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The $L(3,2,1)$-labeling on Bipartite Graphs*

Yuan Wan-lian[1], Zhai Ming-qing[1,2] and Lü Chang-hong[2]
(1. Department of Mathematics, Chuzhou University, Chuzhou, Anhui, 239012)
(2. Department of Mathematics, East China Normal University, Shanghai, 200241)

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Abstract: An $L(3,2,1)$-labeling of a graph G is a function from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u)-f(v)| \geq 3$ if $d_G(u,v)=1$, $|f(u)-f(v)| \geq 2$ if $d_G(u,v)=2$, and $|f(u)-f(v)| \geq 1$ if $d_G(u,v)=3$. The $L(3,2,1)$-labeling problem is to find the smallest number $\lambda_3(G)$ such that there exists an $L(3,2,1)$-labeling function with no label greater than it. This paper studies the problem for bipartite graphs. We obtain some bounds of λ_3 for bipartite graphs and its subclasses. Moreover, we provide a best possible condition for a tree T such that $\lambda_3(T)$ attains the minimum value.

Key words: channel assignment problems, $L(2,1)$-labeling, $L(3,2,1)$-labeling, bipartite graph, tree

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1 Introduction

The problem of vertex labeling with a condition at distance two arises from the channel assignment problem introduced by Hale[1]. For a given graph G, an $L(2,1)$-labeling is defined as a function

$$f: \ V(G) \to \{0,1,2,\cdots\}$$

such that

$$|f(u)-f(v)| \geq \begin{cases} 2, & d_G(u,v)=1; \\ 1, & d_G(u,v)=2, \end{cases}$$

where $d_G(u,v)$, the distance between u and v, is the minimum length of a path between u and v. A k-$L(2,1)$-labeling is an $L(2,1)$-labeling such that no integer is greater than k. The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$, is the smallest number k such that G has a

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k-$L(2,1)$-labeling. The $L(2,1)$-labeling problem has been extensively studied in recent years (see [2]–[9]).

Shao and Liu[10] extend $L(2,1)$-labeling problem to $L(3,2,1)$-labeling problem. For a given graph G, a k-$L(3,2,1)$-labeling is defined as a function

$$f: \ V(G) \to \{0, 1, 2, \cdots k\}$$

such that

$$|f(u) - f(v)| \geq 4 - d_G(u, v), \qquad d_G(u, v) \in \{1, 2, 3\}.$$

The $L(3,2,1)$-labeling number of G, denoted by $\lambda_3(G)$, is the smallest number k such that G has a k-$L(3,2,1)$-labeling. Clearly,

$$\lambda_3(G) \geq 2\Delta(G) + 1$$

for any non-empty graph G. It was showed that

$$\lambda_3(G) \leq \Delta^3 + 2\Delta$$

for any graph G and

$$\lambda_3(T) \leq 2\Delta + 3$$

for any tree T (see [11]). This paper focuses on bipartite graphs. In Section 2, we obtain some bounds of λ_3 for bipartite graphs and its subclasses, where the bound for bipartite graphs is $O(\Delta^2)$. In Section 3 we provide a best possible condition for a tree T with $\Delta(T) \geq 5$ and such that $\lambda_3(T)$ attains the minimum value, that is, $\lambda_3(T) = 2\Delta + 1$ if the distance between any two vertices of maximum degree is not in $\{2, 4, 6\}$.

