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Spectral Method for a Class of Cahn-Hilliard Equation with Nonconstant Mobility*

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Abstract: In this paper, we propose and analyze a full-discretization spectral approximation for a class of Cahn-Hilliard equation with nonconstant mobility. Convergence analysis and error estimates are presented and numerical experiments are carried out.

Key words: Cahn-Hilliard equation, spectral method, error estimate

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1 Introduction

The Cahn-Hilliard (C-H for short) equation was originally proposed by Cahn and Hilliard to simulate binary alloys. It has subsequently been adopted to model many physical situations such as phase transitions and interface dynamics in multi-phase fluids (see [1]). In this paper, we consider an initial-boundary value problem for a class of C-H equation which is of the form

$$\frac{\partial u}{\partial t} + D[m(x,t)(D^3u - DA(u))] = 0, \qquad (t,x) \in Q_T = (0,T] \times (0,1), \tag{1.1}$$

$$Du(x,t) = D^3u(x,t) = 0, \qquad x = 0,1, \quad 0 \le t \le T, \tag{1.2}$$

$$u(x,0) = u_0, \qquad x \in (0,1), \tag{1.3}$$

where $D = \dfrac{\partial}{\partial x}$ and typically

$$A(s) = -s + \gamma_1 s^2 + \gamma_2 s^3, \qquad \gamma_2 > 0.$$

Here $u(x,t)$ represents a relative concentration of one component in binary mixture. The function $m(x,t)$ is the mobility, which restricts diffusion of both components to the interfacial region only. Throughout this paper, we assume that

$$0 < m_0 \le m(x,t) \le M_0, \quad |m'_x(x,t)| \le M_1, \qquad \forall (x,t) \in Q_T, \tag{1.4}$$

where m_0, M_0 and M_1 are positive constants.

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In the past years, the C-H equation with constant mobility has been intensively studied, and there have been many outstanding results concerning the existence, regularity and special properties of the solution in [2] and [3]. Numerical methods for C-H equation can be found in [3]–[10]. These numerical computation techniques contained the finite element methods (see [5] and [11]), the finite difference methods (see [6] and [8]), and spectral the formulations (see [4] and [10]). In recent years, the equation with concentration dependent mobility has also caused much attention.

The layout of the paper is as follows: in Section 2, we consider a full-discretization implicit scheme and some corresponding estimates. In Section 3, we study the convergence property of this fully discrete spectral approximation. Finally, we perform some numerical experiments which illustrate our results in Section 4.

2 Full-discretization Spectral Method and Some Estimates

In this section we set up a full-discretization scheme for equation (1.1) and analyze the boundedness of its solution. Let $\|\cdot\|_k$ and $|\cdot|_k$ be the norm and semi-norm of the Soboliv spaces $H^k(0,1)(k\in\mathbb{N})$, respectively. Let $(\,\cdot\,,\,\cdot\,)$ be the standard L^2 inner product over $(0,1)$. Define

$$L^\infty(0,1)=\{v;\ \|v\|_\infty=\operatorname*{esssup}_{x\in(0,1)}|u|<+\infty\},$$
$$H_E^2(0,1)=\{v\in H^2(0,1);\ Dv\big|_{x=0,1}=0\},$$
$$L^2(0,T;H^m(0,1))=\{u\in H^m(0,1);\int_0^T\|u\|_m^2\mathrm{d}t<+\infty\}.$$

Denote by

$$S_N=\operatorname{span}\{\cos k\pi x, k=0,1,2,\cdots,N\}$$

for any integer $N>0$. Define an orthogonal projection operator $P_N: H_E^2\mapsto S_N$ by

$$(P_Nu,v_N)=(u,v_N),\qquad \forall v_N\in S_N. \tag{2.1}$$

The weak solution for the initial boundary value problem (1.1)–(1.3) is equivalent to the solution of the following equations

$$(u_t,v)+(D^2u-A(u),D(mDv))=0,\qquad \forall v\in H_E^2(0,1), \tag{2.2}$$
$$(u(\,\cdot\,,0),v)=(u_0,v),\qquad \forall v\in H_E^2(0,1). \tag{2.3}$$

Moreover, the existence of the weak solution of this problem was introduced and some boundedness about the weak solution was given in [12].

