :\ k = 1, 2, \cdots, m\}$ be an orthonormal basis of $H^2(\Gamma_\omega) \ominus \varphi(\omega)H^2(\Gamma_\omega)$, and*

$$e_j(z, \omega) = \frac{\omega^j + \omega^{j-1}z + \cdots + z^j}{\sqrt{j+1}} \qquad (j = 0, 1, \cdots).$$

Let

$$E_{k,j} = \lambda_k(\omega)e_j(z,\ \varphi(\omega)).$$

Then $\{E_{k,j} :\ k = 1, 2, \cdots, m;\ j = 0, 1, \cdots\}$ is an orthonormal basis for N_φ.

Lemma 2.2[2] *There exists a unitary operator U,*

$$U :\ N_\varphi \longrightarrow (H^2(\Gamma_\omega) \ominus \varphi(\omega)H^2(\Gamma_\omega)) \otimes L_a^2(D),$$
$$E_{k,j} \longmapsto \lambda_k(\omega)\sqrt{j+1}\xi^j$$

such that

$$US_z = (I \bigotimes M_z)U,$$

where I is an identity map on $H^2(\Gamma_\omega)\ominus\varphi(\omega)H^2(\Gamma_\omega)$.

Lemma 2.3[1] *Suppose that*

$$\varphi(\omega) = \omega, \qquad \psi(z) = z\prod_{l=1}^{N-1}\frac{z-\alpha_l}{1-\bar{\alpha}_l z} \quad (|\ \alpha_l\ | > 0,\ \alpha_l \neq \alpha_k (\forall l \neq k),\ 1 \leq l, k \leq N-1).$$

Then there exists a unique unit vector e such that

$$e \in \ker T^*_{\psi(z)} \cap \ker T^*_{\psi(\omega)} \cap N_\varphi = \ker S^*_{\psi(z)} \cap \ker S^*_{\psi(\omega)}, \tag{2.1}$$

$$(\psi(z) + \psi(\omega))e \in N_\varphi. \tag{2.2}$$

Lemma 2.4[3] *Suppose that φ is the inner function. Then the boundary value of φ is the measurable transformation on T, $m\varphi^{-1}$ is the measure on T. And the Radon-Nikodym derivative of $m\varphi^{-1}$ is equal to poisson's kernel, i.e.,*

$$\frac{\mathrm{d}m(\varphi^{-1}(t))}{\mathrm{d}m(t)} = p_a(t) = \mathrm{Re}\Big(\frac{t+a}{t-a}\Big) \qquad \Big(a = \int_0^{2\pi} \varphi(e^{i\theta})\mathrm{d}m(\theta)\Big).$$

Lemma 2.5 *Suppose that $\lambda \in D$ and $\eta_\lambda = \dfrac{\lambda - z}{1 - \overline{\lambda} z}$. Then the Toeplitz operator S_{η_λ} on N_φ is unitary equivalent to S_z, i.e., $S_{\eta_\lambda} \cong S_z$.*

Proof. There exists a unitary transformation (see [2]),

$$W_1: \ L^2_a(D) \longrightarrow L^2_a(D),$$

$$W_1(h) = (1 - |\lambda|^2) h \circ \eta_\lambda \cdot \widetilde{k}_\lambda \qquad \Big(\widetilde{k}_\lambda = \frac{1}{(1 - \overline{\lambda} z)^2}\Big)$$

such that

$$W_1 M_{\eta_\lambda} W_1^* = M_z.$$

Let

$$W_2 = I \otimes W_1.$$

Then it is clear that W_2 is the unitary transformation on $(H^2(\Gamma_\omega) \ominus \varphi(\omega) H^2(\Gamma_\omega)) \otimes L^2_a(D)$. What's more,

$$\begin{aligned} W_2(I \otimes M_{\eta_\lambda}) &= (I \otimes W_1)(I \otimes M_{\eta_\lambda}) \\ &= I \otimes (W_1 M_{\eta_\lambda}) \\ &= I \otimes (M_z W_1) \\ &= (I \otimes M_z)(I \otimes W_1) \\ &= (I \otimes M_z) W_2. \end{aligned}$$

