Reducing Subspaces of Toeplitz Operators on $N_\phi$-type Quotient Modules on the Torus

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Abstract: In this paper, we prove that the Toeplitz operator with finite Blaschke product symbol $S_\psi(z)$ on $N_\phi$ has at least $m$ non-trivial minimal reducing subspaces, where $m$ is the dimension of $H^2(\Gamma_\omega) \ominus \varphi(\omega)H^2(\Gamma_\omega)$. Moreover, the restriction of $S_\psi(z)$ on any of these minimal reducing subspaces is unitary equivalent to the Bergman shift $M_z$.

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1 Introduction

Let $D$ denote the open unit disk in the complex plane $\mathbb{C}$ and $T^2$ be cartesian product of two copies of $T$, where $T$ is the unit circle. It is well known that $T^2$, as usually is endowed with the rotation invariant Lebesgue measure, is the distinguished boundary of $D^2$. Let $dm(z)dm(\omega)$ denote the normalized Lebesgue measure on $T$ and $dm(z)dm(\omega)$ be the product measure on the torus $T^2$. The Bergman space is denoted by $L^2_\omega(D)$ and Bergman shift is denoted by $M_z$. Let $H^2(I^2)$ be the Hardy space on the two dimensional torus $T^2$. We denote by $z$ and $\omega$ the coordinate functions. Shift operators $T_z$ and $T_\omega$ on $H^2(I^2)$ are defined by $T_zf = zf$ and $T_\omega f = \omega f$ for $f \in H^2(I^2)$. Clearly, both $T_z$ and $T_\omega$ have infinite multiplicity. A closed subspace $M$ of $H^2(I^2)$ is called a submodule (over the algebra $H^\infty(D^2)$), if it is invariant under multiplications by functions $H^\infty(D^2)$. Equivalently, $M$ is a submodule if it is invariant for both $T_z$ and $T_\omega$. The quotient space $N : H^2(I^2) \ominus M$ is called a quotient module. Clearly, $T_z^* N \subset N$ and $T_\omega^* N \subset N$. In the study here, it is necessary to distinguish the classical Hardy space in the variable $z$ and that in the variable $\omega$, for which we denote

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Lemma 2.2 Let \( \varphi(\omega) \) be a one variable non-constant inner function and \( \{ \lambda_k(\omega) : k = 1, 2, \cdots, m \} \) be an orthonormal basis of \( H^2(\Gamma_\omega) \odot \varphi(\omega)H^2(\Gamma_\omega) \), and

\[
\epsilon_j(z, \omega) = \frac{\omega^j + \omega^{j-1}z + \cdots + z^j}{\sqrt{j+1}} \quad (j = 0, 1, \cdots).
\]

Let

\[
E_{k,j} = \lambda_k(\omega)\epsilon_j(z, \varphi(\omega)).
\]

Then \( \{ E_{k,j} : k = 1, 2, \cdots, m; j = 0, 1, \cdots \} \) is an orthonormal basis for \( N_\varphi \).

Lemma 2.3 Suppose that

\[
\varphi(\omega) = \omega, \quad \psi(z) = z \prod_{l=1}^{N-1} \frac{z - \alpha_l}{1 - \alpha_l z} \quad (|\alpha_l| > 0, \ \alpha_l \neq \alpha_k \forall l \neq k, \ 1 \leq l, k \leq N - 1).
\]
Then there exists a unique unit vector $e$ such that
\[ e \in \ker T_{\psi(z)}^* \cap \ker T_{\psi(\omega)}^* \cap N_\varphi = \ker S_{\psi(z)}^* \cap \ker S_{\psi(\omega)}^*, \quad (2.1) \]
\[ (\psi(z) + \psi(\omega))e \in N_\varphi. \quad (2.2) \]

Lemma 2.4 \[3\] Suppose that $\varphi$ is the inner function. Then the boundary value of $\varphi$ is the measurable transformation on $T$, $m_\varphi^{-1}$ is the measure on $T$. And the Radon-Nikodym derivative of $m_\varphi^{-1}$ is equal to Poisson’s kernel, i.e.,
\[ \frac{dm(\varphi^{-1}(t))}{dm(t)} = p_\alpha(t) = \text{Re} \left( \frac{t + a}{t - a} \right) \quad \left( a = \int_0^{2\pi} \varphi(e^{i\theta})dm(\theta) \right). \]

Lemma 2.5 Suppose that $\lambda \in D$ and $\eta_\lambda = \frac{\lambda - z}{1 - \lambda z}$. Then the Toeplitz operator $S_{\eta_\lambda}$ on $N_\varphi$ is unitary equivalent to $S_z$, i.e., $S_{\eta_\lambda} \cong S_z$.