Theorem 2.1[12] *Assume that $u_0\in H_E^2(0,1)$ and* (1.4) *is satisfied. Then there exists a unique weak solution $u\in H^{4,1}(Q_T)$ of the initial-boundary value problem* (1.1)–(1.3). *Furthermore, we have*

$$\|Du(x,t)\|\le C,\quad \|u\|_\infty\le C,\qquad 0\le t\le T, \tag{2.4}$$
$$\|D^2u(x,t)\|^2\le C,\quad \int_0^t\|D^4u(x,t)\|^2\mathrm{d}s\le C,\qquad 0\le t\le T, \tag{2.5}$$

where $C = C(m, u_0, \gamma_1, \gamma_2) > 0$ *is a constant.*

Let $\triangle t = T/k$ for a positive integer k. The full-discretization spectral method for equations (1.1)–(1.3) is read as: find $U_N^j(x) \in S_N$ $(j = 0, 1, 2, \cdots, k)$ such that for any $v \in S_N$ there hold

$$\Big(\frac{U_N^{j+1} - U_N^j}{\triangle t}, v\Big) + \Big(\frac{D^2U_N^{j+1} + D^2U_N^j}{2} - P_N\tilde{A}(U_N^{j+1}, U_N^j), D(m^{j+\frac{1}{2}}Dv)\Big) = 0, \tag{2.6}$$

$$(U_N^0, v) = (u_0, v), \tag{2.7}$$

where

$$m^{j+\frac{1}{2}} = m(x, t_{j+\frac{1}{2}}), \qquad t_{j+\frac{1}{2}} = \frac{1}{2}(t_{j+1} + t_j), \qquad t_j = j\triangle t$$

and

$$\tilde{A}(\phi, \varphi) = \frac{\gamma_2}{4}(\phi^3 + \varphi^3 + \phi^2\varphi + \phi\varphi^2) + \frac{\gamma_1}{3}(\phi^2 + \phi\varphi + \varphi^2) - \frac{1}{2}(\phi + \varphi). \tag{2.8}$$

The solution U_N^j has the following property:

Lemma 2.1 *Assume that* $U_N^j \in S_N$ $(j = 1, 2, \cdots, k)$ *is a solution of the equations* (2.6) *and* (2.7). *Then there exists a positive constant* $C = C(u_0, T)$ *such that*

$$\|DU_N^j\| \le C, \qquad \|U_N^j\|_\infty \le C. \tag{2.9}$$

Proof. Define a discrete energy function at time t_j by

$$F(j) = \frac{1}{2}\|DU_N^j\|^2 + (H(U_N^j), 1).$$

Noticing that

$$\begin{aligned}\frac{1}{\triangle t}(F(j+1) - F(j)) &= \Big(-\frac{1}{2}D^2(U_N^{j+1} + U_N^j) + P_N\tilde{A}(U_N^{j+1}, U_N^j), \frac{U_N^{j+1} - U_N^j}{\triangle t}\Big)\\ &\le -m_0\Big\|\frac{1}{2}D^3(U_N^{j+1} + U_N^j) - DP_N\tilde{A}(U_N^{j+1}, U_N^j)\Big\|^2\\ &\le 0.\end{aligned}$$

Then by recursion, we obtain

$$F(j) \le F(0) = C,$$

where C is a constant depending on u_0. Applying Young inequality, we get

$$(U_N^j)^2 \le \varepsilon(U_N^j)^4 + C_{1\varepsilon}, \qquad |U_N^j|^3 \le \varepsilon(U_N^j)^4 + C_{2\varepsilon}.$$