Thus

$$I \otimes M_{\eta_\lambda} \cong I \otimes M_z.$$

By Lemma 2.2, there exists a unitary operator U such that

$$U S_z = (I \otimes M_z) U.$$

By the function calculus, it is well known that

$$\begin{aligned} U S_{\eta_\lambda} U^* &= U \eta_\lambda(S_z) U^* \\ &= \eta_\lambda(U S_z U^*) \\ &= \eta_\lambda(I \otimes M_z) \\ &= I \otimes M_{\eta_\lambda}. \end{aligned}$$

Let

$$W_3 = U^* W_2 U.$$

Then

$$\begin{aligned} W_3 S_{\eta_\lambda} W_3^* &= U^* W_2 U S_{\eta_\lambda} U^* W_2^* U \\ &= U^* W_2 (I \otimes M_{\eta_\lambda}) W_2^* U \\ &= U^* (I \otimes M_z) U \\ &= S_z. \end{aligned}$$

Therefore

$$S_{\eta_\lambda} \cong S_z.$$

The proof is completed.

Lemma 2.6 *Suppose that ψ is a finite Blaschke product and $\psi_\lambda = \psi \circ \eta_\lambda$. If S_{ψ_λ} has at least a non-trivial reducing subspace on which the restriction of S_{ψ_λ} is unitary equivalent to the Bergman shift, then S_ψ also has at least a non-trivial reducing subspace on which the restriction of S_ψ is unitary equivalent to the Bergman shift.*

Proof. Let M be the non-trivial reducing subspace of S_{ψ_λ} and there exists a unitary transformation $W\colon\ M \longrightarrow L_a^2(D)$ such that

$$W S_{\psi_\lambda}|_M = M_z W.$$

Because

$$\eta_\lambda \circ \eta_\lambda(\omega) = \omega,$$

we have

$$\psi = \psi_\lambda \circ \eta_\lambda.$$

By Lemma 2.5,

$$W_3 S_{\eta_\lambda} W_3^* = S_z.$$

By the function calculus,

$$\begin{aligned} W_3 S_\psi W_3^* &= W_3 S_{\psi_\lambda} \circ \eta_\lambda W_3^* \\ &= W_3 \psi_\lambda (S_{\eta_\lambda}) W_3^* \\ &= \psi_\lambda (W_3 S_{\eta_\lambda} W_3^*) \\ &= \psi_\lambda (S_z) \\ &= S_{\psi_\lambda}, \end{aligned}$$

i.e.,

$$S_\psi \cong S_{\psi_\lambda}.$$

Let

$$M_1 = W_3^* M.$$

Then M_1 is the non-trivial reducing subspace of S_ψ. Let

$$W_4 = W W_3.$$

It is easy to prove that

$$W_4 S_\psi|_{M_1} = M_z W_4,$$

i.e.,

$$S_\psi|_{M_1} \cong M_z.$$

The proof is completed.

Lemma 2.7[1] *Suppose that $\psi(z)$ is the finite Blaschke product having zeros with multiplicity greater than one and $\eta_\lambda = \dfrac{\lambda - z}{1-\overline{\lambda}z}$. Let $\psi_\lambda(z) = (\eta_\lambda \circ \psi)(z)$. Then there exists a $\lambda \in D$ such that $\psi_\lambda(z)$ has distinct zeros.*

3 Principal Results and Proofs

In this section we give our main results.

Theorem 3.1 *Suppose that $\varphi(\omega)$ be a one variable non-constant inner function, and*

$$\psi(z) = z\prod_{l=1}^{N-1}\frac{z-\alpha_l}{1-\bar{\alpha}_l z} \qquad (|\alpha_l|>0,\ \alpha_l \neq \alpha_k\ (\forall l \neq k),\ 1 \leq l,k \leq N-1).$$