Proof. There exists a unitary transformation (see [2]),
\[ W_1 : L_2^2(D) \rightarrow L_2^2(D), \]
\[ W_1(h) = (1 - |\lambda|^2)h \circ \eta_\lambda \cdot \tilde{k}_\lambda \quad \left( \tilde{k}_\lambda = \frac{1}{(1 - \lambda z)^2} \right) \]
such that
\[ W_1 M_{\eta_\lambda} W_1^* = M_z. \]

Let
\[ W_2 = I \otimes W_1. \]

Then it is clear that $W_2$ is the unitary transformation on $(H^2(\Gamma_\omega) \oplus \varphi(\omega)H^2(\Gamma_\omega)) \otimes L_2^2(D)$. What’s more,
\[ W_2(I \otimes M_{\eta_\lambda}) = (I \otimes W_1)(I \otimes M_{\eta_\lambda}) \]
\[ = I \otimes (W_1 M_{\eta_\lambda}) \]
\[ = I \otimes (M_z W_1) \]
\[ = (I \otimes M_z)(I \otimes W_1) \]
\[ = (I \otimes M_z) W_2. \]

Thus
\[ I \otimes M_{\eta_\lambda} \cong I \otimes M_z. \]

By Lemma 2.2, there exists a unitary operator $U$ such that
\[ US_z = (I \otimes M_z)U. \]

By the function calculus, it is well known that
\[ US_{\eta_\lambda} U^* = U_{\eta_\lambda}(S_z)U^* \]
\[ = \eta_\lambda(U S_z U^*) \]
\[ = \eta_\lambda(I \otimes M_z) \]
\[ = I \otimes M_{\eta_\lambda}. \]
Let
\[ W_3 = U^*W_2U. \]
Then
\[
W_3S_{\eta_\lambda}W_3^* = U^*W_2US_{\eta_\lambda}U^*W_2^*U \\
= U^*W_2(I \otimes M_{\eta_\lambda})W_2^*U \\
= U^*(I \otimes M_{\eta_\lambda})U \\
= S_z.
\]
Therefore
\[ S_{\eta_\lambda} \cong S_z. \]
The proof is completed.

**Lemma 2.6** Suppose that \( \psi \) is a finite Blaschke product and \( \psi_\lambda = \psi \circ \eta_\lambda \). If \( S_{\psi_\lambda} \) has at least a non-trivial reducing subspace on which the restriction of \( S_{\psi_\lambda} \) is unitary equivalent to the Bergman shift, then \( S_{\psi} \) also has at least a non-trivial reducing subspace on which the restriction of \( S_{\psi} \) is unitary equivalent to the Bergman shift.

**Proof.** Let \( M \) be the non-trivial reducing subspace of \( S_{\psi_\lambda} \) and there exists a unitary transformation \( W: M \rightarrow L^2_\alpha(D) \) such that
\[ WS_{\psi_\lambda}|_M = MZW. \]
Because
\[ \eta_\lambda \circ \eta_\lambda(\omega) = \omega, \]
we have
\[ \psi = \psi_\lambda \circ \eta_\lambda. \]
By Lemma 2.5,
\[ W_3S_{\eta_\lambda}W_3^* = S_z. \]
By the function calculus,
\[
W_3S_{\psi}W_3^* = W_3S_{\psi_\lambda \circ \eta_\lambda}W_3^* \\
= W_3\psi_\lambda(S_{\eta_\lambda})W_3^* \\
= \psi_\lambda(W_3S_{\eta_\lambda}W_3^*) \\
= \psi_\lambda(S_z) \\
= S_{\psi_\lambda},
\]
i.e.,
\[ S_{\psi} \cong S_{\psi_\lambda}. \]
Let
\[ M_1 = W_3^*M. \]
Then \( M_1 \) is the non-trivial reducing subspace of \( S_{\psi} \). Let
\[ W_4 = WW_3. \]
It is easy to prove that
\[ W_4S_\psi|_{M_1} = M_zW_4, \]
i.e.,
\[ S_\psi|_{M_1} \equiv M_z. \]
The proof is completed.

