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Critical Quenching Exponents for Heat Equations Coupled with Nonlinear Boundary Flux*

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Abstract: We discuss the quenching phenomena for a system of heat equations coupled with nonlinear boundary flux. We determine a critical value for the exponents in the boundary flux, such that only in the super critical case the simultaneous quenching can happen for any solution.

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1 Introduction

This paper is devoted to discussing the quenching phenomena for the following parabolic system with the nonlinear boundary flux of negative exponents

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, \quad \dfrac{\partial v}{\partial t} = \Delta v, & \text{for } (x,t) \in B \times (0,T), \\ \dfrac{\partial u}{\partial \eta} = -v^{-p}, \quad \dfrac{\partial v}{\partial \eta} = -u^{-q}, & \text{for } (x,t) \in \partial B \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & \text{for } x \in B, \end{cases} \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^n$, η is the unit outward normal to ∂B, $p, q > 0$, and $u_0(x)$, $v_0(x)$ are radially symmetric, positive, smooth and satisfy some suitable compatibility conditions on the boundary.

Quenching phenomena has been studied by many authors for a variety of problems (see for instance [1]–[10] and the references therein). In [1]–[3], for the one-dimensional case, the quenching phenomena for the system of heat equations with coupled nonlinear boundary sources and nonlinear inner sources have been studied respectively. Many works are devoted to investigating the quenching phenomena for a single equation (see [4]–[10]). In terms of a system of equations, simultaneous or non-simultaneous quenching phenomena, as far as we know, has been referred by only a few authors.

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The purpose of the present paper is to study the critical quenching exponents of the problem (1.1). This is motivated by the recent work [1], in which the authors investigated quite an interesting simultaneous and non-simultaneous quenching phenomena under some convexity assumptions on the initial data. Roughly speaking, for the simultaneous quenching phenomenon, we mean that for some time $T > 0$, each component of the solution (u, v) vanishes as $t \to T^-$, while the time derivatives blow-up at the same time. For the non-simultaneous quenching phenomenon, we mean that only one component vanishes. What we want to know is that, for fixed exponents p and q, whether the simultaneous quenching happens for all solutions with any initial datum. Precisely speaking, we are interested in seeking a subset $Q \subset \mathbb{R}_+ \times \mathbb{R}_+$, such that for any fixed $(p, q) \in Q$, the simultaneous quenching phenomenon happens for any solution (u, v), while for any fixed $(p, q) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus Q$, non-simultaneous quenching phenomenon happens for at least one solution (u, v). In fact, it shows that

$$Q = \{(p, q);\ p \geq 1,\ q \geq 1\}.$$

In other words, unconditionally simultaneous quenching phenomenon happens for all solutions if and only if $p \geq 1$ and $q \geq 1$, namely, $p_c = 1$, $q_c = 1$ are the critical values of the exponents p and q. It should be noticed that, to establish such a result, the method used in the previous works could not be directly applied to our problem, since we must remove the convexity assumptions on the initial data.

This paper is organized as follows. In Section 2, we present our main result and give some auxiliary lemmas. The proof will be divided into several propositions in the subsequent section.

2 The Main Result and Auxiliary Lemmas

Let u_0, v_0 be radially symmetric. Then the corresponding radial problem for the original problem (1.1) can be given by the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{n-1}{r}\dfrac{\partial u}{\partial r}, \quad \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial r^2} + \dfrac{n-1}{r}\dfrac{\partial v}{\partial r}, & \text{for } (r,t) \in (0,1) \times (0,T), \\ \dfrac{\partial u}{\partial r}(1,t) = -v^{-p}(1,t), \quad \dfrac{\partial v}{\partial r}(1,t) = -u^{-q}(1,t), & \text{for } \ t \in (0,T), \\ \dfrac{\partial u}{\partial r}(0,t) = 0, \quad \dfrac{\partial v}{\partial r}(0,t) = 0, & \text{for } t \in (0,T), \\ u(r,0) = u_0(r), \quad v(r,0) = v_0(r), & \text{for } r \in (0,1). \end{cases} \tag{2.1}$$

Here, for the sake of the simplicity of notations, we still use $(u(r, t),\, v(r, t))$ to denote a solution, although it is a function with two variables while a solution of (1.1) is a function with $n + 1$ variables.