All graphs considered here are non-empty, undirected, finite, simple graphs. For a graph G, we denote its vertex set, edge set and maximum degree by $V(G)$, $E(G)$ and $\Delta(G)$, respectively. For a vertex $v \in V(G)$, let

$$N_G^k(v) = \{u | d_G(u, v) = k\}, \qquad N_G[v] = N_G(v) \cup \{v\},$$

and $d_G(v)$ be the degree of v in G. A vertex of degree k is called a k-vertex. Especially, a 1-vertex of a tree is called a leaf or a pendant vertex. Let

$$D_\Delta(G) = \{d_G(u, v) | u, v \text{ are two } \Delta\text{-vertices}\}.$$

If there are no confusions in the context, we use V, Δ, λ_3, $N^k(v)$, $N[v]$, $d(v)$, $d(u, v)$ and D_Δ to denote $V(G)$, $\Delta(G)$, $\lambda_3(G)$, $N_G^k(v)$, $N_G[v]$, $d_G(v)$, $d_G(u, v)$ and $D_\Delta(G)$, respectively. And we use k-labeling to denote k-$L(3,2,1)$-labeling.

2 Bounds of λ_3 on Bipartite Graphs

First, we summarize some easy observations into the following lemma.

Lemma 2.1 *For any graph G,*

(i) *if $\lambda_3 = 2\Delta + 1$ and f is a $(2\Delta + 1)$-labeling, then $f(u) \in \{0, 2\Delta + 1\}$ for any Δ-vertex u;*

(ii) *if f is a k-labeling of G, then $k - f$ is a k-labeling of G;*

(iii) *if G is connected and its diameter $d \in \{1, 2, 3\}$, then $\lambda_3 \geq (|V| - 1)(4 - d)$.*

Lemma 2.2 *For the complete bipartite graph $K_{r,s}$, $\lambda_3 = 2r + 2s - 1$.*

Proof. First, we show that

$$\lambda_3(K_{r,s}) \geq 2r + 2s - 1$$

by induction on $r + s$. The equality holds clearly if $r = 1$ or $s = 1$. Let $r, s > 1$. Since $K_{r,s}$ is of diameter at most 2, by Lemma 2.1(iii),

$$\lambda_3 \geq 2r + 2s - 2.$$

Assume that there is a $(2r + 2s - 2)$-labeling f of $K_{r,s}$ and

$$f(u) = 2r + 2s - 2, \qquad \text{for some } u \in V.$$

Since $K_{r,s} - u$ is isomorphic to $K_{r-1,s}$ or $K_{r,s-1}$, by induction hypothesis,

$$\lambda_3(K_{r,s} - u) \geq 2r + 2s - 3.$$

Hence, there is a vertex $v \in V\backslash\{u\}$ such that $f(v) \in \{2r+2s-3, 2r+2s-2\}$. This implies that

$$|f(u) - f(v)| \leq 1,$$

which contradicts

$$d(u, v) \leq 2.$$

Thus

$$\lambda_3(K_{r,s}) \geq 2r + 2s - 1.$$

Now we have to give a $(2r + 2s - 1)$-labeling of $K_{r,s}$. Let

$$K_{r,s} = (V_1,\ V_2,\ E),$$

where $|V_1| = r$. We can label the vertices in V_1 by $\{0, 2, \cdots, 2r - 2\}$ and label the vertices in V_2 by $\{2r + 1,\ 2r + 3,\ \cdots,\ 2r + 2s - 1\}$, respectively.

Theorem 2.1 $\lambda_3 \leq 2|V| - 1$ *for any bipartite graph G. The equality holds if and only if G is a complete bipartite graph.*

Proof. Note that $\lambda_3(H) \leq \lambda_3(G)$ for any subgraph H of G. By Lemma 2.2, we need only to prove that $\lambda_3 < 2|V| - 1$ if G is a non-complete bipartite graph. We next give a stronger result.

Claim 2.1 $\lambda_3 \leq 2|V| - 3$ *if G is a non-complete bipartite graph.*

Let

$$G = (V_1,\ V_2,\ E),$$

where

$$|V_1| = r, \qquad |V_2| = s.$$

Since G is non-complete, there are two vertices u and v such that $u \in V_1$, $v \in V_2$ and $uv \notin E(G)$. Thus we can label the vertices in $V_1\backslash\{u\}$ by $\{0, 2, 4, \cdots, 2r - 4\}$, u by $2r - 2$, v by $2r - 1$ and the vertices in $V_2\backslash\{v\}$ by $\{2r + 1, 2r + 3, 2r + 5, \cdots, 2r + 2s - 3\}$.