**Lemma 2.7** [1] Suppose that \( \psi(z) \) is the finite Blaschke product having zeros with multiplicity greater than one and \( \eta_\lambda = \frac{\lambda - z}{1 - \lambda z} \). Let \( \psi_\lambda(z) = (\eta_\lambda \circ \psi)(z) \). Then there exists a \( \lambda \in D \) such that \( \psi_\lambda(z) \) has distinct zeros.

## 3 Principal Results and Proofs

In this section we give our main results.

**Theorem 3.1** Suppose that \( \varphi(\omega) \) be a one variable non-constant inner function, and
\[
\psi(z) = z \prod_{l=1}^{N-1} \frac{z - \alpha_l}{1 - \alpha_l z}, \quad (|\alpha_l| > 0, \alpha_l \neq \alpha_k \ (\forall l \neq k), \ 1 \leq l, k \leq N - 1).
\]
Then there exists a unique unit vector \( e' \) such that
\[
e' \in \ker T_{\psi(z)}^* \cap \ker T_{\varphi(\omega)}^* \cap N_\varphi = \ker S_{\psi(z)}^* \cap \ker S_{\varphi(\omega)}^*.
\]
(3.1)

**Proof.** Picking the unit vector \( e \) in Lemma 2.3, then we have
\[ e \in H^2(T^2) \ominus [z-\omega] = N_\omega. \]
By Lemma 2.1, \( \{e_j(z, \omega) : j \geq 0\} \) is an orthonormal basis for \( H^2(T^2) \ominus [z-\omega] \). Then there exists a sequence of constant numbers \( \{k_j\} \), such that
\[
e_j(z, \omega) = \frac{z - \alpha_j}{1 - \alpha_j z}.
\]
Let
\[
e'(z, \omega) = \lambda_1(\omega)e(z, \varphi(\omega)).
\]
Then obviously
\[
e'(z, \omega) = \sum_{j=0}^{\infty} k_j(\lambda_1(\omega)e_j(z, \varphi(\omega))) = \sum_{j=0}^{\infty} k_jE_{\varphi(\omega)} \in N_\varphi
\]
(3.3)
and
\[ \|e'\|^2 = \sum_{j=0}^{\infty} |k_j|^2 = \|e\|^2 = 1. \]
Because
\[ e \in \ker T_{\psi(z)}^* \iff T_{\psi(z)}^*e(z, \omega) = 0, \]
i.e.,
\[ \int_T \int_T |T_{\psi(z)}^*e(z, \omega)|^2 dm(z) dm(\omega) = 0, \]
then
\[
\|T^*_\psi(z) e(z, \varphi(\omega))\|^2 = \int_T \int_T |T^*_\psi(z) e(z, \varphi(\omega))|^2 dm(z) dm(\omega) \quad \text{(let } t = \varphi(\omega))
\]
\[
= \int_T \int_T |T^*_\psi(z, t)|^2 \frac{dm(\varphi^{-1}(t))}{dm(t)} dm(z) dm(t).
\]
Let
\[a = \int_0^{2\pi} \varphi(e^{i\theta}) dm(\theta).
\]
Then by Lemma 2.4,
\[
\left| \frac{dm(\varphi^{-1}(t))}{dm(t)} \right| = |p_a(t)| = |\operatorname{Re}\left( \frac{t + a}{t - a} \right)|
\]
\[
\leq \left| \frac{t + a}{t - a} \right| \leq \frac{1 + |a|}{1 - |a|}
\]
\[
\leq \int_T \int_T |T^*_\psi(z, t)|^2 \frac{1 + |a|}{1 - |a|} dm(z) dm(t)
\]
\[
\leq \frac{1 + |a|}{1 - |a|} \int_T \int_T |T^*_\psi(z, t)|^2 dm(z) dm(t)
\]
\[
= 0.
\]
Thus
\[
T^*_\psi(z) e(z, \varphi(\omega)) = 0.
\]
Then
\[
T^*_\psi(z) e'(z, \omega) = T^*_\psi(z) (\lambda_1(\omega) e(z, \varphi(\omega))) = \lambda_1(\omega) T^*_\psi(z) e(z, \varphi(\omega)) = 0. \quad (3.4)
\]
By (3.3) and (3.4),
\[e' \in \ker T^*_\psi(z) \cap N_{\varphi}.
\]
We have
\[
T^*_\psi(z) |_{N_{\varphi}} = T^*_\psi(\varphi(\omega)) |_{N_{\varphi}}.
\]
In fact, because \(\psi \in A(D)\), it is easy to prove that
\[
\psi(z) - \psi(\varphi(\omega)) \in [z-\varphi(\omega)] = M_{\varphi},
\]
and it is well known that \(\psi \in H^\infty(D^2)\). Then for any \(g \in H^2(T^2)\), \((\psi(z) - \psi(\varphi(\omega)))g \in M_{\varphi}\).
Therefore,
\[
\langle (T^*_\psi(z) - T^*_\psi(\varphi(\omega))) f, g \rangle = \langle f, (\psi(z) - \psi(\varphi(\omega)))g \rangle = 0, \quad \forall f, g \in N_{\varphi},
\]
i.e.,
\[
T^*_\psi(z) |_{N_{\varphi}} = T^*_\psi(\varphi(\omega)) |_{N_{\varphi}}.
\]
Then
\[e' \in \ker T^*_\psi(z) \cap \ker T^*_\psi(\varphi(\omega)) \cap N_{\varphi}.
\]
Let
\[ \psi_0(z) = \prod_{i=1}^{N-1} z - \alpha_i. \]

