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A KAM-type Theorem for Generalized Hamiltonian Systems*

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Abstract: In this paper we mainly concern the persistence of lower-dimensional invariant tori in generalized Hamiltonian systems. Here the generalized Hamiltonian systems refer to the systems which may admit a distinct number of action and angle variables. In particular, system under consideration can be odd dimensional. Under the Rüssmann type non-degenerate condition, we proved that the majority of the lower-dimension invariant tori of the integrable systems in generalized Hamiltonian system are persistent under small perturbation. The surviving lower-dimensional tori might be elliptic, hyperbolic, or of mixed type.

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1 Introduction and Result

The classical KAM theory, established by Kolmogorov[1], Arnold[2] and Moser[3] in the last century, is a landmark of the development of Hamiltonian systems. It gives a reasonable explanation for the stability of solar system and brings a new method into the study of Hamiltonian systems. The classical KAM theory established on $2n$-dimensional smoothly symplectic manifold asserts that the majority of non-resonant tori of integrable systems can survive small perturbations under the Kolmogorov non-degenerate condition. In 1967, Melnikov formulated a KAM type persistence result for lower-dimensional elliptic tori of nearly integrable Hamiltonian systems. But the complete proof was not carried out until fifteen years later it was provided by Eliasson[4], Kuksin[5], Pöschel[6]. The persistence of hyperbolic

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type lower dimensional invariant tori was treated in [7]–[12]. Parasyuk[13] foremost studied the existence of invariant tori for Hamiltonian systems with distinct numbers of action-angle variables (coisotropic).

As the generalization of the traditional Hamiltonian systems which defined on a symplectic manifold, the generalized Hamiltonian systems are defined on a Poisson manifold which can be odd dimensional and structurally degenerate. The generalized Hamiltonian systems can describe more general mathematical models, so in the study of the generalized Hamiltonian systems there are some practical meanings. The symplectic structure brings some special properties for the classical Hamiltonian systems. Since there is no symplectic structure for odd-dimensional systems, some results in classical Hamiltonian systems no longer hold. Hence the development of the KAM theory for odd-dimensional system has been considered as a challenging problem (see [14]–[16]). The theory of KAM type has been developed for volume preserving flows in [17] and [18]. For the case of diffeomorphism which is volume preserving or satisfies the intersection property, it was treated by Cheng and Sun[19], and Xia[20].

In paper [21], the authors established a KAM type theorem for the generalized Hamiltonian systems. In paper [22], the authors formulated a KAM type persistence result for lower-dimensional hyperbolic tori of nearly integrable generalized Hamiltonian systems. In this paper, we proved that the majority of the lower-dimensional invariant tori in the generalized Hamiltonian system are persistent under small perturbations, the surviving lower-dimensional tori might be elliptic, hyperbolic, or of mixed type.

Consider the Poisson manifold $(G\times T^n\times R^{2m}, \omega^2)$, where $G\subset R^l$ is a bounded, connected, closed region, T^n is the standard n-torus, and l, n, m are positive integers. The structure matrix

$$I = (A_{ij}) : G \times T^n \times R^{2m} \to R^{(l+n+2m)\times(l+n+2m)}$$

associated with 2-form ω^2 is a real analytic, anti-symmetric, matrix valued function and satisfies the following two conditions:

(i) rank $I > 0$;

(ii) Jacobi identity

$$\sum_{t=1}^{l+n+2m} A_{it}\frac{\partial A_{jk}}{\partial w_t} + A_{jt}\frac{\partial A_{ki}}{\partial w_t} + A_{kt}\frac{\partial A_{ij}}{\partial w_t} = 0$$

holds for all $w = (y, x, z) \in G \times T^n \times R^{2m}$, $i, j, k = 1, 2, \cdots, l + n + 2m$.

On the Poisson manifold $(G \times T^n \times R^{2m}, \omega^2)$, we consider the generalized Hamiltonian system

$$H(y, x, z) = h(y) + \frac{\delta}{2}\langle z,\ M(y)z\rangle + \varepsilon P(y, x, z), \tag{1.1}$$

where $x \in T^n$, $y \in G \subset R^l$, $z \in R^{2m}$, G is a bounded closed region, δ and ε are small parameters satisfying $\varepsilon \ll \delta$, $h(y)$, $M(y)$ and $P(y, x, z)$ are real analytic functions respectively, and $M(y)$ is a symmetric matrix.

The 2-form ω^2 is required to be invariant relative to T^n. Suppose that the unperturbed system of (1.1) is completely integrable, i.e.,

$$y = (y_1, \cdots, y_l)^\top \in G$$

satisfies the involution conditions

$$\{y_i, y_j\} = 0, \qquad i, j = 1, 2 \cdots l.$$

And suppose that the part corresponding to the variable z of the structure matrix I is of constant coefficients. That is, in coordinates (y, x, z) the Poisson structure I satisfies

$$\{y_j, y_k\} = \{y_j, z_l\} = \{x_i, z_l\} = 0$$

and is standard symplectic in the normal z-direction, while

$$\{x_i, y_k\} = B_{ik}(y) \quad \text{and} \quad \{x_i, x_j\} = C_{ij}(y)$$

are given y-dependent matrices. So the structure matrix I have the following form

$$I(y) = \begin{pmatrix} E(y) & O_1 \\ O_2 & J \end{pmatrix},$$

where

$$E(y) = \begin{pmatrix} O & B(y) \\ -B(y)^T & C(y) \end{pmatrix},$$

$$O = O_{l\times l}, \quad O_1 = O_{(l+n)\times 2m}, \quad O_2 = O_{2m\times(l+n)},$$

$$B(y) = B_{l\times n}(y), \quad C(y) = C_{n\times n}(y), \quad C^T = -C,$$

and J is the standard $2m \times 2m$ symplectic matrix.

For the Poisson manifold $(G \times T^n \times R^{2m}, \omega^2)$, the 2-form ω^2 or the structure matrix I is degenerate for all $y \in G$ when $l > n$ or $l + n$ is odd. This kind of singularity shows an essential difference between a generalized Hamiltonian system and a standard one.

The equation of motions of (1.1) associated to the 2-form ω^2 reads as

$$\begin{pmatrix} \dot{y} \\ \dot{x} \\ \dot{z} \end{pmatrix} = I(y)\nabla(h(y) + \frac{\delta}{2}\langle z,\ M(y)z\rangle + \varepsilon P(y, x, z)). \tag{1.2}$$

Let

$$E(y)\mathrm{grad}^T_{(y,x)}h(y) = (\overbrace{0, 0, \cdots, 0}^{l}, \omega(y))^T,$$

where $\mathrm{grad}^T_{(y,x)}h$ denotes the gradient vector of the function h with respect to (y, x). When $\varepsilon = 0$, the system (1.2) obviously has invariant torus

$$T_{y_0} = \{(y, x, z) :\ y = y_0,\ x \in T^n,\ z = 0\}, \qquad y_0 \in G,$$

carrying parallel flows

$$x = x_0 + \omega(y_0)t.$$

Remark 1.1 There are some backgrounds for studying the system (1.1). When we study the invariant tori on resonant surface, by some reasonable restrictions on the perturbation, the system can be reduced to the form of (1.1), where δ and ε are small parameters satisfying $\varepsilon \ll \delta$ (see [23]).