We now introduce a special $L(3,2,1)$-labeling. An $L(3,2,1)$-labeling f of G is said to be regular if $f(x)$ and $f(y)$ have different parity for any $xy \in E(G)$. Clearly, G has a regular labeling if and only if G is a bipartite graph.

Theorem 2.2 $\lambda_3 \leq 2(\Delta^2 + \Delta)$ *for any bipartite graph* G.

Proof. Let
$$G = (V_1,\ V_2,\ E).$$
We apply induction on $|V_2|$ to prove that G has a regular $2(\Delta^2 + \Delta)$-labeling such that all the vertices in V_1 get odd labels. By Lemma 2.2,
$$\lambda_3(K_{\Delta,1}) = 2\Delta + 1.$$
Therefore, the conclusion holds for $|V_2| = 1$. Now assume that $|V_2| > 1$ and $v \in V_2$. By induction hypothesis, $G - v$ has a regular $2(\Delta^2 + \Delta)$-labeling f such that $f(x)$ is odd for each $x \in V_1$. We observe that each vertex in $N(v)$ forbids two even labels for v and each vertex in $N^2(v)$ forbids one even label for v. Thus there are at most $\Delta^2 + \Delta$ even labels cannot be used for v and hence v can select an even label.

A connected graph without cycle is a tree. A connected graph is said unicyclic if it contains exactly one cycle. It is known that
$$2\Delta + 1 \leq \lambda_3 \leq 2\Delta + 3$$
for any tree (see [11]). Next we extend this result to a more general subclass of bipartite graphs.

Lemma 2.3[11] *Let* C_n *be a cycle of length* n. *If* n *is even, then* $\lambda_3 = 7$ *and* C_n *has a regular* 7*-labeling.*

Theorem 2.3 *Let* G *be a bipartite graph with each connected component either a tree or a unicyclic graph. Then* $2\Delta + 1 \leq \lambda_3 \leq 2\Delta + 3$.

Proof. Note that $\lambda_3(G) = \lambda_3(H)$ for some connected component H of G. It suffices to consider the case when H is unicyclic. We now use induction on $|V(H)|$ to show that H has a regular $(2\Delta + 3)$-labeling. If $|V(H)| = 4$, then $H \cong C_4$, since H contains no cycle of length odd. By Lemma 2.3, H has a regular 7-labeling. Let $|V(H)| > 4$. If H itself is a cycle, then by Lemma 2.3, H has a regular 7-labeling. Otherwise, let x be a 1-vertex and $N_H(y) = \{x, x_1, x_2 \cdots, x_k\}$. By induction hypothesis, $H - x$ has a regular $(2\Delta + 3)$-labeling f. We assume, without loss of generality, that $f(y)$ is even. Then $f(x_i)$ is odd for each $i \in \{1, 2, \cdots, k\}$. Since $k \leq \Delta - 1$, there exists at least an odd label in $\{1, 3, \cdots, 2\Delta + 3\} \backslash \{f(y) \pm 1, f(x_i) | i = 1, 2, \cdots, k\}$ for x to use. Thus we obtain a regular $(2\Delta + 3)$-labeling of H.

3 Minimizing λ_3 Number for Trees

A star (generalized star) is a tree containing at most one (two, respectively) vertices of degree great than 1. A major handle (weak major handle) of a tree is a Δ-vertex adjacent to exactly one (two, respectively) vertices of degree greater than 1. This section gives several conditions for a tree such that $\lambda_3 = 2\Delta + 1$. Since the values of λ_3 for paths have been given in [11], we next let $\Delta \geq 3$. The following result is clear.