By the fact that
\[ T_z^* e' = T_{\varphi(\omega)}^* e', \]
moreover the conclusion (3.2) is equivalent to the following:
\[ [\psi_0(z) - \psi_0(\varphi(\omega))] e' = [\psi(z) - \psi(\varphi(\omega))] T_z^* e'. \]  
(3.5)

In fact,
\[ (\psi(z) + \psi(\varphi(\omega))) e' \in N_0 \]
\[ \iff (T_z^* - T_{\varphi(\omega)}^*)(\psi(z) + \psi(\varphi(\omega))) e' = 0 \]
\[ \iff [\psi_0(z) - \psi_0(\varphi(\omega))] e' = [\psi(z) - \psi(\varphi(\omega))] T_z^* e'. \]

Similarly, by (2.2), we have
\[ [\psi_0(z) - \psi_0(\omega)] e(z, \omega) = [\psi(z) - \psi(\omega)] T_z^* e(z, \omega). \]

So
\[ \|[\psi_0(z) - \psi_0(\omega)] e(z, \omega) - [\psi(z) - \psi(\omega)] T_z^* e(z, \omega)\|^2 \]
\[ = \int_T \int_T \|[\psi_0(z) - \psi_0(\omega)] e(z, \omega) - [\psi(z) - \psi(\omega)] T_z^* e(z, \omega)\|^2 \, dm(z) \, dm(\omega) \]
\[ = 0. \]

Then
\[ \|[\psi_0(z) - \psi_0(\varphi(\omega))] e(z, \varphi(\omega)) - [\psi(z) - \psi(\varphi(\omega))] T_z^* e(z, \varphi(\omega))\|^2 \]
\[ = \int_T \int_T \|[\psi_0(z) - \psi_0(\varphi(\omega))] e(z, \varphi(\omega)) - [\psi(z) - \psi(\varphi(\omega))] T_z^* e(z, \varphi(\omega))\|^2 \, dm(z) \, dm(\omega) \]
(by t = \varphi(\omega))
\[ = \int_T \int_T \|[\psi_0(z) - \psi_0(t)] e(z, t) - [\psi(z) - \psi(t)] T_z^* e(z, t)\|^2 \, dm(t) \, dm(z) \, dm(t) \]
\[ = \int_T \int_T \|[\psi_0(z) - \psi_0(t)] e(z, t) - [\psi(z) - \psi(t)] T_z^* e(z, t)\|^2 p_0(t) \, dm(z) \, dm(t) \]
\[ \leq \frac{1 + |a|}{1 - |a|} \int_T \int_T \|[\psi_0(z) - \psi_0(t)] e(z, t) - [\psi(z) - \psi(t)] T_z^* e(z, t)\|^2 \, dm(z) \, dm(t) \]
\[ = 0. \]