Then there exists a unique unit vector e' such that

$$e' \in \ker T^*_{\psi(z)} \cap \ker T^*_{\psi(\varphi(\omega))} \cap N_\varphi = \ker S^*_{\psi(z)} \cap \ker S^*_{\psi(\varphi(\omega))}, \tag{3.1}$$

$$(\psi(z)+\psi(\varphi(\omega)))e' \in N_\varphi. \tag{3.2}$$

Proof. Picking the unit vector e in Lemma 2.3, then we have

$$e \in H^2(T^2) \ominus [z\text{-}\omega] = N_\omega.$$

By Lemma 2.1, $\{e_j(z,\ \omega):\ j \geq 0\}$ is an orthonormal basis for $H^2(T^2)\ominus[z\text{-}\omega]$. Then there exsits a sequence of constant numbers $\{k_j\}$, such that

$$e = \sum_{j=0}^{\infty} k_j e_j(z,\ \omega).$$

Let

$$e'(z,\ \omega) = \lambda_1(\omega)e(z,\ \varphi(\omega)).$$

Then obviously

$$e'(z,\ \omega) = \sum_{j=0}^{\infty} k_j(\lambda_1(\omega)e_j(z,\ \varphi(\omega))) = \sum_{j=0}^{\infty} k_j E_{1,j} \in N_\varphi \tag{3.3}$$

and

$$\|e'\|^2 = \sum_{j=0}^{\infty}|k_j|^2 = \|e\|^2 = 1.$$

Because

$$e \in \ker T^*_{\psi(z)} \Longleftrightarrow T^*_{\psi(z)}e(z,\ \omega) = 0,$$

i.e.,

$$\int_T\int_T |T^*_{\psi(z)}e(z,\ \omega)|^2 \mathrm{d}m(z)\mathrm{d}m(\omega) = 0,$$

then

$$\begin{aligned} &\|T^*_{\psi(z)}e(z,\ \varphi(\omega))\|^2 \\ = &\int_T\int_T |T^*_{\psi(z)}e(z,\ \varphi(\omega))|^2 \mathrm{d}m(z)\mathrm{d}m(\omega) \quad (\text{let } t=\varphi(\omega)) \\ = &\int_T\int_T |T^*_{\psi(z)}e(z,\ t)|^2 \frac{\mathrm{d}m(\varphi^{-1}(t))}{\mathrm{d}m(t)}\mathrm{d}m(z)\mathrm{d}m(t). \end{aligned}$$

Let

$$a=\int_0^{2\pi}\varphi(e^{i\theta})\mathrm{d}m(\theta).$$

Then by Lemma 2.4,

$$\begin{aligned} \Big|\frac{\mathrm{d}m(\varphi^{-1}(t))}{\mathrm{d}m(t)}\Big| &= |p_a(t)| \\ &= \Big|\mathrm{Re}\Big(\frac{t+a}{t-a}\Big)\Big| \\ &\le \Big|\frac{t+a}{t-a}\Big| \\ &\le \frac{1+|a|}{1-|a|} \\ &\le \int_T\int_T |T^*_{\psi(z)}e(z,t)|^2\frac{1+|a|}{1-|a|}\mathrm{d}m(z)\mathrm{d}m(t) \\ &\le \frac{1+|a|}{1-|a|}\int_T\int_T |T^*_{\psi(z)}e(z,t)|^2\mathrm{d}m(z)\mathrm{d}m(t) \\ &= 0. \end{aligned}$$

Thus

$$T^*_{\psi(z)}e(z,\ \varphi(\omega))=0.$$

Then

$$T^*_{\psi(z)}e'(z,\ \omega)=T^*_{\psi(z)}(\lambda_1(\omega)e(z,\ \varphi(\omega)))=\lambda_1(\omega)T^*_{\psi(z)}e(z,\ \varphi(\omega))=0. \tag{3.4}$$

By (3.3) and (3.4),

$$e'\in\ker T^*_{\psi(z)}\cap N_\varphi.$$

We have

$$T^*_{\psi(z)}|_{N_\varphi}=T^*_{\psi(\varphi(\omega))}|_{N_\varphi}.$$

In fact, because $\psi\in A(D)$, it is easy to prove that

$$\psi(z)-\psi(\varphi(\omega))\in[z\text{-}\varphi(\omega)]=M_\varphi,$$

and it is well known that $\psi\in H^\infty(D^2)$. Then for any $g\in H^2(T^2)$, $(\psi(z)-\psi(\varphi(\omega)))g\in M_\varphi$. Therefore,

$$\langle(T^*_{\psi(z)}-T^*_{\psi(\varphi(\omega))})f,\ g\rangle=\langle f,\ (\psi(z)-\psi(\varphi(\omega)))g\rangle=0,\qquad \forall f,g\in N_\varphi,$$