Lemma 3.1 *If one of the following is satisfied by a tree T, then T has a regular $(2\Delta+1)$-labeling.*

(i) *T is a generalized star*;

(ii) *T contains a leaf v which is adjacent to a vertex of degree less than Δ and $T-v$ has a regular $(2\Delta(T)+1)$-labeling*;

(iii) *T contains a major handle x_1 with non-pendant neighbor x_2 and $T-(N(x_1)\backslash\{x_2\})$ has a regular $(2\Delta(T)+1)$-labeling f such that $f(x_1)\in\{0,2\Delta+1\}$.*

Lemma 3.2 *Let T be a tree with $\Delta\geq 4$ and $2,4,6\notin D_{\Delta}$. If T is not a generalized star, then T contains one of the following configurations*:

(C1) *A leaf v adjacent to a vertex u with $d(u)<\Delta$*;

(C2) *A path $x_1x_2x_3x_4x_5$ such that $d(x_2)=d(x_3)=2$, x_4 is either a major handle or a weak major handle, and x_1 is a major handle*;

(C3) *A path $x_1x_2x_3x_4x_5$ such that $d(x_2)=d(x_3)=d(x_4)=2$, and x_1 is a major handle*;

(C4) *A path $x_1x_2x_3x_4y_1y_2$ such that $d(x_2)=d(x_3)=d(y_1)=2$, $d(x_4)=3$ and x_1,y_2 are both major handles*;

(C5) *A path $x_1x_2x_3x_4x_5$ such that $d(x_3)=d(x_4)=2$, $d(x_5)\leq\Delta-1$, x_1 is a major handle and x_2 is a weak major handle*;

(C6) *A path $x_1x_2x_3x_4x_5$ such that $d(x_2)=d(x_4)=2$, $d(x_5)\leq\Delta-1$, $d(x_3)=3$, x_1 and another neighbor y of x_3 are major handles.*

Proof. Suppose that T does not contain (C1), (C3), (C4), (C5) and (C6). We have to show that T contains (C2). Let $P_1=x_0x_1x_2\cdots x_m$ be a longest path in T. Since T contains no (C1), x_1 and x_{m-1} are both major handles. Since T is not a generalized star, $m\geq 4$. Furthermore $m>4$; otherwise, $d(x_3)=\Delta$ and $d(x_1,x_3)=2$, which contradicts $2\notin D_{\Delta}$.

Claim 3.1 $d(x_2)=2$.

Suppose that $d(x_2)>2$. Since P_1 is the longest and T contains no (C1), x_2 is a weak major handle. Since $2,4,6\notin D_{\Delta}$, we immediately have $d(x_3)=d(x_4)=2$ and $d(x_5)\neq\Delta$. Thus T contains (C5), a contradiction.

Claim 3.2 $d(x_3)=2$.

Clearly $d(x_3),d(x_5)\neq\Delta$. Suppose that $d(x_3)>2$ and let $P_2=x_3y_1y_2\cdots y_k$ be a longest path starting from x_3 and not along P_1. Since P_1 is the longest and T contains no (C1), $2\leq k\leq 3$ and y_{k-1} is a major handle. Moreover, $k\neq 3$ since $d(y_2,x_1)=4$. That is, $k=2$ and y_1 is a major handle. And hence $d(x_4)=2$, since $2,4,6\notin D_{\Delta}$. Now T contains (C6), a contradiction.