Now we study the persistence of the invariant tori under small perturbation. We make the following hypothesis:

i) For all $x \in G$, $\text{rank}\Big\{\dfrac{\partial^i \omega(y)}{\partial y^i} : |i| \le n-1\Big\} = n$, where $i \in Z_+^n$, $|i| = \sum\limits_{j=1}^n |i_j|$;

ii) There exists a constant $d > 0$ such that $|\det M(y)| \ge d$ for all $y \in G$.

Our main result is as follows:

Theorem A *Consider* (1.1) *and assume the non-degenerate conditions* i) *and* ii). *Then there are sufficiently small* $\varepsilon_0, \delta_0 > 0$ $(\varepsilon_0 \ll \delta_0)$ *and a family of nonempty Cantor sets* $G_\varepsilon \subset G$ *such that when* $0 < \varepsilon < \varepsilon_0$, $0 < \delta < \delta_0$, *the following holds*:

1) *For any* $y_0 \in G_\varepsilon$, *the invariant torus*

$$T_{y_0} = \{(y, x, z) : \ y = y_0, \ x \in T^n, \ z = 0\}$$

of the unperturbed system persists and gives rise to an analytic, invariant torus of the perturbed system whose total frequency is of Diophantine type (γ, τ), *where*

$$0 < \gamma \le \varepsilon^{\frac{1}{24\times 4m^2}}, \qquad \tau > \max\{0, l-1, n-1\}.$$

Moreover, the perturbed tori forms a Whitney smooth family;

2) *The Lebesgue measure*

$$|G \backslash G_\varepsilon| = O(\varepsilon^{\frac{1}{24\times 4m^2(a-1)}}) \to 0 \qquad (\varepsilon \to 0),$$

where

$$a = \begin{cases} 2, & n = 1; \\ \max(l, n), & n > 1. \end{cases}$$

Remark 1.2 Theorem A shows that for the generalized Hamiltonian systems (1.1), we only require that $M(y)$ is a non-degenerate matrix, so the surviving tori might be of elliptic, hyperbolic or mixed type.

Remark 1.3 If δ is not small, say $\delta = 1$, to get a similar result, one has to require more restrictions involving the normal frequencies such as: there exists a constant $K > 0$ such that for $0 < |k| \le K$,

$$|\{y \in G : \sqrt{-1}\langle k, \ \omega(y)\rangle - \lambda_i(y) - \lambda_j(y) = 0\}| = 0, \qquad 1 \le i, j \le 2m,$$

where $\lambda_i(y)$, $\lambda_j(y)$, $1 \le i, j \le 2m$ are $2m$ eigenvalues of $M(y)J$ (see (2.23), Lemma 5.3).

Throughout the paper, we shall use the symbol $|\cdot|$ to denote norm of vectors, matrices, absolute value of functions and the Lebesgue measure of sets, etc., and use $|\cdot|_D$ to denote the supremum norm of functions on a domain D. They will have obvious meaning unless it is specified otherwise. Also, for any two complex column vectors ξ, η of the same dimension, $\langle \xi, \ \eta\rangle$ always stands for $\xi^T \eta$.

Let us outline the proof of Theorem A. In Section2, we describe one cycle of KAM steps. In Section 3 we provide an iteration lemma, which shows the validity of each step. In Section 4, we give the proof of Theorem A.

2 KAM Step

For any $y_0 \in G$, the Taylor expansion of Hamiltonian (1.1) about y_0 reads as

$$\begin{aligned} H(y, y_0, x, z) &= h(y_0) + \langle \Omega(y_0),\ y - y_0\rangle + \frac{\delta}{2}\langle z,\ M(y_0)z\rangle \\ &\quad + (O(|y-y_0|^2) + \delta O(|z|^2|y-y_0|) + \varepsilon P(y, x, z)) \\ &= N_0 + P_0, \end{aligned} \tag{2.1}$$

where

$$\Omega(y_0) = \frac{\partial h(y)}{\partial y}\Big|_{y=y_0}.$$

Using the transformation $y - y_0 \to y$, we have

$$\begin{aligned} H(y, y_0, x, z) &= h(y_0) + \langle \Omega(y_0)\ , y\rangle + \frac{\delta}{2}\langle z,\ M(y_0)z\rangle \\ &\quad + (O(|y|^2) + \delta O(|z|^2|y|) + \varepsilon P(y + y_0, x, z)) \\ &= N_0 + P_0, \end{aligned} \tag{2.2}$$

where

$$N_0 = h(y_0) + \langle \Omega(y_0),\ y\rangle + \frac{\delta}{2}\langle z,\ M(y_0)z\rangle.$$

Thus, as $\varepsilon = 0$, for any $y_0 \in G$, the invariant torus

$$T_{y_0} = \{(y_0, x, 0),\ x \in T^n\}$$

of (2.1) corresponds to the invariant torus

$$T_0 = \{(0, x, 0),\ x \in T^n\}$$

of (2.2).

For convenience, in the following we study the Hamiltonian (2.2) on

$$\mathcal{D}(s, r) \times G = \{|y| < s^2,\ |\mathrm{Im}\, x| < r,\ |z| < s\} \times G,$$

where $M(y_0)$ is a real analytic function defined on G and $P(y, y_0, x, z)$, $I(y + y_0)$ are real analytic functions defined on $D(s, r) \times G$ respectively, and $M(y)$ is a symmetric matrix. To begin with the induction, we initially set $\mathcal{O}_0 = G$, $M_0 = M$, $\beta_0 = s_0$, $s_0 = \varepsilon^{\frac{1}{4}}$, $\delta = \varepsilon^{\frac{1}{12}}$, $\mu_0 = \varepsilon^{\frac{1}{8}}$, $\gamma_0 = \varepsilon^{\overline{\frac{1}{24\times 4m^2}}}$. Without loss of generality, we assume that $0 < \gamma_0, \beta_0, \mu_0 < 1$.

Clearly, as ε is a sufficiently small parameter we have

$$|P_0|_{\mathcal{D}(s_0, r_0)\times\mathcal{O}_0} \le \delta\gamma_0^{4m^2} s_0^2 \mu_0.$$

Now suppose at a KAM step, say the ν-step, we have arrived at a Hamiltonian

$$H = H_\nu = N_\nu + P_\nu, \tag{2.3}$$

$$N = h_\nu(y_0) + \langle \Omega_\nu(y_0),\ y\rangle + \frac{\delta}{2}\langle z,\ M_\nu(y_0)z\rangle,$$

where $(y, x) \in \mathcal{D}_\nu = \mathcal{D}(s_\nu, r_\nu)$, $r_\nu \le r_0$, $s_\nu \le s_0$, $y_0 \in \mathcal{O}_\nu$, and $h_\nu(y_0)$, $M_\nu(y_0)$ are real analytic functions defined on $\mathcal{O}_\nu$, and moreover

$$|P_\nu|_{\mathcal{D}_\nu\times\mathcal{O}_\nu} \le \delta\gamma_\nu^{4m^2} s_\nu^2 \mu_\nu. \tag{2.4}$$