i.e.,

$$T^*_{\psi(z)}|_{N_\varphi}=T^*_{\psi(\varphi(\omega))}|_{N_\varphi}.$$

Then

$$e'\in\ker T^*_{\psi(z)}\cap\ker T^*_{\psi(\varphi(\omega))}\cap N_\varphi.$$

Let

$$\psi_0(z) = \prod_{l=1}^{N-1} \frac{z-\alpha_l}{1-\bar{\alpha_l} z}.$$

By the fact that

$$T_z^* e' = T_{\varphi(\omega)}^* e',$$

moreover the conclusion (3.2) is equivalent to the following:

$$[\psi_0(z) - \psi_0(\varphi(\omega))]e' = [\psi(z) - \psi(\varphi(\omega))]T_z^* e'. \tag{3.5}$$

In fact,

$$\begin{aligned} &(\psi(z) + \psi(\varphi(\omega)))e' \in N_\varphi \\ \Longleftrightarrow\ &(T_z^* - T_{\varphi(\omega)}^*)[(\psi(z) + \psi(\varphi(\omega)))e'] = 0 \\ \Longleftrightarrow\ &[\psi_0(z) - \psi_0(\varphi(\omega))]e' = [\psi(z) - \psi(\varphi(\omega))]T_z^* e'. \end{aligned}$$

Similarly, by (2.2), we have

$$[\psi_0(z) - \psi_0(\omega)]e(z,\ \omega) = [\psi(z) - \psi(\omega)]T_z^* e(z,\ \omega).$$

So

$$\begin{aligned} &\|[\psi_0(z) - \psi_0(\omega)]e(z,\ \omega) - [\psi(z) - \psi(\omega)]T_z^* e(z,\ \omega)\|^2 \\ =\ &\int_T \int_T |[\psi_0(z) - \psi_0(\omega)]e(z,\ \omega) - [\psi(z) - \psi(\omega)]T_z^* e(z,\ \omega)|^2 \mathrm{d}m(z)\mathrm{d}m(\omega) \\ =\ &0. \end{aligned}$$

Then

$$\begin{aligned} &\|[\psi_0(z) - \psi_0(\varphi(\omega))]e(z,\ \varphi(\omega)) - [\psi(z) - \psi(\varphi(\omega))]T_z^* e(z,\ \varphi(\omega))\|^2 \\ =\ &\int_T \int_T |[\psi_0(z) - \psi_0(\varphi(\omega))]e(z,\ \varphi(\omega)) - [\psi(z) - \psi(\varphi(\omega))]T_z^* e(z,\ \varphi(\omega))|^2 \mathrm{d}m(z)\mathrm{d}m(\omega) \\ &(\text{let } t = \varphi(\omega)) \\ =\ &\int_T \int_T |[\psi_0(z) - \psi_0(t)]e(z,\ t) - [\psi(z) - \psi(t)]T_z^* e(z,\ t)|^2 \frac{\mathrm{d}m(\varphi^{-1}(t))}{\mathrm{d}m(t)} \mathrm{d}m(z)\mathrm{d}m(t) \\ =\ &\int_T \int_T |[\psi_0(z) - \psi_0(t)]e(z,\ t) - [\psi(z) - \psi(t)]T_z^* e(z,\ t)|^2 p_a(t)\mathrm{d}m(z)\mathrm{d}m(t) \\ \le\ &\frac{1+|a|}{1-|a|} \int_T \int_T |[\psi_0(z) - \psi_0(t)]e(z,\ t) - [\psi(z) - \psi(t)]T_z^* e(z,\ t)|^2 \mathrm{d}m(z)\mathrm{d}m(t) \\ =\ &0. \end{aligned}$$

Therefore,

$$[\psi_0(z) - \psi_0(\varphi(\omega))]e(z,\ \varphi(\omega)) = [\psi(z) - \psi(\varphi(\omega))]T_z^* e(z,\ \varphi(\omega)).$$

Multiplied by $\lambda_1(\omega)$, we can obtain the conclusion (3.5). The proof is completed.

Remark It is different from Lemma 2.3, e' in the theorem is not unique. We can let

$$e' = \lambda_k(\omega)e(z,\ \varphi(\omega)),$$

where $\lambda_k(\omega)$ is any element of the orthonormal basis of $H^2(\mathit{\Gamma}_\omega) \ominus \varphi(\omega)H^2(\mathit{\Gamma}_\omega)$ in Lemma 2.1.