Claim 3.3 *x_4 is either a major handle or a weak major handle.*

Since T contains no (C3), $d(x_4)>2$. Let $P_3=x_4y_1y_2\cdots y_k$ be a longest path starting from x_4 and not along P_1. First, assume that $k\neq 1$. Since P_1 is the longest and T contains no (C1), $2 \le k \le 4$ and y_{k-1} is a major handle. Moreover $k \notin \{2, 4\}$, since $d(y_1, x_1) = 4$ and $d(y_3, x_1) = 6$. That is, $k = 3$ and y_2 is a major handle. And hence $d(y_1) = 2$. Now, if $d(x_4) = 3$, then T contains (C4), a contradiction. If $d(x_4) > 3$, we denote by $P_4 = x_4 z_1 z_2 \cdots z_t$ a longest path starting from x_4 and not going along P_1 and P_3. Then $t \ne 1$ (Otherwise, $d(x_4) = \Delta$ and $d(x_4, y_2) = 2$, which contradicts $2 \notin D_\Delta$.). Similar to $k \ne 1$, we have $t = 3$ and z_2 is a major handle. However, $d(y_2,\ z_2) = 4$, which contradicts $4 \notin D_\Delta$. So $k = 1$. Since T contains no (C1), $d(x_4) = \Delta$. Thus x_4 is either a major handle or a weak major handle.

Let T_1 be a subtree of a tree T. T_1 is called a Δ-subtree of T if $\Delta(T_1) = \Delta(T)$. Lemma 3.2 and its proof indicate the following result. It is necessary to the induction proofs of our main theorem.

Lemma 3.3 *Let T be a tree that contains no* (C1).

(i) *If T contains* (C2)*, then $T - N[x_1]$ is a Δ-subtree of T.*

(ii) *If T contains* (C3) *or* (C4)*, then $T - (N[x_1] \cup \{x_3\})$ is a Δ-subtree of T.*

(iii) *If T contains* (C5)*, then $T - (N(x_1) \cup N(x_2) \cup \{x_4\})$ is a Δ-subtree of T.*

(iv) *If T contains* (C6)*, then $T - (N[x_1] \cup N[y] \cup \{x_4\})$ is a Δ-subtree of T.*

Lemma 3.4 *Let T be a tree with $\Delta \ge 4$. If one of the following is satisfied, then T has a regular* $(2\Delta + 1)$*-labeling.*

(i) *T contains* (C2) *and $T - N[x_1]$ has a regular* $(2\Delta + 1)$*-labeling;*

(ii) *T contains* (C3) *or* (C4) *and $T - (N[x_1] \cup \{x_3\})$ has a regular* $(2\Delta + 1)$*-labeling.*

Proof. (i) Let f be a regular $(2\Delta + 1)$-labeling of $T - N[x_1]$. By Lemma 2.1(ii), we may assume, without loss of generality, that $f(x_4)$ is even. That is, $f(x_4) = 0$, according to Lemma 2.1(i). This implies $\{f(x)|x \in N(x_4)\} = \{3, 5, 7, \cdots, 2\Delta + 1\}$. Let u be a leaf in $N(x_4)$. We can exchange $f(x_3)$ with $f(u)$ (if necessary) such that

$$f(x_3) \ne 2\Delta + 1.$$

Now we can define

$$f(x_1) = 2\Delta + 1.$$

And x_2 can select an even label in $\{2, 4, \cdots, 2\Delta - 2\} \backslash \{f(x_3) \pm 1\}$.

(ii) Let f be a regular $(2\Delta+1)$-labeling of $T-(N[x_1] \cup \{x_3\})$. Similarly, we may assume that $f(x_4)$ is even. Then $f(x_1)$ can be defined as $2\Delta + 1$. Now let

$$A = \{f(x_1), f(x_4) \pm 1, f(x)|x \in N(x_4)\backslash\{x_3\}\}.$$

Suppose that T contains (C3). Since $d(x_4) = 2$, $|A| \le 4$. It follows from $\Delta \ge 4$ that x_3 can select an odd label in $\{1, 3, \cdots, 2\Delta + 1\}\backslash A$.