The main result of this paper is the following theorem.

Theorem 2.1 *If $p, q \geq 1$, then simultaneous quenching must happen for any solution of the problem* (2.1). *Otherwise, if* $\min\{p,\, q\} < 1$, *then for every such* $(p,\, q)$, *there exists at* *least one initial datum such that non-simultaneous quenching happens for the corresponding solution.*

The result in the theorem will be shown by combining several propositions. On the one hand, the result follows from these propositions immediately. On the other hand, there are still some independent interests included in these propositions. For this reason, we will only present some auxiliary lemmas and then prove these propositions in the subsequent section.

Firstly, we denote by

$$U(t) = \min_{r\in[0,1]} u(r,t), \qquad V(t) = \min_{r\in[0,1]} v(r,t).$$

Using the maximum principle, we have

$$u \le \|u_0\|_\infty \equiv M, \qquad v \le \|v_0\|_\infty \equiv N.$$

Multiplying the first equation of (2.1) by r^{n-1}, integrating over $(0,1)$, and then combining with the initial and boundary value conditions, we obtain that

$$\int_0^1 r^{n-1} u_t(r,t)\mathrm{d}r = -v^{-p}(1,t), \tag{2.2}$$

$$\int_0^1 r^{n-1} v_t(r,t)\mathrm{d}r = -u^{-q}(1,t), \tag{2.3}$$

moreover,

$$\frac{1}{n}U(t) \le \int_0^1 r^{n-1} u(r,t)\mathrm{d}r \le M/n - N^{-p}t, \tag{2.4}$$

$$\frac{1}{n}V(t) \le \int_0^1 r^{n-1} v(r,t)\mathrm{d}r \le N/n - M^{-q}t. \tag{2.5}$$

Hence, for any initial datum of the system (1.1), quenching always happens.

Firstly, because of the absorbing-type condition on the boundary, it is intuitive that for any initial datum, the solution may decrease near the boundary. We have the following result.

Lemma 2.1 *Let (u,v) be a pair of radial solution of the problem* (2.1)*. Then there exist $r_0 \in [0,1)$ and $C > 0$ such that*

$$u_r(r,t) \le -C, \quad v_r(r,t) \le -C, \qquad \forall (r,t) \in [r_0,1]\times[0,T). \tag{2.6}$$

Proof. Noticing that

$$\frac{\partial u}{\partial r}(1,t) = -v^{-p}(1,t) \le -N^{-p},$$

and using the continuity of u_r, we see that there exist $r(t) < 1$ and $\varepsilon(t) > 0$ such that

$$u_r(r,s) \le -\frac{1}{2}N^{-p}, \qquad \forall (r,s) \in [r(t),1]\times(t-\varepsilon(t), t+\varepsilon(t)).$$

Note that the above inequality also holds as $t \to T^-$, and here we take the open neighborhood by $(T-\varepsilon(T), T)$. By using the finite covering theorem on $[0, T - \frac{1}{2}\varepsilon(T)]$, we can choose such finite neighborhoods such that

$$[0,T) = \bigcup_{i=1,\cdots,K} \{(t_i - \varepsilon_i(t),\ t_i + \varepsilon_i(t))\}.$$

Let

$$r_0 = \max_{i=1,\cdots,K}\{r(t_i)\} < 1.$$

Then we have

$$u_r(r,t) \le -\frac{1}{2}N^{-p}, \qquad \forall (r,t) \in [r_0,1]\times[0,T).$$

The proof is complete.

Consequently, we have the following lemma without any hypotheses of convexity or monotonicity on initial data.