We will construct a generalized canonical transformation $\Phi_{\nu+1}$, which transforms the Hamiltonian (2.3) in a smaller domain to the desired Hamiltonian into the nest KAM cycle (the $(\nu + 1)$-th KAM step):

$$\Phi_{\nu+1} : \mathcal{D}_{\nu+1} \times \mathcal{O}_{\nu+1} \to \mathcal{D}_\nu, \mathcal{D}_{\nu+1} \subset \mathcal{D}_\nu,$$

$$\begin{aligned} H_{\nu+1} &= H_\nu \circ \Phi_{\nu+1} \\ &= h_{\nu+1}(y_0) + \langle \Omega_{\nu+1}(y_0),\ y\rangle + \frac{\delta}{2}\langle z,\ M_{\nu+1}(y_0)z\rangle + P_{\nu+1}(y, y_0, x, z) \\ &= N_{\nu+1} + P_{\nu+1}, \end{aligned}$$

and prove

$$|P_{\nu+1}|_{D_{\nu+1}\times\mathcal{O}_{\nu+1}} \le \delta\gamma_{\nu+1}^{4m^2} s_{\nu+1}^2 \mu_{\nu+1}. \tag{2.5}$$

Let

$$\tau > \max\{0,\ l(l-1)-1,\ n(n-1)-1\}$$

be fixed. Inductively define

$$r_{\nu+1} = r_0\Big[1 - \frac{1}{8}\sum_{i=1}^{\nu+1}\Big(\frac{7}{8}\Big)^{i+1}\Big], \qquad \gamma_{\nu+1} = \frac{\gamma_\nu}{2} + \frac{\gamma_0}{4}, \qquad s_{\nu+1} = \alpha_\nu s_\nu,$$

$$\beta_{\nu+1} = \frac{\beta_\nu}{2} + \frac{\beta_0}{4},\quad \beta_0 = s_0,\quad \mu_{\nu+1} = \alpha_\nu^{\frac{1}{2}}\mu_\nu,\quad \alpha_\nu = \mu_\nu^{\frac{1}{3}},\quad \mu_\nu = s_\nu^{\frac{1}{2}},$$

$$K_\nu = \Big[\Big(\log\frac{1}{\mu_\nu}\Big) + 1\Big]^3, \qquad \Gamma_\nu(r_\nu - r_{\nu+1}) = \sum_{0<|k|\le K_+} |k|^{\tau+2} \mathrm{e}^{-\frac{r_\nu - r_{\nu+1}}{16}},$$

$$\mathcal{D}_* = \mathcal{D}\Big(\frac{s}{4},\ r_+ + \frac{5}{8}(r - r_+)\Big), \qquad \mathcal{D}_{**} = \mathcal{D}\Big(\frac{s}{2},\ r_+ + \frac{6}{8}(r - r_+)\Big),$$

$$\mathcal{D}_{***} = \mathcal{D}(s,\ r_+ + \frac{7}{8}(r - r_+)), \qquad \mathcal{D}_\nu = \mathcal{D}(s_\nu,\ r_\nu),$$

$$\tilde{\mathcal{D}}(\beta_+) = \mathcal{D}(\beta_+,\ r_{\nu+1} + \frac{5}{8}(r_\nu - r_{\nu+1})), \qquad c = \max\{1, c_1, \cdots, c_8, c_0\},$$

$$\mathcal{D}_i = \mathcal{D}(i\alpha_\nu s_\nu,\ r_{\nu+1} + \frac{i-1}{8}(r_\nu - r_{\nu+1})), \qquad i = 1, 2, \cdots, 8.$$

2.1 Truncation

Consider the Taylor-Fourier series of $P(y, y_0, x, z)$:

$$P(y, y_0, x, z) = \sum_{i\in Z_+^l,\ j\in Z_+^{2m},\ k\in Z^n} P_{kij} y^i z^j \mathrm{e}^{\sqrt{-1}\langle k,\ x\rangle}. \tag{2.6}$$

Let

$$\begin{aligned} R &= \sum_{|k|\le K_+,\ 2|i|+|j|<3} P_{kij} y^i z^j \mathrm{e}^{\sqrt{-1}\langle k,\ x\rangle} \\ &= \sum_{|k|\le K_+,\ 2|i|+|j|<3} (P_{k00} + \langle P_{k10},\ y\rangle + \langle P_{k01},\ z\rangle + \langle z,\ P_{k02} z\rangle) \mathrm{e}^{\sqrt{-1}\langle k,\ x\rangle} \end{aligned} \tag{2.7}$$

be the truncation of P, where K_+ is the truncation order in x. Let

$$P - R = \Big(\sum_{|k|>K_+} + \sum_{|k|\le K_+,\ 2|i|+|j|\ge 3}\Big) P_{kij} y^i z^j \mathrm{e}^{\sqrt{-1}\langle k,\ x\rangle}. \tag{2.8}$$

Lemma 2.1 *Assume that*

H1) $\alpha < \dfrac{1}{32}$;

H2) $\displaystyle\int_{K_+}^{\infty} \lambda^n \mathrm{e}^{-\lambda\frac{r-r_+}{16}} \mathrm{d}\lambda \le \mu$.

Then there exists a constant c_1 such that

$$|P - R|_{D_8} \le c_1 \delta \gamma^{4m^2} s^2 \mu^2. \tag{2.9}$$

Proof. Let

$$\begin{aligned}
\mathrm{I} &= \sum_{|k|>K_+} P_{kij} y^i z^j \mathrm{e}^{\sqrt{-1}\langle k,\ x\rangle}, \\
\mathrm{II} &= \sum_{|k|\le K_+,\ 2|i|+|j|\ge 3} P_{kij} y^i z^j \mathrm{e}^{\sqrt{-1}\langle k,\ x\rangle} \\
&= \int \frac{\partial^q}{\partial u^q} \sum_{|k|\le K_+,\ 2|i|+|j|\ge 3} P_{kij} y^i z^j \mathrm{e}^{\sqrt{-1}\langle k,\ x\rangle} \mathrm{d}u,
\end{aligned}$$

where $u=(y,z)$, $\int$ is the q-order anti-derivative of $\dfrac{\partial^q}{\partial u^q}$ with $|q|=3$. Obviously,

$$P - R = \mathrm{I} + \mathrm{II}.$$

By the Cauchy estimate, we have

$$\Big| \sum_{i\in Z_+^l,\ j\in Z_+^{2m}} P_{kij} y^i z^j \Big| \le |P|_{\mathcal{D}(s,r)} \mathrm{e}^{-|k|r} \le \delta\gamma^{4m^2} s^2 \mu \mathrm{e}^{-|k|r}. \tag{2.10}$$

By H2), we have

$$\begin{aligned}
|\mathrm{I}|_{\mathcal{D}_{***}} &\le \sum_{|k|>K_+} \delta\gamma^{4m^2} s^2 \mu \mathrm{e}^{-|k|r} \mathrm{e}^{|k|(r_+ + \frac{7}{8}(r-r_+))} \\
&\le \delta\gamma^{4m^2} s^2 \mu \sum_{|\lambda|\ge K_+} |\lambda|^n \mathrm{e}^{-|\lambda|\frac{r-r_+}{8}} \\
&\le \delta\gamma^{4m^2} s^2 \mu \int_{K_+}^{\infty} \lambda^n \mathrm{e}^{-\lambda\frac{r-r_+}{16}} \mathrm{d}\lambda \\
&\le \delta\gamma^{4m^2} s^2 \mu^2.
\end{aligned} \tag{2.11}$$