Therefore,
\[ [\psi_0(z) - \psi_0(\varphi(\omega))] e(z, \varphi(\omega)) = [\psi(z) - \psi(\varphi(\omega))] T_z^* e(z, \varphi(\omega)). \]

Multiplied by \( \lambda_1(\omega) \), we can obtain the conclusion (3.5). The proof is completed.

**Remark** It is different from Lemma 2.3, \( e' \) in the theorem is not unique. We can let
\[ e' = \lambda_1(\omega) e(z, \varphi(\omega)), \]
where \( \lambda_1(\omega) \) is any element of the orthonormal basis of \( H^2(I_\omega) \oplus \varphi(\omega) H^2(I_\omega) \) in Lemma 2.1.
Theorem 3.2 Suppose that $\varphi(\omega)$ be a one variable non-constant inner function, and 
\[ \psi(z) = z \prod_{l=1}^{N-1} \frac{z - \alpha_l}{1 - \alpha_l z} \quad (|\alpha_l| > 0, \alpha_l \neq \alpha_k (\forall l \neq k), \, 1 \leq l, k \leq N - 1). \]

Pick $e'$ in Theorem 3.1. Then 
\[ M_{e'} = \text{span}\{p'_n(\psi)e' : n \geq 0\}, \]
where 
\[ p'_n(\psi) = \psi^n(z) + \psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi(z)\psi^{n-1}(\varphi(\omega)) + \psi^n(\varphi(\omega)) \]
is a non-trivial minimal reducing subspace of $S_{\psi(\omega)}$. Moreover $S_{\psi(\omega)}|_{M_{e'}}$ is unitary equivalent to Bergman shift $M_z$.

Proof.
\[ T^*_n p'_n(\psi)e' - T^*_0 p'_n(\psi)e' \]
\[ = T^*_n [\psi^n(z) + \psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi(z)\psi^{n-1}(\varphi(\omega)) + \psi^n(\varphi(\omega))]e' \]
\[ - T^*_0 [\psi^n(z) + \psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi(z)\psi^{n-1}(\varphi(\omega)) + \psi^n(\varphi(\omega))]e' \]
\[ = [\psi_0(z)\psi^n(z) + \psi_0(z)\psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi_0(z)\psi(z)\psi^{n-1}(\varphi(\omega)) + \psi_0(z)\psi^n(\varphi(\omega))]T^*_n e' \]
\[ - [\psi^n(z)T^*_n e' + \psi^{n-1}(z)\psi(\varphi(\omega))e' + \cdots + \psi(z)\psi_0(\varphi(\omega))\psi^{n-1}(\varphi(\omega))e'] \]
\[ = \psi_0(z)\psi^n(z)T^*_n e' + \psi^n(z)\psi^{n-1}(z)\psi(\varphi(\omega))e' + \cdots + \psi(z)\psi_0(\varphi(\omega))\psi^{n-1}(\varphi(\omega))e'] \]
\[ = p'_n(\psi)(\psi_0(z) - \psi_0(\varphi(\omega)))e' + (\psi^n(\varphi(\omega)) - \psi^n(z))T^*_n e' \]
(by (3.5))
\[ = p'_n(\psi)(\psi(z) - \psi(\varphi(\omega)))T^*_n e' + (\psi^n(\varphi(\omega)) - \psi^n(z))T^*_n e' \]
\[ = (\psi^n(z) - \psi^n(\varphi(\omega)))T^*_n e' + (\psi^n(\varphi(\omega)) - \psi^n(z))T^*_n e' \]
\[ = 0. \]
We have 
\[ (T^*_n - T^*_0 p'_n(\psi)e' = 0. \]
So 
\[ p'_n(\psi)e' \in N_{\varphi}. \]