Lemma 2.2 *Let (u_0, v_0) be the pair of initial datum which is smooth enough and satisfy some suitable compatibility conditions. Then there exists a sufficiently large $m_0 > 0$ such that for any $m \ge m_0$*

$$r^m \le -u_0'(r)v_0^p(r), \qquad r \in [r_0,1], \tag{2.7}$$

$$r^m \le -v_0'(r)u_0^p(r), \qquad r \in [r_0,1], \tag{2.8}$$

where r_0 is defined as Lemma 2.1*.*

Proof. Let

$$f(r) = -u_0'(r)v_0^p.$$

Clearly $f(r)$ is smooth enough. Denote

$$\lim_{r\to 1^-} f'(r) = C < \infty.$$

Note that

$$\lim_{r\to 1^-}\frac{1-f(r)}{1-r^m} = \lim_{r\to 1^-}\frac{f'(r)}{mr^{m-1}} = \frac{C}{m}.$$

Then there exists a constant $m_1 > 0$ which is sufficiently large such that

$$\frac{C}{m} < 1.$$

Then there exist $0 < \delta < 1$ such that

$$r^{m_1} < f(r), \qquad r \in (\delta,\, 1).$$

If $\delta < r_0$, then the proof is complete. Otherwise, we note that

$$\lim_{m\to\infty} r^m = 0$$

uniformly for any $r \in [r_0,\, \delta]$. Moreover, recalling Lemma 2.1, we see that

$$u_0'(r) < -C, \qquad r \in [r_0,\, 1].$$

Then we can choose a sufficiently large $m > m_1$ such that

$$r^m \le -u_0'(r)v_0^p(r), \qquad r \in [r_0,\, 1].$$

The inequality (2.8) can be obtained by using a parallel process of arguments, and so we omit it here.

Lemma 2.3 *Let (u,v) be the pair of radial solution of the problem* (2.1)*. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} u_t(1,t) &\ge -Cv^{-p-1}(1,t)u^{-q}(1,t),\\ v_t(1,t) &\ge -Cu^{-q-1}(1,t)v^{-p}(1,t). \end{aligned} \tag{2.9}$$

In addition, the quenching can only happen on the boundary ∂B.

Proof. Define the following auxiliary functions W and Q:

$$W(r,t) = r^{n-1}u_r + r^m v^{-p}, \qquad Q(r,t) = r^{n-1}v_r + r^m u^{-q}.$$

Then we have that for $m > n$

$$\begin{aligned} W_t - W_{rr} + \frac{n-1}{r}W_r &= m(n-m)v^{-p} - p(1+p)r^m v^{-p-2}(v_r)^2 \\ &\quad + 2p(m-n+1)r^{m-1}v^{-p-1}v_r \\ &\le 0 \qquad \text{for } \ r \in [r_0,\, 1]. \end{aligned}$$

Moreover, for a sufficiently large $m > 0$, combining with Lemma 2.2, we also have

$$W(r_0,t) \le 0, \qquad W(1,t) = 0,$$
$$W(r,0) = r^{n-1}u_0'(r) + r^m v_0^{-p}(r) \le 0 \qquad \text{for } \ r \in [r_0,\, 1].$$

Recalling the maximum principle, we conclude that

$$W(r,t) \le 0 \qquad \text{for } \ r \in [r_0,\, 1].$$

Noticing that $W(1,t) = 0$, then we have $W_r(1,t) \ge 0$, namely

$$W_r(1,t) = (n-1)u_r + u_{rr} + mv^{-p} + pv^{-p-1}u^{-q} \ge 0,$$

which implies that

$$\begin{aligned} u_t(1,t) &= (n-1)u_r + u_{rr} \\ &\ge -mv^{-p} - pv^{-p-1}u^{-q} \\ &= -(mu^q v + p)v^{-p-1}u^{-q} \\ &\ge -Cv^{-p-1}(1,t)u^{-q}(1,t). \end{aligned}$$

Similar to the proof above, employing a similar analysis for Q, we shall get the second inequality in (2.9). Since $W(r,t) \le 0$, it is not difficult to check that

$$u_r \le -r^{m-n+1}v^{-p}.$$

Integrating from r to 1 gives

$$\begin{aligned} u(r,t) &\ge u(1,t) + \int_r^1 s^{m-n+1}v^{-p}\mathrm{d}s \\ &\ge u(1,t) + \frac{C}{m+2-n}(1 - r^{m+2-n}), \end{aligned}$$

and the same thing is true for v, which implies that the quenching only happens on the boundary. The proof is complete.