So

$$|P - I|_{\mathcal{D}_{***}} \le |P|_{\mathcal{D}(s,r)} + |I|_{\mathcal{D}_{***}} < 2\delta\gamma^{4m^2} s^2 \mu. \tag{2.12}$$

By H1) and $\mathcal{D}_8 \subset \mathcal{D}_{***}$, it follows from the Cauchy estimate on $\mathcal{D}_{***}$ that

$$\begin{aligned}
|II|_{\mathcal{D}_8} &\le \Big| \int \frac{\partial^q}{\partial u^q} \sum_{|k|\le K_+,\ 2|i|+|j|\ge 3} P_{kij} y^i z^j \mathrm{e}^{\sqrt{-1}\langle k,\ x\rangle} \mathrm{d}u \Big|_{\mathcal{D}_8} \\
&\le \Big| \int \Big| \frac{\partial^q}{\partial u^q} (P - I) \Big|_{\mathcal{D}_{***}} \mathrm{d}u \Big|_{\mathcal{D}_8} \\
&\le 2\delta\gamma s^2 \mu \Big(\frac{1}{s - \alpha s} \Big)^3 \Big| \int \mathrm{d}u \Big|_{\mathcal{D}_8} \\
&\le 2\delta\gamma s^2 \mu \Big(\frac{1}{s - 8\alpha s} \Big)^3 (8\alpha s)^3 \\
&\le 2\delta 8^3 \Big(\frac{4}{3}\Big)^3 \gamma^{4m^2} s^2 \mu^2,
\end{aligned} \tag{2.13}$$

where $u=(y,z)$, $|q|=3$. From the estimate above we have

$$|P - R|_{\mathcal{D}_8} \le c_1 \delta\gamma^{4m^2} s^2 \mu^2. \tag{2.14}$$

2.2 Modified Homology Equation

In the following we will find a Generalized Hamiltonian F such that, under the transformation of the time-1 map Φ_+ generated by X_F, we can eliminate all resonant terms in

$$R = \sum_{i\in Z_+^l,\ j\in Z_+^{2m},\ k\in Z^n} P_{kij} y^i z^j \mathrm{e}^{\sqrt{-1}\langle k,\ x\rangle}, \qquad 0<|k|\le K_+, \quad 2|i|+|j|<3.$$

We first construct a generalized Hamiltonian F of the form:

$$F = \sum_{k\in Z^n,\ 0<|k|\le K_+,} (F_{k00} + \langle F_{k10},\ y\rangle + \langle F_{k01},\ z\rangle + \langle z,\ F_{k02} z\rangle)\mathrm{e}^{\sqrt{-1}\langle k,\ x\rangle} + \langle F_{001},\ z\rangle, \tag{2.15}$$

such that F satisfies

$$\{N,\ F\} + R - [R] + \langle P_{001},\ z\rangle - Q = 0, \tag{2.16}$$

where

$$\begin{aligned} [R] &= \frac{1}{(2\pi)^n}\int_{T^n} R(y,x,z)\mathrm{d}x, \\ Q &= \sum_{0<|k|\le K_+} \sqrt{-1}\langle k,\ (B^T(y+y_0) - B^T(y_0))\Omega(y_0)\rangle \\ &\quad + (F_{k00}\langle F_{k10},\ y\rangle + \langle F_{k01},\ z\rangle + \langle z,\ F_{k02} z\rangle)\mathrm{e}^{\sqrt{-1}\langle k,\ x\rangle}. \end{aligned}$$

Putting (2.7), (2.15) into (2.16) and comparing the coefficients we obtain

$$\langle k,\ \omega(y_0)\rangle F_{k00} = -P_{k00}, \tag{2.17}$$

$$\langle k,\ \omega(y_0)\rangle F_{k10} = -P_{k10}, \tag{2.18}$$

$$\langle k,\ \omega(y_0)\rangle F_{k01} + \delta M(y_0) J F_{k01} = -P_{k01}, \tag{2.19}$$

$$\langle k,\ \omega(y_0)\rangle F_{k02} + \delta M(y_0) J F_{k02} - \delta F_{k02}^T J M(y_0) = -P_{k02}, \tag{2.20}$$

$$\delta M(y_0) J F_{001} = -P_{001}, \tag{2.21}$$

where

$$\omega(y_0) = -B^T(y_0)\Omega(y_0).$$

(2.19) is equivalent to

$$[\sqrt{-1}\langle k,\ \omega(y_0)\rangle I_{2m} + \delta M(y_0) J] F_{k01} = -P_{k01}. \tag{2.22}$$

(2.20) is equivalent to

$$[\sqrt{-1}\langle k,\ \omega(y_0)\rangle I_{4m^2} + \delta M(y_0) J \otimes I_{2m} + \delta I_{2m} \otimes (M(y_0)J)] F_{k02} = -P_{k02}, \tag{2.23}$$

where $\otimes$ denote the Kronecker product of two matrices. The above linear systems are solvable if the coefficient matrices are nonsingular. Let D^1 and D^2 be the coefficient matrices of (2.22) and (2.23), respectively. Consider the following set:

$$\begin{aligned} \mathcal{O}_+ = &\{y_0 \in \mathcal{O} : |\langle k,\ \omega(y_0)\rangle| > \frac{\gamma}{|k|^\tau}, \\ &|\mathrm{det} D^1| > \Big(\frac{\gamma}{|k|^\tau}\Big)^{2m},\ |\mathrm{det} D^2| > \Big(\frac{\gamma}{|k|^\tau}\Big)^{4m^2},\ k\in Z^n,\ 0<|k|\le K_+\}. \end{aligned}$$

Obviously, (2.17)–(2.21) are solvable on $\mathcal{O}_+$, and the solutions are unique and real analytic. And by (2.17)–(2.21) we have found the generalized Hamiltonian F and by the argument above we know that F is real analytic for all $y_0 \in \mathcal{O}_+$, $(y,x)\in D$.

Let $\Phi_+ = \Phi_F^1$ be the time-1 map of the equation of motion associated to F, i.e.,

$$\begin{pmatrix} \dot{y} \\ \dot{x} \\ \dot{z} \end{pmatrix} = I(y+y_0)\nabla(F(y,x,z). \tag{2.24}$$

According to the theory of generalized Hamiltonian, Φ_+ is a generalized canonical transformation, and moreover

$$\begin{aligned} H_+ &= H \circ \phi_+ \\ &= H \circ \phi_F^1 \\ &= (N+R) \circ \phi_F^1 + (P-R) \circ \phi_F^1 \\ &= N + R + \{N,\ F\} + \int_0^1 \{R_t, F\} \circ \phi_F^t \mathrm{d}t + (P-R) \circ \phi_F^1 \\ &= (N + [R] - \langle P_{001},\ z\rangle) + (\{N,\ F\} + R - [R] + \langle P_{001},\ z\rangle - Q) \\ &\quad + \int_0^1 \{R_t,\ F\} \circ \phi_F^t \mathrm{d}t + (P-R) \circ \phi_F^1 + Q \\ &= N + [R] - \langle P_{001},\ z\rangle + \int_0^1 \{R_t, F\} \circ \phi_F^t \mathrm{d}t + (P-R) \circ \phi_F^1 + Q, \end{aligned} \tag{2.25}$$

where

$$R_t = \{(1-t)N, F\} + R.$$

Let

$$h_+ = h + P_{000}, \tag{2.26}$$

$$\Omega_+ = \Omega + P_{010}, \qquad P_{010} \le \delta\gamma^{4m^2}\mu, \tag{2.27}$$

$$\omega_+ = -B^T \Omega_+, \tag{2.28}$$

$$M_+ = M + \frac{2}{\delta} P_{002}, \qquad P_{002} \le \delta\gamma^{4m^2}\mu, \tag{2.29}$$

$$N_+ = N + [R] - \langle P_{001},\ z\rangle = h_+(y_0) + \langle \Omega_+(y_0),\ y\rangle + \frac{\delta}{2}\langle z,\ M_+(y_0) z\rangle, \tag{2.30}$$

$$P_+ = \int_0^1 \{R_t, F\} \circ \phi_F^t \mathrm{d}t + (P-R) \circ \phi_F^1 + Q = P_1 + P_2 + P_3. \tag{2.31}$$

Then

$$H_+ = N_+ + P_+$$

is the new generalized Hamiltonian with normal form N_+.