Theorem 3.2 *Suppose that* $\varphi(\omega)$ *be a one variable non-constant inner function, and*

$$\psi(z) = z\prod_{l=1}^{N-1}\frac{z-\alpha_l}{1-\bar{\alpha}_l z} \qquad (|\alpha_l|>0,\ \alpha_l\neq\alpha_k\ (\forall l\neq k),\ 1\leq l,k\leq N-1).$$

Pick e' *in Theorem* 3.1*. Then*

$$M_{e'} = \overline{\mathrm{span}}\{p'_n(\psi)e' :\ n\geq 0\},$$

where

$$p'_n(\psi) = \psi^n(z) + \psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi(z)\psi^{n-1}(\varphi(\omega)) + \psi^n(\varphi(\omega))$$

is a non-trivial minimal reducing subspace of $S_{\psi(z)}$*. Moreover* $S_{\psi(z)}|_{M_{e'}}$ *is unitary equivalent to Bergman shift* M_z*.*

Proof.

$$\begin{aligned}
&T_z^* p'_n(\psi)e' - T^*_{\varphi(\omega)}p'_n(\psi)e'\\
&= T_z^*[\psi^n(z) + \psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi(z)\psi^{n-1}(\varphi(\omega)) + \psi^n(\varphi(\omega))]e'\\
&\quad - T^*_{\varphi(\omega)}[\psi^n(z) + \psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi(z)\psi^{n-1}(\varphi(\omega)) + \psi^n(\varphi(\omega))]e'\\
&= [\psi_0(z)\psi^{n-1}(z)e' + \psi_0(z)\psi^{n-2}(z)\psi(\varphi(\omega))e' + \cdots + \psi_0(z)\psi^{n-1}(\varphi(\omega))e' + \psi^n(\varphi(\omega))T_z^*e']\\
&\quad - [\psi^n(z)T^*_{\varphi(\omega)}e' + \psi^{n-1}(z)\psi_0(\varphi(\omega))e' + \cdots\\
&\qquad + \psi(z)\psi_0(\varphi(\omega))\psi^{n-2}(\varphi(\omega))e' + \psi_0(\varphi(\omega))\psi^{n-1}(\varphi(\omega))e']\\
&= [\psi_0(z)\psi^{n-1}(z)e' + \psi_0(z)\psi^{n-2}(z)\psi(\varphi(\omega))e' + \cdots + \psi_0(z)\psi^{n-1}(\varphi(\omega))e' + \psi^n(\varphi(\omega))T_z^*e']\\
&\quad - [\psi^n(z)T_z^*e' + \psi^{n-1}(z)\psi_0(\varphi(\omega))e' + \cdots\\
&\qquad + \psi(z)\psi_0(\varphi(\omega))\psi^{n-2}(\varphi(\omega))e' + \psi_0(\varphi(\omega))\psi^{n-1}(\varphi(\omega))e']\\
&= p'_{n-1}(\psi)(\psi_0(z) - \psi_0(\varphi(\omega)))e' + (\psi^n(\varphi(\omega)) - \psi^n(z))T_z^*e'\\
&\quad \text{(by (3.5))}\\
&= p'_{n-1}(\psi)(\psi(z) - \psi(\varphi(\omega)))T_z^*e' + (\psi^n(\varphi(\omega)) - \psi^n(z))T_z^*e'\\
&= (\psi^n(z) - \psi^n(\varphi(\omega)))T_z^*e' + (\psi^n(\varphi(\omega)) - \psi^n(z))T_z^*e'\\
&= 0.
\end{aligned}$$