Lemma 2.4 *Assume that*

$$u_0''(r) + \frac{n-1}{r}u_0'(r) < 0, \qquad v_0''(r) + \frac{n-1}{r}v_0'(r) < 0,$$

and

$$c_0 v_0^{-(1+p)/2}(1)\Big(v_0''(r) + \frac{n-1}{r}v_0'(r)\Big) \le u_0^{-(1+q)/2}(1)\Big(u_0''(r) + \frac{n-1}{r}u_0'(r)\Big).$$

If $0 < p \le q < 1$, $v_0^{1-p}(r) \le Cu_0^{1-q}(r)$, *then*

$$v^{1-p}(r,t) \le Cu^{1-q}(r,t), \qquad \forall (r,t) \in [0,1] \times [0,T) \tag{2.10}$$

and

$$u^{-(1+q)/2}(1,t)u_t(r,t) \ge c_0 v^{-(1+p)/2}(1,t)v_t(r,t), \tag{2.11}$$

where

$$0 < C \leq (1-p)/(1-q)$$

and

$$\sqrt{p/q} < c_0 < (1+p)/(\sqrt{C}(1+q)).$$

While if $0 < p < 1 \leq q$, *then the inequality* (2.11) *is valid too, provided that*

$$c_0 = \frac{p}{q}$$

and

$$N^{p-1} \geq \left(\frac{p(q+1)}{q(p+1)}\right)^2 M^{q-1}.$$

Proof. According to these assumptions, together with the comparison principle, it is easy to see that u, v are decreasing in t, namely

$$u_t(r,t) \leq 0, \qquad v_t(r,t) \leq 0.$$

We only show the first conclusion, and the second one can be proved similarly. Let

$$\Phi(r,t) = v^{1-p}(r,t) - Cu^{1-q}(r,t).$$

Then we have

$$\begin{aligned}&\Phi_t - \Phi_{rr} - \Big(\frac{n-1}{r} + pv^{-1}v_r + qu^{-1}u_r\Big)\Phi_r + q(1-p)u^{-1}v^{-1}u_r v_r \Phi \\ =&C(p-q)v^{-1}u^{-q}u_r v_r \leq \ 0, \qquad \forall (r,t) \in (0,1)\times(0,T),\end{aligned}$$

and

$$\begin{aligned}&\Phi_r(0,t) = 0, \qquad \forall t \in (0,T),\\ &\Phi_r(1,t) = (C(1-q)-(1-p))u^{-q}(1,t)v^{-p}(1,t) \leq 0, \qquad \forall t \in (0,T),\\ &\Phi(r,0) = v_0^{1-p}(r) - Cu_0^{1-q}(r) \leq 0, \qquad \forall r \in [0,1].\end{aligned}$$

By the maximum principle, we arrive at $\Phi(r,t) \leq 0$, that is

$$v^{1-p}(r,t) \leq Cu^{1-q}(r,t), \qquad \forall (r,t) \in [0,1]\times[0,T).$$

Let

$$J(r,t) = u^{-(1+q)/2}(1,t)u_t(r,t) - c_0 v^{-(1+p)/2}(1,t)v_t(r,t).$$

Then it is clearly that

$$\begin{aligned}J(r,0) =& u_0^{-(1+q)/2}(1)\Big(u_0''(r) + \frac{n-1}{r}u_0'(r)\Big) - c_0 v_0^{-(1+p)/2}(1)\Big(v_0''(r) + \frac{n-1}{r}v_0'(r)\Big)\\ \geq& 0.\end{aligned}$$