2.3 Estimate on the New Hamiltonian

We now give some estimates on F and its derivatives, which are vital in proving the convergence of the transformation sequence and in estimating the new perturbation.

Lemma 2.2 *There are constants $c_2 > 0$, $c_3 > 0$ such that*

1) *on* $\mathcal{O}_+$,

$$|F_{k00}| \le c_2|k|^\tau \delta s^2 \mu \mathrm{e}^{-|k|r},$$
$$|F_{k10}| \le c_2|k|^\tau \delta \mu \mathrm{e}^{-|k|r},$$
$$|F_{k01}| \le c_2|k|^\tau \delta s \mu \mathrm{e}^{-|k|r},$$
$$|F_{k02}| \le c_2|k|^\tau \delta \mu \mathrm{e}^{-|k|r},$$
$$|F_{001}| \le c_2 \delta s \mu;$$

2) *on* $\mathcal{D}_{**} \times \mathcal{O}_+$, $\tilde{\mathcal{D}}(\beta_+) \times \mathcal{O}_+$,

$$|F|,\ (r-r_+)|F_x|,\ s^2|F_y|,\ s|F_z|,\ s^2|F_{zz}| \le c_3\delta s^2\mu\Gamma(r-r_+) + c_3\delta s^2\mu.$$

Proof. According to (2.17)–(2.21) and Cauchy's estimate, we can get the conclusion.

Lemma 2.3 *Assume that*

H3) $\dfrac{c_3\delta s^2\mu\Gamma(r-r_+) + c_3\delta s^2\mu}{r-r_+} < 7\alpha^2 s^2$;

H4) $\dfrac{c_3\delta s^2\mu\Gamma(r-r_+) + c_3\delta s^2\mu}{r-r_+} + c_3\delta\mu\Gamma(r-r_+) + c_3\delta\mu) < \dfrac{r-r_+}{8}$;

H5) $c_3\delta s\mu\Gamma(r-r_+) + c_3\delta s\mu < \alpha s$;

H6) $c_3\delta\mu\Gamma(r-r_+) + c_3\delta\mu < \beta - \beta_+$.

Then the following hold:

1) *Let* ϕ_F^t *be the flow generated by equation* (2.26). *Then*

$$\phi_F^t : \mathcal{D}_3 \to \mathcal{D}_4, \qquad 0 \le t \le 1;$$

2) $\Phi_+ : \mathcal{D}_+ \to \mathcal{D}(s,r)$, $\tilde{\mathcal{D}}(\beta-\beta_+) \to \tilde{\mathcal{D}}(\beta)$;

3) *There is a constant* c_4 *such that*

$$|\phi_F^t - \mathrm{id}|_{\tilde{\mathcal{D}}_+\times\mathcal{O}_+} \le c_4\delta\mu\Gamma(r-r_+) + c_4\delta\mu,$$
$$|D\phi_F^t - \mathrm{Id}|_{\tilde{\mathcal{D}}_+\times\mathcal{O}_+} \le c_4\delta\mu\Gamma(r-r_+) + c_4\delta\mu,$$

where $0 \le |t| \le 1$;

4)

$$|\Phi_+ - \mathrm{id}|_{\tilde{\mathcal{D}}_+\times\mathcal{O}_+} \le c_4\delta\mu\Gamma(r-r_+),$$
$$|D\Phi_+ - \mathrm{Id}|_{\tilde{\mathcal{D}}_+\times\mathcal{O}_+} \le c_4\delta\mu\Gamma(r-r_+).$$

To save space, we omitted the proof of the Lemma.

2.4 Estimation on the New Perturbation

Lemma 2.4 *There is a constant* $c_5 > 0$, *such that when* ε *is sufficiently small the following hold*:

$$|h_+ - h|_{\mathcal{O}_+} \le c_5\delta\gamma^{4m^2} s^2\mu,$$
$$|\Omega_+ - \Omega|_{\mathcal{O}_+} \le c_5\delta\gamma^{4m^2}\mu,$$
$$|\omega_+ - \omega|_{\mathcal{O}_+} \le c_5\delta\gamma^{4m^2}\mu,$$
$$|M_+ - M|_{\mathcal{O}_+} \le c_5\gamma^{4m^2}\mu.$$

Proof. The above inequalities obviously come from (2.26)–(2.29).

Lemma 2.5 *Assume that*

H7) $c_5\gamma_0^{4m^2}\delta\mu < \dfrac{\gamma-\gamma_+}{K_+^{\tau+1}}$.

Then for any $y_0 \in \mathcal{O}_+$, $0 < |k| \le K_+$,

$$|\langle k,\ \omega_+(y_0)\rangle| \ge \frac{\gamma_+}{|k|^\tau}, \quad |\det D^1_{\nu+1}| \ge \left(\frac{\gamma_+}{|k|^\tau}\right)^{2m}, \quad |\det D^2_{\nu+1}| \ge \left(\frac{\gamma_+}{|k|^\tau}\right)^{4m^2}.$$

Proof. By H7) and Lemma 2.4 we have

$$|\langle k,\ \omega_+(y_0)\rangle| \ge |\langle k,\ \omega(y_0)\rangle| - c_5\delta\gamma_0^{4m^2}\mu K_+ \ge \frac{\gamma_+}{|k|^\tau}.$$

Similarly we can get the other two inequalities.

By the definition of P_+, we have

$$|P_+|_{\mathcal{D}_+\times\mathcal{O}_+} \le \sum_{i=1}^{3}|P_i|_{\mathcal{D}_+\times\mathcal{O}_+}.$$

Lemma 2.6 *Assume that*

H8) $\sum\limits_{i=1}^{3}|P_i|_{\mathcal{D}_+\times\mathcal{O}_+} \le \delta\gamma_+^{4m^2}s_+^2\mu_+$.

Then $|P_+|_{\mathcal{D}_+\times\mathcal{O}_+} \le \delta\gamma_+^{4m^2}s_+^2\mu_+$.