Choosing ε such that

$$\Big(\frac{1}{3}|\gamma_1| + \frac{1}{2}\Big)\varepsilon = \frac{\gamma_2}{8},$$

we have

$$\begin{aligned}\int_0^1 H(U_N^j)\mathrm{d}x &\ge \int_0^1 \Big(\frac{1}{4}\gamma_2(U_N^j)^4 - \frac{1}{3}|\gamma_1(U_N^j)^3| - \frac{1}{2}(U_N^j)^2\Big)\mathrm{d}x\\ &\ge \frac{\gamma_2}{8}\int_0^1 (U_N^j)^4\mathrm{d}x - K_1,\end{aligned}$$

where K_1 is a positive constant depending on γ_1 and γ_2. Then we have

$$\frac{1}{2}\|DU_N^j\|^2 + \frac{\gamma_2}{8}\int_0^1 (U_N^j)^4\mathrm{d}x \le F(0) + K_1.$$

Therefore

$$\|DU_N^j\| \le C, \qquad \|U_N^j\| \le C. \tag{2.10}$$

From the embedding theorem it follows that

$$\|U_N^j\|_\infty \le C,$$

where $C = C(m, u_0, \gamma_1, \gamma_2) > 0$ is a constant.

3 The Convergence of the Full-discretization Scheme

In this section, we analyze the error estimates between the numerical solution U_N^j and the exact solution $u(t_j)$. According to the properties of the projection operator P_N, we only need to analyze the error between $P_N u(t_j)$ and U_N^j. Denoted by $u^j = u(t_j)$, $e^j = P_N u^j - U_N^j$ and $\eta^j = u^j - P_N u^j$. Then

$$u^j - U_N^j = \eta^j + e^j.$$

If no confusion occurs, we denote the average of the two instant errors e^j and e^{j+1} by $\bar{e}^{j+\frac{1}{2}}$, and $\bar{e}^{j+\frac{1}{2}} = \dfrac{e^j + e^{j+1}}{2}$. Firstly we prove the following error estimates for the full-discretization scheme.

Lemma 3.1

$$\begin{aligned}\|e^{j+1}\|^2 \le \|e^j\|^2 &+ 2\triangle t\Big(u_t(t_{j+\frac{1}{2}}) - \frac{U_N^{j+1} - U_N^j}{\triangle t}, \bar{e}^{j+\frac{1}{2}}\Big) \\ &+ \frac{1}{320}\triangle t^4 \int_{t_j}^{t_{j+1}} \|u_{ttt}\|^2 \mathrm{d}t + \triangle t \|\bar{e}^{j+\frac{1}{2}}\|^2. \end{aligned} \tag{3.1}$$

Proof. Applying Taylor expansion about $t_{j+\frac{1}{2}}$, we have

$$u^j = u^{j+\frac{1}{2}} - \frac{\triangle t}{2} u_t^{j+\frac{1}{2}} + \frac{\triangle t^2}{8} u_{tt}^{j+\frac{1}{2}} - \frac{1}{2}\int_{t_j}^{t_{j+\frac{1}{2}}} (t - t_j)^2 u_{ttt} \mathrm{d}t, \tag{3.2}$$

$$u^{j+1} = u^{j+\frac{1}{2}} + \frac{\triangle t}{2} u_t^{j+\frac{1}{2}} + \frac{\triangle t^2}{8} u_{tt}^{j+\frac{1}{2}} + \frac{1}{2}\int_{t_{j+\frac{1}{2}}}^{t_{j+1}} (t_{j+1} - t)^2 u_{ttt} \mathrm{d}t. \tag{3.3}$$

Then

$$u_t^{j+\frac{1}{2}} - \frac{u^{j+1} - u^j}{\triangle t} = -\frac{1}{2\triangle t}\Big(\int_{t_{j+\frac{1}{2}}}^{t_{j+1}} (t_{j+1} - t)^2 u_{ttt} \mathrm{d}t + \int_{t_j}^{t_{j+\frac{1}{2}}} (t - t_j)^2 u_{ttt} \mathrm{d}t\Big).$$