Also, 
\[ S_{\psi(\omega)}(p'_n(\psi)e') \]
\[ = q(\psi)p'_n(\psi)e' \]
\[ = q(\psi)\psi^n(z) + \psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi(z)\psi^{n-1}(\varphi(\omega)) + \psi^n(\varphi(\omega))e' \]
\[ = q(\psi^{n+1}(z) + \psi^n(z)\psi(\varphi(\omega)) + \cdots + \psi(z)\psi^{n-1}(\varphi(\omega)) + \psi(z)\psi^n(\varphi(\omega)))e' \]
of these minimal reducing subspaces is unitary equivalent to the Bergman shift
\[\dim(H)\]
Hence by (3.6) and (3.7),
\[M_{e'} = \frac{n+1}{n+2} p_{n+1}'(\psi)e' \in M_{e'}, \]
and
\[S_{\psi(z)}(p_{n}'(\psi)e') = q\psi(z)p_{n}'(\psi)e'\]
\[= q\psi(z)\psi^n(z) + \psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi(\varphi(\omega))e' \]
\[= q\psi(z)\psi^n(z) + \psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi(\varphi(\omega))e' + \psi^n(\varphi(\omega))T_{\psi(z)}^*e' \]
(by (3.1))
\[= q\psi(z)\psi^n(z) + \psi^{n-1}(z)\psi(\varphi(\omega)) + \cdots + \psi(\varphi(\omega))e' \]
\[= p_{n-1}'(\psi)e' \in M_{e'}. \] (3.7)
Hence by (3.6) and (3.7), \(M_{e'}\) is the non-trivial reducing subspace of \(S_{\psi(z)}\). Because \(|\psi(z)| = |\psi(\varphi(\omega))| = 1\ a.e. on \(T^2\),
then
\[p_{n}'(\psi)p_{n}'(\psi) = \left\{ \begin{array}{ll}
\sum_{k+l=n-m, k \leq n} c_{k,l} \psi^k(z)\psi^l(\varphi(\omega)), & \text{if } m > n; \\
\sum_{-n \leq k \leq n, k \neq 0} c_k \psi^k(z)\psi^{-k}(\varphi(\omega)) + (n+1), & \text{if } m = n \\end{array} \right. \text{ a.e. on } T^2. \]
Since \(e' \in \ker T_{\psi(z)}^* \cap \ker T_{\psi(\varphi(\omega))}^* \cap N_{e'}\), it is easy to check
\[(p_{n}'(\psi)e', p_{m}'(\psi)e') = \left\{ \begin{array}{ll}
0, & \text{if } m \neq n; \\
n+1, & \text{if } m = n. \end{array} \right. \]
Therefore, \(\left\{ \frac{p_{n}'(\psi)e'}{\sqrt{n+1}} : n = 0, 1, \cdots \right\} \) is an orthonormal basis for \(M_{e'}\). By (3.6) we can define a unitary transformation
\[W_1 : M_{e'} \to L^2_0(D), \]
\[\frac{p_{n}'(\psi)e'}{\sqrt{n+1}} \to \sqrt{n+1}z^n \]
such that
\[W_1S_{\psi(z)}|_{M_{e'}} = M_nW_1. \]
Hence
\[S_{\psi(z)}|_{M_{e'}} \cong M_n. \]
The proof is completed.

Corollary 3.1 Suppose that \(\varphi(\omega)\) be a one variable non-constant inner function, and
\[\psi(z) = z \prod_{|a_l| > 0, \alpha_l \neq \alpha_k (\forall l \neq k), 1 \leq l, k \leq N-1} \frac{z - a_l}{1 - \alpha_l z} \]
Then the Toeplitz operator \(S_{\psi(z)}\) has at least \(m\) non-trivial minimal reducing subspaces \((m = \dim(H^2(\Gamma_0) \otimes \varphi(\omega)H^2(\Gamma_0))\) and \(m\) may be \(+\infty\). Moreover, the restriction of \(S_{\psi(z)}\) on any of these minimal reducing subspaces is unitary equivalent to the Bergman shift \(M_n\).
Theorem 3.3 Suppose that \( \psi(z) \) is a common finite Blaschke product. Then \( S_\psi(z) \) has at least a non-trivial minimal reducing subspace on which the restriction of \( S_\psi(z) \) is unitary equivalent to the Bergman shift.