3 Iteration Lemma

In this section, we shall prove an iteration Lemma which guarantees the inductive construction of all the transformation in all KAM steps. Let $r_0,\ s_0,\ \mu_0,\ \mathcal{O}_0,\ H_0,\ N_0,\ e_0,\ \varOmega_0,\ M_0,\ P_0$ be given as at the beginning of Section 2, $\mathcal{D}_0 = \mathcal{D}(s_0, r_0)$, $K_0 = 0$, and $\varPhi_0 = \mathrm{id}$. We define the following sequences as in Section 2 inductively for all $\nu = 1, 2, \cdots$:

$$r_\nu,\ s_\nu,\ \mu_\nu,\ K_\nu,\ \mathcal{O}_\nu,\ \mathcal{D}_\nu,\ \tilde{\mathcal{D}}_\nu,\ H_\nu,\ N_\nu,\ e_\nu,\ \varOmega_\nu,\ M_\nu,\ P_\nu,\ \varPhi_\nu, \qquad \nu = 1, 2, \cdots,$$

where $(y, x, z) \in \mathcal{D}_\nu$, $y_0 \in \mathcal{O}_\nu$, $\omega_\nu(y_0) = -B^T(y_0)\varOmega_\nu(y_0)$, $e_\nu(y_0)$, $\varOmega_\nu(y_0)$, $M(y_0)$ are real analytic functions defined on $\mathcal{O}_\nu$ respectively, and $P_\nu(y, y_0, x, z)$ is a real analytic function defined on $\mathcal{D}_\nu \times \mathcal{O}_\nu$.

Lemma 3.1(Iteration Lemma) *Let*

$$\tilde{\mu} = \mu_0^{1-\sigma}, \qquad \sigma \ll 1.$$

If $\mu_0(\varepsilon_0)$ *is sufficiently small, then for all* $\nu = 0, 1, \cdots$ *the following hold:*

1)

$$|h_\nu - h_0|_{\mathcal{O}_\nu} \le 2\delta\gamma_0\tilde{\mu}, \qquad |h_{\nu+1} - h_\nu|_{\mathcal{O}_{\nu+1}} \le \frac{\delta\gamma_0\tilde{\mu}}{2^{\nu+1}},$$

$$|\varOmega_\nu - \varOmega_0|_{\mathcal{O}_\nu} \le 2\delta\gamma_0\tilde{\mu}, \qquad |\varOmega_{\nu+1} - \varOmega_\nu|_{\mathcal{O}_{\nu+1}} \le \frac{\delta\gamma_0\tilde{\mu}}{2^{\nu+1}},$$

$$|\omega_\nu - \omega_0|_{\mathcal{O}_\nu} \le 2\delta\gamma_0\tilde{\mu}, \qquad |\omega_{\nu+1} - \omega_\nu|_{\mathcal{O}_{\nu+1}} \le \frac{\delta\gamma_0\tilde{\mu}}{2^{\nu+1}},$$

$$|M_\nu - M_0|_{\mathcal{O}_\nu} \le 2\gamma_0\tilde{\mu}, \qquad |M_{\nu+1} - M_\nu|_{\mathcal{O}_\nu} \le \frac{\gamma_0\tilde{\mu}}{2^{\nu+1}},$$

$$|P_\nu|_{\mathcal{D}_\nu\times\mathcal{O}_\nu} \le \delta\gamma_\nu^{4m^2}s_\nu^2\mu_\nu.$$

2) *$\Phi_{\nu+1} : \tilde{\mathcal{D}}_{\nu+1} \times \mathcal{O}_{\nu+1} \to \tilde{\mathcal{D}}_\nu$ is a generalized canonical transformation, and is real analytic with respect to $(y,x,z) \in \tilde{\mathcal{D}}_{\nu+1}$, $y_0 \in \mathcal{O}_{\nu+1}$; moreover,*

$$H_{\nu+1} = H_\nu \circ \Phi_{\nu+1},$$

and on $\tilde{D}_{\nu+1} \times \mathcal{O}_{\nu+1}$, we have

$$|\Phi_{\nu+1} - \mathrm{id}|,\ |D\Phi_{\nu+1} - \mathrm{Id}| \leq \frac{\tilde{\mu}}{2^{\nu+1}}. \tag{3.1}$$

3)

$$\begin{aligned}\mathcal{O}_{\nu+1} =& \{y_0 \in \mathcal{O}_\nu : |\langle k,\ \omega_\nu(y_0)\rangle| > \frac{\gamma_\nu}{|k|^\tau},\\ & |\mathrm{det} D_\nu^1| > \Big(\frac{\gamma_\nu}{|k|^\tau}\Big)^{2m},\ |\mathrm{det} D_\nu^2| > \Big(\frac{\gamma_\nu}{|k|^\tau}\Big)^{4m^2},\ K_\nu < |k| \leq K_{\nu+1}\}.\end{aligned}$$

Proof. To prove parts 1), 2) of Lemma 3.1 we only need to verify the conditions H1)–H8) in Section 2 for all $\nu = 0, 1, \cdots$. By the definition of μ_ν, s_ν, we have

$$r_\nu - r_{\nu+1} = \frac{1}{8}\Big(\frac{7}{8}\Big)^2\Big(\frac{7}{8}\Big)^\nu,$$

$$s_{\nu+1} = \alpha_\nu s_\nu = \mu_\nu^{\frac{1}{3}} s_\nu = (s_0)^{(\frac{7}{6})^\nu},$$

$$\mu_{\nu+1} = (\mu_0)^{(\frac{7}{6})^\nu}.$$

Notice that

$$\begin{aligned}\Gamma_\nu(r_\nu - r_{\nu+1}) &= \sum_{0<|k|\leq K_{\nu+1}} |k|^{\tau+2} \mathrm{e}^{-\frac{r_\nu - r_{\nu+1}}{16}}\\ &\leq \int_1^\infty \lambda^{\tau+2+n} \mathrm{e}^{-\frac{r_\nu - r_{\nu+1}}{16}} \mathrm{d}\lambda\\ &\leq ([\tau]+3+n)!\Big[16 \cdot 8\Big(\frac{8}{7}\Big)^2\Big(\frac{8}{7}\Big)^\nu\Big]^{([\tau]+3+n)}\\ &\leq c_1^* \cdot (c_2^*)^\nu,\end{aligned} \tag{3.2}$$

where

$$c_1^* = ([\tau]+3+n)!\Big[16 \cdot 8\Big(\frac{8}{7}\Big)^2\Big]^{([\tau]+3+n)}, \qquad c_2^* = \Big(\frac{8}{7}\Big)^{([\tau]+3+n)}.$$

For any $a > 0$, choose $\lambda \gg 1$, such that

$$\mu_0 < \frac{1}{\lambda^{6\frac{1}{a}}} \ll 1.$$

Then

$$\begin{aligned}\mu_1 &= \mu_0^{\frac{1}{6}} \mu_0 < \frac{1}{\lambda^{\frac{1}{a}}} \mu_0,\\ \mu_2 &= \mu_1^{\frac{1}{6}} \mu_1 < \mu_0^{\frac{1}{6}} \mu_1 < \frac{1}{\lambda^{\frac{1}{a}}} \frac{1}{\lambda^{\frac{1}{a}}} \mu_0,\\ &\vdots\\ \mu_\nu &= \mu_{\nu-1}^{\frac{1}{6}} \mu_{\nu-1} < \mu_0^{\frac{1}{6}} \mu_{\nu-1} < \cdots < \frac{1}{\lambda^{\frac{\nu}{a}}} \mu_0,\end{aligned}$$

and moreover

$$\mu_\nu^a \quad < \quad \frac{1}{\lambda^\nu} \mu_0^a, \qquad \nu = 1, 2, \cdots \tag{3.3}$$

By the definition of α_ν, H1) is obvious as ε is sufficiently small.