From Hölder inequality it follows that

$$\begin{aligned} &\Big\| u_t^{j+\frac{1}{2}} - \frac{u^{j+1} - u^j}{\triangle t} \Big\|^2 \\ &\le \frac{1}{2\triangle t^2}\Big(\Big\|\int_{t_{j+\frac{1}{2}}}^{t_{j+1}} (t_{j+1} - t)^2 u_{ttt} \mathrm{d}t\Big\|^2 + \Big\|\int_{t_j}^{t_{j+\frac{1}{2}}} (t - t_j)^2 u_{ttt} \mathrm{d}t\Big\|^2\Big) \\ &\le \frac{\triangle t^3}{320}\int_{t_j}^{t_{j+1}} \|u_{ttt}\|^2 \mathrm{d}t. \end{aligned}$$

Noticing that for any $v \in S_N$, we have

$$\Big(u_t^{j+\frac{1}{2}} - \frac{U_N^{j+1} - U_N^j}{\triangle t}, v\Big) = \Big(u_t^{j+\frac{1}{2}} - \frac{u^{j+1} - u^j}{\triangle t}, v\Big) + \frac{1}{\triangle t}(e^{j+1} - e^j, v). \tag{3.4}$$

Taking $v = 2\bar{e}^{j+\frac{1}{2}}$ in (3.4), we obtain

$$\begin{aligned}\|e^{j+1}\|^2 &= \|e^j\|^2 + 2\triangle t\Big(u_t^{j+\frac{1}{2}} - \frac{U_N^{j+1} - U_N^j}{\triangle t}, \bar{e}^{j+\frac{1}{2}}\Big) - 2\triangle t(u_t^{j+\frac{1}{2}} - \frac{u^{j+1} - u^j}{\triangle t}, \bar{e}^{j+\frac{1}{2}}) \\ &\le \|e^j\|^2 + 2\triangle t\Big(u_t^{j+\frac{1}{2}} - \frac{U_N^{j+1} - U_N^j}{\triangle t}, \bar{e}^{j+\frac{1}{2}}\Big) \\ &\quad + \frac{1}{320}\triangle t^4 \int_{t_j}^{t_{j+1}} \|u_{ttt}\|^2 \mathrm{d}t + \triangle t\|\bar{e}^{j+\frac{1}{2}}\|^2.\end{aligned}$$

Taking $v = \bar{e}^{j+\frac{1}{2}}$ in (2.2) and $v_N = \bar{e}^{j+\frac{1}{2}}$ in (2.6), respectively, and comparing these two equations at time $t = t_{j+\frac{1}{2}}$, we have

$$\begin{aligned}\Big(u_t^{j+\frac{1}{2}} - \frac{U_N^{j+1} - U_N^j}{\triangle t}, \bar{e}^{j+\frac{1}{2}}\Big) = &- \Big(D^2u^{j+\frac{1}{2}} - \frac{D^2U_N^{j+1} + D^2U_N^j}{2}, D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\Big) \\ &+ \Big(A(u^{j+\frac{1}{2}}) - P_N\tilde{A}(U_N^{j+1}, U_N^j), D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\Big).\end{aligned}$$

Now we investigate the error estimates of the two items in the right-hand side of previous equation.

Lemma 3.2 *Assume that* $u \in L^2(0, T; H^4(0, 1))$ *is a solution of the equations* (1.1)–(1.3), *then there exists a constant* $C_1 = C_1(m, u_0) > 0$ *such that*

$$\begin{aligned}&- \Big(D^2u^{j+\frac{1}{2}} - \frac{D^2U_N^{j+1} + D^2U_N^j}{2}, D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\Big) \\ \le &- \frac{m_0}{2}\|D^2\bar{e}^{j+\frac{1}{2}}\|^2 + C_1(N^{-4} + \|e^j\|^2 + \|e^{j+1}\|^2 + \triangle t^3 \int_{t_j}^{t_{j+1}} \|D^2u_{tt}\|^2\mathrm{d}t),\end{aligned} \tag{3.5}$$