Proof. Suppose that \( \psi(z) \) is a finite Blaschke product of order \( N \). If \( \psi(z) \) is the finite Blaschke product having zero with multiplicity greater than one, then, by Lemma 2.7, there exists a \( \lambda_0 \in D \) such that \( \psi_{\lambda_0}(z) \) has distinct zeros, where

\[
\psi_{\lambda_0}(z) = (\eta_{\lambda_0} \circ \psi)(z), \quad \eta_{\lambda_0}(z) = \frac{\lambda_0 - z}{1 - \lambda_0 z}.
\]

If \( \psi_{\lambda_0}(0) \neq 0 \), let \( \psi_{\lambda_1}(z) = (\psi_{\eta_{\lambda_0}})(z) \). Suppose that \( \lambda_1 \) satisfies the condition

\[
\psi_{\lambda_0}(\lambda_1) = 0.
\]

Then

\[
\psi_{\lambda_1}(0) = \psi_{\lambda_0}(\eta_{\lambda_1}(0)) = \psi_{\lambda_0}(1) = 0.
\]

Hence \( \psi_{\lambda_1}(z) \) is the case in Theorem 3.2. Therefore, \( S_\psi_{\lambda_1}(z) \) has at least a reducing subspace, denoted by \( M \) and

\[
W_1S_{\psi_{\lambda_0}}|_M = MzW_1.
\]

By \( \eta_{\lambda} \circ \eta_{\lambda}(\omega) = \omega \) and function calculus, one has

\[
S_{\psi(z)} = S_{\psi_{\eta_{\lambda_0}} \circ \psi_{\lambda_0}(z)} = \eta_{\lambda_0}(S_{\psi_{\lambda_0}}(z)) = \frac{\lambda_0 - S_{\psi_{\lambda_0}}(z)}{1 - \lambda_0 S_{\psi_{\lambda_0}}(z)}.
\]

So \( M \) is the reducing subspace of \( S_{\psi(z)} \). We have

\[
W_1S_{\psi(z)}W_1^* = W_1S_{\psi_{\eta_{\lambda_0}} \circ \psi_{\lambda_0}(z)}W_1^* = W_1 \eta_{\lambda_0}(S_{\psi_{\lambda_0}})W_1^* = \eta_{\lambda_0}(W_1S_{\psi_{\lambda_0}}W_1^*) = \eta_{\lambda_0}(Mz) = M_{\eta_{\lambda_0}}.
\]

By [1], there exists a unitary transformation \( W_2 \) such that

\[
W_2M_{\eta_{\lambda_0}}W_2^* = Mz.
\]

Define a unitary transformation:

\[
W : M \to L^2_\omega(D)W = W_2W_1.
\]

Therefore,

\[
WS_{\psi(W)^*} = W_2W_1S_{\psi(W)^*W_1^*} = W_2M_{\eta_{\lambda_0}}W_2^* = Mz,
\]

i.e.,

\[
S_{\psi}|_M \cong Mz.
\]

Corollary 3.2 Suppose that \( \varphi(\omega) \) be a one variable non-constant inner function and \( \psi(z) \) is a common finite Blaschke product. Then \( S_{\psi(z)} \) has at least \( m \) non-trivial minimal reducing
subspaces \((m = \dim(H^2(\Gamma_\omega) \ominus \varphi(\omega)H^2(\Gamma_\omega)))\) and \(m\) may be \(+\infty\). Moreover, the restriction of \(S_{\psi(z)}\) on any of these minimal reducing subspaces is unitary equivalent to the Bergman shift \(M_z\).

**Proof.** It can be easily obtained by Corollary 3.1 and Theorem 3.3.

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**References**