Moreover, by virtue of $p \leq c_0^2 q$, we also see that for any $t \in (0,T)$,

$$\begin{aligned}J_r(0,t) &= u^{-(1+q)/2}(1,t)(u_r(0,t))_t - c_0 v^{-(1+p)/2}(1,t)(v_r(0,t))_t\\ &= 0\end{aligned}$$

and

$$\begin{aligned}&J_r(1,t) + c_0 q v^{-(1+p)/2}(1,t)u^{-(1+q)/2}(1,t)J(1,t)\\ =& (p - c_0^2 q)v^{-(1+p)}(1,t)u^{-(1+q)/2}(1,t)v_t(1,t)\\ \geq& 0.\end{aligned}$$

Moreover, noticing that $c_0 < (1+p)/(\sqrt{C}(1+q))$, we have

$$\begin{aligned}
&J_t - J_{rr} - \frac{n-1}{r}J_r + \frac{1+q}{2}u^{-1}(1,t)u_t(r,t)J(1,t)\\
&+\frac{c_0(1+q)}{2}u^{(q-1)/2}(1,t)v^{-(p+1)/2}(1,t)v_t(1,t)J(r,t)\\
=&\frac{c_0}{2}v^{-(1+p)}(1,t)v_t(1,t)v_t(r,t)\\
&\cdot\Big((1+p)v^{(p-1)/2}(1,t)-c_0(1+q)u^{(q-1)/2}(1,t)\Big)\\
\geq&0,\qquad \forall (r,t)\in(0,1)\times(0,T).
\end{aligned}$$

By the maximum principle (see, e.g., Lemma 2.1 of [12]), one has $J(r,t)\geq 0$.

3 The Proof of the Main Result

We are now in a position to show the proof of the main result, which is given by the following propositions.

Proposition 3.1 *If $p, q \geq 1$, then simultaneous quenching must happen for any solution of the problem* (2.1).

Proof. Using the reduction to absurdity method, we prove that the non-simultaneous quenching happens for some solution (u, v). Without loss of generality, we assume that v does not quench, then there exists a constant $c_0 > 0$ such that $v(1,t)\geq c_0$. According to Lemma 2.3, we have

$$u_t(1,t)\geq -Cu^{-q}(1,t).$$

A direct integrating from t to T yields

$$u(1,t)\leq C(T-t)^{1/(q+1)}.$$

Let $G(x,t)$ be the fundamental solution of the heat equation, namely

$$G(x,t)=\frac{1}{(4\pi t)^{n/2}}\exp\Big\{-\frac{|x|^2}{4t}\Big\}.$$

From Green's second identity we obtain

$$\begin{aligned}
v(x,t)=&\int_B G(x-y,t)v_0(y)\mathrm{d}y\\
&-\int_0^t\int_{\partial B}u^{-q}(1,\tau)G(x-y,t-\tau)\mathrm{d}s_y\mathrm{d}\tau\\
&-\int_0^t\int_{\partial B}v(1,\tau)\frac{\partial G}{\partial\eta_y}(x-y,t-\tau)\mathrm{d}s_y\mathrm{d}\tau.
\end{aligned}$$

Letting x tend to the point of the boundary ∂B, according to the jump relation (see [11]), and noticing that $u>0$, $v>0$ are radial, we arrive at

$$\begin{aligned}
\frac{1}{2}v(1,t)\leq&-\int_0^t u^{-q}(1,\tau)\mathrm{d}\tau\int_{\partial B}G(x-y,t-\tau)\mathrm{d}s_y+N\int_B G(x-y,t)\mathrm{d}y\\
&-\int_0^t v(1,\tau)\int_{\partial B}\frac{\partial G}{\partial\eta_y}(x-y,t-\tau)\mathrm{d}s_y\mathrm{d}\tau.
\end{aligned}$$

Since ∂B is smooth enough, for any fixed $0 < \theta < 1$, we choose μ such that

$$1 - \theta/2 < \mu < 1.$$

From the property of fundamental solution, we have (see [11])

$$\left| \frac{\partial G}{\partial \eta_y}(x-y, t-\tau) \right| \le C \frac{1}{|t-\tau|^{\mu}} \cdot \frac{1}{|x-y|^{n+1-2\mu-\theta}}. \tag{3.1}$$