Proof. By Taylor expansion and Hölder inequality, we obtain

$$\begin{aligned}&\Big\|D^2(u^{j+\frac{1}{2}} - \frac{1}{2}(u^j + u^{j+1}))\Big\|^2 \\ = &\frac{1}{4}\Big\|D^2\Big(\int_{t_j}^{t_{j+\frac{1}{2}}} (t - t_j)u_{tt}\mathrm{d}t + \int_{t_{j+\frac{1}{2}}}^{t_{j+1}} (t_{j+1} - t)u_{tt}\mathrm{d}t\Big)\Big\|^2 \\ \le &\frac{\triangle t^3}{96} \int_{t_j}^{t_{j+1}} \|D^2u_{tt}\|^2\mathrm{d}t.\end{aligned} \tag{3.6}$$

Therefore,

$$\begin{aligned}&- \Big(D^2(u^{j+\frac{1}{2}} - \frac{U_N^{j+1} + U_N^j}{2}), D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\Big) \\ \le &\Big\|D^2(u^{j+\frac{1}{2}} - \frac{u^{j+1} + u^j}{2})\Big\| \cdot \Big\|D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\Big\| \\ &- \Big(D^2(\bar{\eta}^{j+\frac{1}{2}} + \bar{e}^{j+\frac{1}{2}}), D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\Big) \\ \le &\Big\{\frac{\triangle t^3}{96} \int_{t_j}^{t_{j+1}} \|D^2u_{tt}\|^2\mathrm{d}t\Big\}^{\frac{1}{2}} \cdot \Big(\|m^{j+\frac{1}{2}}D^2\bar{e}^{j+\frac{1}{2}}\| + \|m_x'(t_{j+\frac{1}{2}})D\bar{e}^{j+\frac{1}{2}}\|\Big) \\ &- \Big(D^2\bar{\eta}^{j+\frac{1}{2}}, m^{j+\frac{1}{2}}D^2\bar{e}^{j+\frac{1}{2}}\Big) - \Big(D^2\bar{\eta}^{j+\frac{1}{2}}, m_x'(t_{j+\frac{1}{2}})D\bar{e}^{j+\frac{1}{2}}\Big) \\ &- \Big(D^2\bar{e}^{j+\frac{1}{2}}, m^{j+\frac{1}{2}}D^2\bar{e}^{j+\frac{1}{2}}\Big) - \Big(D^2\bar{e}^{j+\frac{1}{2}}, m_x'(t_{j+\frac{1}{2}})D\bar{e}^{j+\frac{1}{2}}\Big) \\ \triangleq &I_1^j + I_2^j + I_3^j.\end{aligned}$$

Directly computation gives

$$I_1^j \le \triangle t^3 C_{1\varepsilon} \int_{t_j}^{t_{j+1}} \|D^2 u_{tt}\|^2 \mathrm{d}t + \varepsilon \|D^2 \bar{e}^{j+\frac{1}{2}}\|^2 + \varepsilon \|\bar{e}^{j+\frac{1}{2}}\|^2,$$

$$I_2^j \le C_{2\varepsilon} \|D^2 \bar{\eta}^{j+\frac{1}{2}}\|^2 + 2\varepsilon \|D^2 \bar{e}^{j+\frac{1}{2}}\|^2 + \varepsilon \|\bar{e}^{j+\frac{1}{2}}\|^2,$$

$$I_3^j \le -m_0 \|D^2 \bar{e}^{j+\frac{1}{2}}\|^2 + \varepsilon \|D^2 \bar{e}^{j+\frac{1}{2}}\|^2 + C_{3\varepsilon} \|\bar{e}^{j+\frac{1}{2}}\|^2.$$

Choosing $\varepsilon = \dfrac{m_0}{8}$ and in terms of the properties of the projection operator P_N, we complete the proof of the estimate (3.5).