By the definition of r_ν, s_ν, μ_ν, $K_{\nu+1}$, when ε is sufficiently small we have

$$\frac{(r_\nu - r_{\nu+1})}{16} \log \frac{1}{\mu_\nu} = -\frac{1}{16}\Big(\frac{7}{8}\Big)^2\Big(\frac{7}{8}\Big)^\nu\Big(\frac{7}{6}\Big)^\nu \log \mu_0 \geq 1,$$

$$
\begin{aligned}
&\log(n+1)! + 3n\log[\log\frac{1}{\mu_\nu}+1] - \frac{r_\nu - r_{\nu+1}}{16}[\log\frac{1}{\mu_\nu}+1]^3 \\
&\le \log(n+1)! + 3n\log[\log\frac{1}{\mu_\nu}+1] - 3\Big(\log\frac{1}{\mu_\nu}\Big) \le -\log\frac{1}{\mu_\nu},
\end{aligned}
$$

so

$$
\int_{K_{\nu+1}}^{\infty} \lambda^n e^{-\lambda(r_\nu - r_{\nu+1})/16} d\lambda \le (n+1)! K_{\nu+1}^n e^{-K_{\nu+1}(r_\nu - r_{\nu+1})/16} \le \mu_\nu,
$$

i.e., H6) and H2) hold.

Similarly H3), H4), H5), H7) and H8) can be verified.

It is obvious that 3) holds for $\nu = 0$. Now we suppose that for some $\nu > 0$ 3) holds. Then by Lemma 2.5,

$$
\begin{aligned}
\mathcal{O}_\nu =& \{y_0 \in \mathcal{O}_\nu : |\langle k,\ \omega_\nu(y_0)\rangle| > \frac{\gamma_\nu}{|k|^\tau}, \\
& |\det D_\nu^1| \ge \Big(\frac{\gamma_\nu}{|k|^\tau}\Big)^{2m},\ |\det D_\nu^2| \ge \Big(\frac{\gamma_\nu}{|k|^\tau}\Big)^{4m^2},\ 0<|k|\le K_\nu\}.
\end{aligned}
$$

So

$$
\begin{aligned}
\mathcal{O}_{\nu+1} =& \{y_0 \in \mathcal{O}_\nu : |\langle k,\ \omega_\nu(y_0)\rangle| > \frac{\gamma_\nu}{|k|^\tau}, \\
& |\det D_\nu^1| \ge \Big(\frac{\gamma_\nu}{|k|^\tau}\Big)^{2m},\ |\det D_\nu^2| \ge \Big(\frac{\gamma_\nu}{|k|^\tau}\Big)^{4m^2},\ 0<|k|\le K_{\nu+1}\} \\
=& \{y_0 \in \mathcal{O}_\nu : |\langle k,\ \omega_\nu(y_0)\rangle| > \frac{\gamma_\nu}{|k|^\tau}, \\
& |\det D_\nu^1| \ge \Big(\frac{\gamma_\nu}{|k|^\tau}\Big)^{2m},\ |\det D_\nu^2| \ge \Big(\frac{\gamma_\nu}{|k|^\tau}\Big)^{4m^2},\ K_\nu<|k|\le K_{\nu+1}\},
\end{aligned}
$$

which completes the proof of the Lemma.

4 Proof of the Main Result

Let $\tilde{\mu} = \mu_0^{1-\sigma}$ and $\sigma \ll 1$ sufficiently small. Then Lemma 3.1 holds for all $\nu = 0, 1, 2, \cdots$. Denote

$$
\Psi^\nu = \Phi_1 \circ \Phi_2 \circ \cdots \Phi_\nu, \qquad \nu = 1, 2, \cdots.
$$

By Lemma 3.1 we have

$$
\begin{aligned}
&\mathcal{D}_{\nu+1} \times \mathcal{O}_{\nu+1} \subset \mathcal{D}_\nu \times \mathcal{O}_\nu, \qquad \Psi^\nu : \tilde{\mathcal{D}}_\nu \times \mathcal{O}_{v+1} \to \mathcal{D}_0, \\
&H \circ \Psi^\nu = H_\nu = N_\nu + P_\nu, \qquad N_\nu = e_\nu(y_0) + \langle \Omega_\nu(y_0),\ y\rangle + \langle z,\ M_\nu(y_0)z\rangle.
\end{aligned}
$$

Let

$$
\mathcal{O}_* = \bigcap_{\nu=0}^{\infty} \mathcal{O}_\nu, \qquad \mathcal{D}_0 = \mathcal{D}\Big(\frac{\beta_0}{2}, \frac{15}{64} r_0\Big), \qquad G_* = \mathcal{D}\Big(\frac{\beta_0}{2}\Big).
$$

By Lemma 3.1, it clear that e_ν, Ω_ν, M_ν converge uniformly on $\mathcal{O}_*$, say, to e_∞, Ω_∞, M_∞ respectively. Hence N_ν converges uniformly on $\mathcal{D}_0 \times \mathcal{O}_*$ to

$$
N_\infty = e_\infty(y_0) + \langle \Omega_\infty(y_0),\ y\rangle + \langle z,\ M_\infty(y)z\rangle,
$$

which implies the uniform convergence of Ψ^ν. Let $\Psi^\nu \to \Psi^\infty$. Then

$$
\Psi^\infty = \mathrm{id} + \sum_{i=1}^{\infty} (\Psi^\nu - \Psi^{\nu-1}), \qquad |\Psi^\infty - \mathrm{id}|_{\mathcal{D}_0} = O(\tilde{\mu}) = O(\mu_0^{1-\sigma}).
$$

Thus $\varPsi^{\nu}$ is uniformly close to the identify and is real analytic on $\mathcal{D}\Big(\frac{\beta_0}{2}, \frac{35}{64}r_0\Big)$. Similarly, one can show the uniform convergence of $D\varPsi_{\infty}$. By a standard argument using the Whitney extension theorem, one can further show that $\varPsi^{\infty}$ is Whitney smooth with respect to $y_0 \in \mathcal{O}_*$.

By Lemma 3.1, 2) we have

$$|\varPhi_{\nu} - \mathrm{id}|_{\mathcal{D}_0\times\mathcal{O}_*} \le \frac{\tilde{\mu}}{2^{\nu}}.$$