We have

$$(T_z^* - T^*_{\varphi(\omega)})p'_n(\psi)e' = 0.$$

So

$$p'_n(\psi)e' \in N_\varphi.$$

Also,

$$\begin{aligned}
&S_{\psi(z)}(p'_n(\psi)e')\\
&= q\psi(z)p'_n(\psi)e'\\
&= q\psi(z)[\psi^n(z) + \psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi(z)\psi^{n-1}(\varphi(\omega)) + \psi^n(\varphi(\omega))]e'\\
&= q[\psi^{n+1}(z) + \psi^n(z)\psi(\varphi(\omega)) + \cdots + \psi^2(z)\psi^{n-1}(\varphi(\omega)) + \psi(z)\psi^n(\varphi(\omega))]e'
\end{aligned}$$

$$= q\Big\{\frac{n+1}{n+2}p'_{n+1}(\psi)e' + \frac{1}{n+2}[(\psi^{n+1}(z) - \psi^{n+1}(\varphi(\omega)) + (\psi^n(z) - \psi^n(\varphi(\omega))\psi(\varphi(\omega)) + \cdots$$
$$+ (\psi(z) - \psi(\varphi(\omega))\psi^n(\varphi(\omega))]e'\Big\}$$
$$= \frac{n+1}{n+2}p'_{n+1}(\psi)e' \in M_{e'}, \tag{3.6}$$

and

$$\begin{aligned}
&S^*_{\psi(z)}(p'_n(\psi)e') \\
&= q\overline{\psi(z)}p'_n(\psi)e' \\
&= q\overline{\psi(z)}[\psi^n(z) + \psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi(z)\psi^{n-1}(\varphi(\omega)) + \psi^n(\varphi(\omega))]e' \\
&= q[\psi^{n-1}(z) + \psi^{n-2}(z)\psi(\varphi(\omega)) + \cdots + \psi^{n-1}(\varphi(\omega))]e' + \psi^n(\varphi(\omega))T^*_{\psi(z)}e' \\
&\quad (\text{by } (3.1)) \\
&= q[\psi^{n-1}(z) + \psi^{n-2}(z)\psi(\varphi(\omega)) + \cdots + \psi^{n-1}(\varphi(\omega))]e' \\
&= p'_{n-1}(\psi)e' \in M_{e'}.
\end{aligned} \tag{3.7}$$

Hence by (3.6) and (3.7), $M_{e'}$ is the non-trivial reducing subspace of $S_{\psi(z)}$. Because

$$|\psi(z)| = |\psi(\varphi(\omega))| = 1 \qquad \text{a.e. on } T^2,$$

then

$$p'_n(\psi)\overline{p'_m(\psi)} = \begin{cases} \sum\limits_{k+l=n-m,-n\le k,l\le n} c_{k,l}\psi^k(z)\psi^l(\varphi(\omega)), & \text{if } m > n; \\ \sum\limits_{-n\le k\le n,k\ne 0} c_k\psi^k(z)\psi^{-k}(\varphi(\omega)) + (n+1), & \text{if } m = n \end{cases} \qquad \text{a.e. on } T^2.$$

Since $e' \in \ker T^*_{\psi(z)} \cap \ker T^*_{\psi(\varphi(\omega))} \cap N_\varphi$, it is easy to check

$$\langle p'_n(\psi)e',\ p'_m(\psi)e'\rangle = \begin{cases} 0, & \text{if } m \ne n; \\ n+1, & \text{if } m = n. \end{cases}$$

Therefore, $\Big\{\dfrac{p'_n(\psi)e'}{\sqrt{n+1}} :\ n = 0, 1, \cdots\Big\}$ is an orthonormal basis for $M_{e'}$. By (3.6) we can define a unitary transformation

$$W_1 : M_{e'} \to L^2_a(D),$$
$$\frac{p'_n(\psi)e'}{\sqrt{n+1}} \mapsto \sqrt{n+1}z^n$$

such that

$$W_1 S_{\psi(z)}|_{M_{e'}} = M_z W_1.$$

Hence

$$S_{\psi(z)}|_{M_{e'}} \cong M_z.$$

The proof is completed.

Corollary 3.1 *Suppose that $\varphi(\omega)$ be a one variable non-constant inner function, and*

$$\psi(z) = z\prod_{l=1}^{N-1}\frac{z-\alpha_l}{1-\bar{\alpha}_l z} \qquad (|\alpha_l| > 0,\ \alpha_l \ne \alpha_k\ (\forall l \ne k),\ 1 \le l, k \le N-1).$$

Then the Toeplitz operator $S_{\psi(z)}$ has at least m non-trivial minimal reducing subspaces ($m = \dim(H^2(\Gamma_\omega) \ominus \varphi(\omega)H^2(\Gamma_\omega))$ and m may be $+\infty$). Moreover, the restriction of $S_{\psi(z)}$ on any of these minimal reducing subspaces is unitary equivalent to the Bergman shift M_z.