Moreover, a direct calculation yields that for any $x \in \partial B$, there exist C_1^*, C_1 $(0 < C_1^* \le C_1)$, such that

$$\frac{C_1^*}{|t-\tau|^{1/2}} \le \int_{\partial B} G(x-y, t-\tau)\mathrm{d}s_y \le \frac{C_1}{|t-\tau|^{1/2}}. \tag{3.2}$$

Therefore, we have

$$\frac{1}{2} v(1,t) \le C_0 - C_1 \int_0^t (T-\tau)^{-q/(q+1)} (t-\tau)^{-1/2} \mathrm{d}\tau.$$

Clearly, as $t \to T$, the right hand side goes to $-\infty$ if $q \ge 1$, which is a contradiction since v is bounded from below. The proof is complete.

Proposition 3.2 *If* $\min\{p, q\} < 1$*, then there exists an initial datum* (u_0, v_0) *such that for the corresponding solution* (u, v)*, the quenching is non-simultaneous.*

Proof. We divide the proof into two steps. In the first step, we consider the cases of $\min\{p, q\} < 1$ and $\max\{p, q\} \ge 1$, and in the second step is devoted to considering the case of $\max\{p, q\} < 1$.

Step 1. Without loss of generality, we might as well assume that $0 < p < 1 \le q$. Suppose to the contrary, by Lemma 2.4, we have

$$u^{-(1+q)/2}(1,t) u_t(1,t) \ge c_0 v^{-(1+p)/2}(1,t) v_t(1,t). \tag{3.3}$$

Integrating (3.3) from 0 to t for $q > 1$ yields

$$u^{(1-q)/2}(1,t) \le u_0^{(1-q)/2}(1) - \frac{c_0(q-1)}{1-p} v^{(1-p)/2}(1,t) + \frac{c_0(q-1)}{1-p} v_0^{(1-p)/2}(1).$$

Letting $t \to T$, the right hand side is bounded from above, while the left hand side goes to $+\infty$, which is a contradiction.

Furthermore, if $q = 1$, then integrating (3.3) from 0 to t we have

$$\ln u(1,t) \ge \ln u_0(1) + \frac{2c_0}{1-p} \left(v^{(1-p)/2}(1,t) - v_0^{(1-p)/2}(1) \right)$$

Letting $t \to T$, the right hand side is bounded from below, while the left hand side goes to $-\infty$, which is a contradiction. To sum up the above arguments, we infer that quenching is non-simultaneous.

Step 2. In fact, according to (2.10), we see that v quenches since quenching happens for at least one component. Then integrating (2.11) from 0 to t yields

$$u^{(1-q)/2}(1,t) \ge u_0^{(1-q)/2}(1) + \frac{c_0(1-q)}{1-p} v^{(1-p)/2}(1,t) - \frac{c_0(1-q)}{1-p} v_0^{(1-p)/2}(1).$$

Letting $t \to T$, we obtain

$$\lim_{t \to T^-} u^{(1-q)/2}(1,t) \ge u_0^{(1-q)/2}(1) - \frac{c_0(1-q)}{1-p} v_0^{(1-p)/2}(1).$$

Clearly, u does not quench if u_0 is appropriately large and v_0 is appropriately small. The proof is complete.

In the end of this paper, we now present

The proof of Theorem 2.1. If $p \geq 1$ and $q \geq 1$, then from Proposition 3.1, we see that for any initial datum (u_0, v_0), the simultaneous quenching must happen for the corresponding solution. While if $p \geq 1$ or $q \geq 1$ is not valid, namely $\min\{p,\ q\} < 1$, the conclusion follows from Proposition 3.2.

Remark 3.1 The conclusion of Theorem 2.1 does not imply that in the case of $\min\{p,\ q\} < 1$ all solutions have the non-simultaneous quenching property. In fact, at least in the special case $p = q < 1$, if $u_0(x) \equiv v_0(x)$, then simultaneous quenching happens for the corresponding solution. Of course, if $u_0(x) \not\equiv v_0(x)$, and $u_0(x)$, $v_0(x)$ satisfy some conditions, the corresponding solution might still have the non-simultaneous quenching property.

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