By the definition of $\varPsi^{\nu}$, we have

$$\varPsi^{\nu} = \mathrm{id} + \sum_{i=1}^{\nu}(\varPsi^{\nu} - \varPsi^{\nu-1}), \, \varPsi^1 - \varPsi^0 = \varPhi_1 - \mathrm{id},$$

$$\begin{aligned} |\varPsi^{\nu} - \varPsi^{\nu-1}|_{\mathcal{D}_0\times\mathcal{O}_*} &= |\varPhi_1 \circ \varPhi_2 \cdots \varPhi_{\nu} - \varPhi_1 \circ \varPhi_2 \cdots \varPhi_{\nu-1}|_{\mathcal{D}_0\times\mathcal{O}_*} \\ &= \Big|\int_0^1 D(\varPhi_1 \circ \varPhi_2 \cdots \varPhi_{\nu-1})(\mathrm{id} + \theta(\varPhi_{\nu} - \mathrm{id}))\mathrm{d}\theta(\varPhi_{\nu} - \mathrm{id})\Big|_{\mathcal{D}_0\times\mathcal{O}_*} \\ &\le |D\varPhi_1(\varPhi_2 \circ \cdots \circ \varPhi_{\nu-1})(\mathrm{id} + \theta(\varPhi_{\nu} - \mathrm{id}))|_{\mathcal{D}_0\times\mathcal{O}_*} \cdots \\ &\quad |D\varPhi_{\nu-1}(\mathrm{id} + \theta(\varPhi_{\nu} - \mathrm{id}))|_{\mathcal{D}_0\times\mathcal{O}_*}|\varPhi_{\nu} - \mathrm{id}|_{\mathcal{D}_0\times\mathcal{O}_*} \\ &\le \Big(1 + \frac{\tilde{\mu}}{2}\Big)\cdots\Big(1 + \frac{\tilde{\mu}}{2^{\nu-1}}\Big)\frac{\tilde{\mu}}{2^{\nu}} \\ &\le \mathrm{e}^{\tilde{\mu}}\frac{\tilde{\mu}}{2^{\nu}}, \end{aligned}$$

so $\varPsi^{\nu}$ converge uniformly on $\mathcal{D}_0 \times \mathcal{O}_*$. Let $\varPsi^{\nu} \to \varPsi^{\infty}$. Then

$$\varPsi^{\infty} = \mathrm{id} + \sum_{i=1}^{\infty}(\varPsi^{\nu} - \varPsi^{\nu-1}), \qquad |\varPsi^{\infty} - \mathrm{id}|_{\mathcal{D}_0\times\mathcal{O}_*} = O(\tilde{\mu}) = O(\mu_0^{(1-\sigma)}).$$

This indicates that $\varPsi^{\nu}$ are uniformly close to the identity, and real analytic on $\mathcal{D}_0$. In the same way we can prove the uniform convergence of $D\varPsi^{\nu}$. By the standard Whitney extension theorem, we can prove that, for all $y_0 \in \mathcal{O}_*$, $\varPsi^{\infty}$ are Whitney smooth. Hence

$$P_{\nu} = H \circ \varPsi^{\nu} - N_{\nu}$$

converges uniformly on $\mathcal{D}_0 \times \mathcal{O}_*$, say, to

$$P_{\infty} = H \circ \varPsi^{\infty} - N_{\infty}.$$

By Lemma 3.1, we have

$$|P_{\nu}|_{\mathcal{D}_{\nu}} \le \delta\gamma_{\nu}^{4m^2}s_{\nu}^2\mu_{\nu}.$$

Thus

$$|\partial_z^p\partial_y^l P_{\infty}|_{\mathcal{D}(0,\frac{15r_0}{64})} = 0, \qquad |p| + 2|l| \le 2.$$

So for any $y_0 \in \mathcal{O}_*$, the generalized Hamiltonian

$$H_{\infty} = N_{\infty} + P_{\infty}$$

admits an analytic, quasi-periodic, invariant torus

$$T_{y_0} = \{0\} \times \{0\} \times T^n$$

with Diophantine frequency

$$\omega_{\infty}(y_0) = -B^T(y_0)\varOmega_{\infty}(y_0).$$

Moreover, these invariant tori form a Whitney smooth family.

For the measure estimate 3), it has been proved in detail in [22], and therefore we state it without proof here.

This completes the proof of Theorem A.

Below, we list three technical Lemmas which have been used in the previous sections.

5 Technical Lemmas

Lemma 5.1[24] *Let $\Lambda \subset \mathbb{R}^d$ $(d > 1)$ be a bounded closed region and suppose that $g : \Lambda \to \mathbb{R}^d$ satisfies*

$$\mathrm{rank}\Big\{\frac{\partial^\alpha g}{\partial \lambda^\alpha} : |\alpha| \le d-1\Big\} = d.$$

Then for a fixed $\tau > d(d-1)-1$,

$$\Big|\Big\{\lambda \in \Lambda : | < g(\lambda), k > | \le \frac{\gamma}{|k|^\tau}\Big\}\Big| \le c(\Lambda, d, \tau)\Big(\frac{\gamma}{|k|^{\tau+1}}\Big)^{\frac{1}{d-1}}, \qquad k \in \mathbb{Z}^d \setminus \{0\},\ \gamma > 0.$$

Lemma 5.2 *Let $G \subset R^n$, $\omega(y) : G \to R^n$ is real analytic and for all $y \in G$*

$$\mathrm{rank}\Big\{\frac{\partial^i \omega(y)}{\partial y^i} : |i| \le n-1\Big\} = n, \qquad \text{where } \ i \in Z_+^n,\ |i| = \sum_{j=1}^n |i_j|. \tag{5.1}$$

Denote

$$G^k = \Big\{y : |\langle k,\ \omega(y)\rangle + g(y)| \le \frac{\gamma}{|k|^\tau}\Big\}, \qquad k \in Z^n \backslash \{0\}.$$

Then there exist constants $c > 0$ and $\eta > 0$, such that if

$$|g(y)|_{C^{n-1}} \le \eta,$$

then

$$|G^k| \le c\Big(\frac{\gamma}{|k|^{\tau+1}}\Big)^{\frac{1}{n-1}}.$$

Proof. According to the proof of Lemma 2.4 of [24], Lemma 5.2 can be easily proved.

Lemma 5.3 *Let $\lambda_1(y), \cdots, \lambda_{2m}(y)$ be eigenvalues of $M(y)J$. Then the following hold: for all $k \in Z^n$*

$$\det(\sqrt{-1}\langle k,\ \omega(y)\rangle I_{2m} - M(y)J) = \prod_{i=1}^{2m}(\sqrt{-1}\langle k,\ \omega(y)\rangle - \lambda_i(y)),$$

$$\begin{aligned}&\det[\sqrt{-1}\langle k,\ \omega(y)\rangle I_{4m^2} - (M(y)J) \otimes I_{2m} - I_{2m} \otimes (M(y)J)]\\ =& \prod_{i,j=1}^{2m}(\sqrt{-1}\langle k,\ \omega(y)\rangle - \lambda_i(y) - \lambda_j(y)).\end{aligned}$$

Proof. Since, for any square matrices A, B, C, D of the same dimension, we have

$$(A \otimes B)(C \otimes D) = (AC \otimes BD).$$

It follows that

$$(T^{-1} \otimes T^{-1})((MJ) \otimes I_{2m} + I_{2m} \otimes (MJ))(T \otimes T) = \hat{E} \otimes I_{2m} + I_{2m} \otimes \hat{E}.$$

So

$$\begin{aligned}&\det[\sqrt{-1}\langle k,\ \omega(y)\rangle I_{4m^2} - (M(y)J) \otimes I_{2m} - I_{2m} \otimes (M(y)J)]\\ =& \det[\sqrt{-1}\langle k,\ \omega(y)\rangle I_{4m^2} - \hat{E} \otimes I_{2m} - I_{2m} \otimes \hat{E}]\\ =& \prod_{i,j=1}^{2m}(\sqrt{-1}\langle k,\ \omega(y)\rangle - \lambda_i(y) - \lambda_j(y)).\end{aligned}$$

The other equality can be proved similarly.

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