Theorem 3.3 *Suppose that $\psi(z)$ is a common finite Blaschke product. Then $S_{\psi(z)}$ has at least a non-trivial minimal reducing subspace on which the restriction of $S_{\psi(z)}$ is unitary equivalent to the Bergman shift.*

Proof. Suppose that $\psi(z)$ is a finite Blaschke product of order N. If $\psi(z)$ is the finite Blaschke product having zero with multiplicity greater than one, then, by Lemma 2.7, there exists a $\lambda_0 \in D$ such that $\psi_{\lambda_0}(z)$ has distinct zeros, where

$$\psi_{\lambda_0}(z) = (\eta_{\lambda_0} \circ \psi)(z), \qquad \eta_{\lambda_0}(z) = \frac{\lambda_0 - z}{1 - \overline{\lambda_0} z}.$$

If $\psi_{\lambda_0}(0) \neq 0$, let

$$\psi_{\lambda_1}(z) = (\psi_{\lambda_0} \circ \eta_{\lambda_1})(z).$$

Suppose that λ_1 satisfies the condition

$$\psi_{\lambda_0}(\lambda_1) = 0.$$

Then

$$\psi_{\lambda_1}(0) = \psi_{\lambda_0}(\eta_{\lambda_1}(0)) = \psi_{\lambda_0}(\lambda_1) = 0.$$

Hence $\psi_{\lambda_1}(z)$ is the case in Theorem 3.2. Therefore, $S_{\psi_{\lambda_1}}(z)$ has at least a reducing subspace on which the restriction of $S_{\psi_{\lambda_1}}(z)$ is unitary equivalent to the Bergman shift. By Lemma 2.6, $S_{\psi_{\lambda_0}}(z)$ also has at least a reducing subspace, denoted by M and

$$W_1 S_{\psi_{\lambda_0}}|_M = M_z W_1.$$

By $\eta_\lambda \circ \eta_\lambda(\omega) = \omega$ and function calculus, one has

$$S_{\psi(z)} = S_{\eta_{\lambda_0} \circ \psi_{\lambda_0}(z)} = \eta_{\lambda_0}(S_{\psi_{\lambda_0}(z)}) = \frac{\lambda_0 - S_{\psi_{\lambda_0}(z)}}{1 - \bar{\lambda}_0 S_{\psi_{\lambda_0}}(z)}.$$

So M is the reducing subspace of $S_{\psi(z)}$. We have

$$\begin{aligned} W_1 S_{\psi(z)} W_1^* &= W_1 S_{\eta_{\lambda_0} \circ \psi_{\lambda_0}(z)} W_1^* \\ &= W_1 \eta_{\lambda_0}(S_{\psi_{\lambda_0}}) W_1^* \\ &= \eta_{\lambda_0}(W_1 S_{\psi_{\lambda_0}} W_1^*) \\ &= \eta_{\lambda_0}(M_z) \\ &= M_{\eta_{\lambda_0}}. \end{aligned}$$

By [1], there exists a unitary transformation W_2 such that

$$W_2 M_{\eta_{\lambda_0}} W_2^* = M_z.$$

Define a unitary transformation:

$$W: \ M \to L_a^2(D) W = W_2 W_1.$$

Therefore,

$$W S_\psi W^* = W_2 W_1 S_\psi W_1^* W_2^* = W_2 M_{\eta_{\lambda_0}} W_2^* = M_z,$$

i.e.,

$$S_\psi|_M \cong M_z.$$

Corollary 3.2 *Suppose that $\varphi(\omega)$ be a one variable non-constant inner function and $\psi(z)$ is a common finite Blaschke product. Then $S_{\psi(z)}$ has at least m non-trivial minimal reducing* *subspaces* ($m = \dim(H^2(\Gamma_\omega) \ominus \varphi(\omega)H^2(\Gamma_\omega))$ *and* m *may be* $+\infty$)*. Moreover, the restriction of* $S_{\psi(z)}$ *on any of these minimal reducing subspaces is unitary equivalent to the Bergman shift* M_z.

Proof. It can be easily obtained by Corollary 3.1 and Theorem 3.3.

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