Suppose that T contains (C4). If $f(x_4) = 2\Delta$, $f(x_5) = 2\Delta + 1$ or $f(y_1) = 2\Delta + 1$, then $2\Delta+1$ must occur at least twice in A. Thus $|A| \le 4$, and hence x_3 can select an odd label in $\{1, 3, \cdots, 2\Delta+1\}\backslash A$. Next, let $f(x_4) \in \{0, 2, 4 \cdots, 2\Delta-2\}$ and $f(x_5), f(y_1) \ne 2\Delta+1$. Note that y_2 is a major handle and $f(y_2)$ is even. Hence $f(y_2) = 0$ and there is a leaf $y_3 \in N(y_2)$ such that

$$f(y_3) = 2\Delta + 1.$$

Now we can exchange $f(y_1)$ with $f(y_3)$. Thus y_1 gets the label $2\Delta+1$ and it becomes the case given above.

Since $f(x_1) = 2\Delta+1$ in each case given above, the leaves in $N(x_1)$ can get even labels, by Lemma 3.1(iii).

Lemma 3.5 *Let T be a tree with $\Delta \geq 5$. If one of the following is satisfied, then T has a regular $(2\Delta+1)$-labeling*:

(i) *T contains (C5) and $T - (N(x_1) \cup N(x_2) \cup \{x_4\})$ has a regular $(2\Delta+1)$-labeling*;

(ii) *T contains (C6) and $T - (N[x_1] \cup N[y] \cup \{x_4\})$ has a regular $(2\Delta+1)$-labeling.*

Proof. (i) Let f be a regular $(2\Delta+1)$-labeling of $T-(N(x_1)\cup N(x_2)\cup\{x_4\})$. We assume, without loss of generality, that $f(x_5)$ is even. Then we can define $f(x_1) = 0$ and $f(x_2) = 2\Delta+1$. Thus x_4 can select an odd label in $\{1, 3, \cdots, 2\Delta+1\}\backslash\{f(x_5) \pm 1, f(x)|x \in N(x_5)\backslash\{x_4\}\}$, since $d(x_5) \leq \Delta-1$. And x_3 can select an even label in $\{2, 4, \cdots, 2\Delta-2\}\backslash\{f(x_4)\pm1, f(x_5)\}$, since $\Delta \geq 5$. Note that $f(x_1), f(x_2) \in \{0, 2\Delta+1\}$. It is easy to label the leaves in $N(x_1) \cup N(x_2)$ according to appropriate parity.

(ii) Let f be a regular $(2\Delta+1)$-labeling of $T - (N[x_1] \cup N[y] \cup \{x_4\})$. Similarly, we can assume that $f(x_5)$ is even and define $f(x_1) = 0$ and $f(y) = 2\Delta+1$. Thus x_4 can select an odd label in $\{1, 3, \cdots, 2\Delta+1\}\backslash\{f(x_5) \pm 1, f(x)|x \in N(x_5)\backslash\{x_4\}\}$ since $d(x_5) \leq \Delta-1$, x_3 can select an even label in $\{2, 4, \cdots, 2\Delta-2\}\backslash\{f(x_4) \pm 1, f(x_5)\}$, and x_2 can select an odd label in $\{3, 5, \cdots, 2\Delta-1\}\backslash\{f(x_3) \pm 1, f(x_4)\}$. Note that $f(x_1), f(y) \in \{0, 2\Delta+1\}$. It is easy to label the leaves in $N(x_1) \cup N(y)$.

Theorem 3.1 *Let T be a tree with $\Delta \geq 5$. If $2, 4, 6 \notin D_\Delta$, then $\lambda_3(T) = 2\Delta+1$. Moreover, the condition is the best possible.*

Proof. Let us prove that G has a regular $(2\Delta+1)$-labeling by induction on $|V|$. The case $|V| = 6$ is clear, since now G is isomorphic to the star $K_{1,5}$. Let $|V| > 6$. If T is a generalized star, then by Lemma 3.1, the conclusion holds. If T contains (C1), then $T - v$ has a regular $(2\Delta+1)$-labeling, by induction hypothesis. And by Lemma 3.1, T has a regular $(2\Delta+1)$-labeling. Now assume that T is not a generalized star and contains no (C1). Then T contains some configuration (Ci) ($2 \leq i \leq 6$), according to Lemma 3.2. It follows from Lemmas 3.4, 3.5 and induction hypothesis that T has a regular $(2\Delta+1)$-labeling.

To show the condition is the best possible, we have to construct a tree such that $\lambda_3 > 2\Delta+1$ and 2 (4, 6, respectively) is in D_Δ. Fig. 3.1 gives two trees T_1 and T_2. By Lemma 2.1(i), it is easy to check that

$$\lambda_3(T_i) > 2\Delta+1 \qquad (i = 1, 2).$$

We now construct a tree T_3 with $D_\Delta = \{4, 8\}$ as follows:

(i) give a star $K_{1,\Delta}$ with Δ-vertex u and leaves x_i ($i = 1, 2, \cdots, \Delta$), where $\Delta \geq 5$;

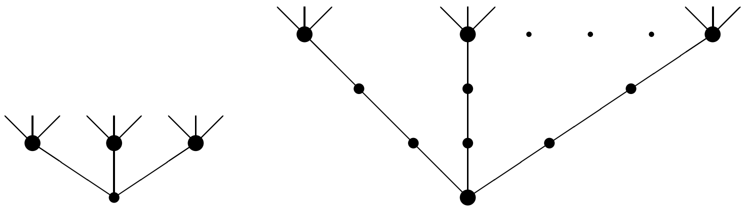
(ii) join a leaf y_i to each $x_i \in N(u)$;

(iii) join $\Delta-2$ leaves to each $y_i \in N^2(u)$;

(iv) join a leaf to each vertex in $N^3(u)$;

(v) join $\Delta - 1$ leaves to each vertex in $N^4(u)$.
It suffices to show that

$$\lambda_3(T_3) > 2\Delta + 1.$$



T_1**:** $D_\Delta = \{2\}$ T_2**:** $D_\Delta = \{3, 6\}$

Fig. 3.1 The biggest vertices in T_1 and T_2 stand for Δ-vertices.

Suppose that T_3 has a $(2\Delta+1)$-labeling f. By Lemma 2.1, we may assume, without loss of generality, that

$$f(u) = 0.$$

Thus

$$\{f(x_i)|i = 1, 2, \cdots, \Delta\} = \{3, 5, 7, \cdots, 2\Delta + 1\}.$$

Let

$$f(x_1) = 3.$$

Then $f(y_1) \in \{6, 8, 10, \cdots, 2\Delta\}$. For each vertex $z \in N(y_1)\backslash\{x_1\}$, $f(z) \neq 0$ since $d(u, z) = 3$; $f(z) \notin \{2, 3, 4\}$ since $d(x_1, z) = 2$; and $f(z) \notin \{f(y_1), f(y_1) \pm 1, f(y_1) \pm 2\}$ since $d(y_1, z) = 1$. Moreover,

$$|f(z) - f(z')| \geq 2$$

for any two different vertices $z, z' \in N(y_1)\backslash\{x_1\}$. These conditions imply that there are at most $\Delta - 3$ labels can be used by vertices in $N(y_1)\backslash\{x_1\}$. However,

$$|N(y_1)\backslash\{x_1\}| = \Delta - 2.$$

It is a contradiction. Thus

$$\lambda_3(T_3) > 2\Delta + 1.$$

By a discussion similar to that given for $\Delta \geq 5$, we can get the following result. The detail of its proof is omitted.

Theorem 3.2 (i) *Let T be a tree with $\Delta = 3$. If $1, 2, \cdots, 7 \notin D_\Delta$, then $\lambda_3 = 2\Delta + 1$.*

(ii) *Let T be a tree with $\Delta = 4$. If $1, 2, 3, 4, 6 \notin D_\Delta$, then $\lambda_3 = 2\Delta + 1$.*

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