Lemma 3.3 *Assume that $u \in L^\infty(0,T;W^{4,\infty}(0,1))$ is a solution of equations* (1.1)–(1.3), *then for any $\varepsilon > 0$, there is a positive constant $C_2 = C_2(m, u_0)$ such that*

$$\begin{aligned} &\Big(A(u^{j+\frac{1}{2}}) - P_N \tilde{A}(U_N^{j+1}, U_N^j), D(m^{j+\frac{1}{2}} D \bar{e}^{j+\frac{1}{2}})\Big) \\ \le\ & \frac{m_0}{4} \|D^2 \bar{e}^{j+\frac{1}{2}}\|^2 + C_2 \Big\{ \|e^{j+1}\|^2 + \|e^j\|^2 + N^{-8} \\ &+ \triangle t^3 \Big(\int_{t_j}^{t_{j+1}} \|u_{tt}\|^2 \mathrm{d}t + \|u_t^{j+\frac{1}{2}}\|_\infty^2 \int_{t_j}^{t_{j+1}} \|u_t\|^2 \mathrm{d}t \Big) \Big\}. \end{aligned} \tag{3.7}$$

Proof. Firstly we consider

$$\begin{aligned} A(u^{j+\frac{1}{2}}) - \tilde{A}(u^{j+1}, u^j) &= \gamma_2 (u^{j+\frac{1}{2}})^3 - \frac{\gamma_2}{4} \Big[(u^{j+1})^3 + (u^{j+1})^2 u^j + u^{j+1} (u^j)^2 + (u^j)^3 \Big] \\ &\quad + \gamma_1 (u^{j+\frac{1}{2}})^2 - \frac{\gamma_1}{3} \Big[(u^{j+1})^2 + u^{j+1} u^j + (u^j)^2 \Big] \\ &\quad - (u^{j+\frac{1}{2}} - \frac{1}{2}(u^{j+1} + u^j)) \\ &\triangleq \gamma_2 \rho_1^j + \gamma_1 \rho_2^j - \rho_3^j. \end{aligned}$$

By directly computation, we obtain

$$\|\rho_3^j\| = \frac{1}{2} \Big\| \int_{t_j}^{t_{j+\frac{1}{2}}} (t - t_j) u_{tt} \mathrm{d}t + \int_{t_{j+\frac{1}{2}}}^{t_{j+1}} (t_{j+1} - t) u_{tt} \mathrm{d}t \Big\| \le \Big(\frac{\triangle t^3}{96} \int_{t_j}^{t_{j+1}} \|u_{tt}\|^2 \mathrm{d}t \Big)^{\frac{1}{2}},$$

$$\begin{aligned} \|\rho_2^j\| &= \Big\| \frac{1}{6} \Big[(2u^{j+\frac{1}{2}})^2 - (u^{j+1} + u^j)^2 \Big] + \frac{1}{6} \Big[2(u^{j+\frac{1}{2}})^2 - (u^{j+1})^2 - (u^j)^2 \Big] \Big\| \\ &\le \frac{1}{6} \Big\| \Big[2u^{j+\frac{1}{2}} - (u^{j+1} + u^j) \Big] \Big[2u^{j+\frac{1}{2}} + (u^{j+1} + u^j) \Big] \Big\| \\ &\quad + \frac{1}{6} \Big\| (u^{j+\frac{1}{2}} - u^{j+1})(u^{j+\frac{1}{2}} + u^{j+1}) + (u^{j+\frac{1}{2}} - u^j)(u^{j+\frac{1}{2}} + u^j) \Big\| \\ &\le \frac{1}{6} \Big\| 2u^{j+\frac{1}{2}} - (u^{j+1} + u^j) \Big\| \cdot \Big\| 2u^{j+\frac{1}{2}} + (u^{j+1} + u^j) \Big\|_\infty \\ &\quad + \frac{1}{6} \Big\| (\frac{-\triangle t}{2} u_t^{j+\frac{1}{2}} - \int_{t_{j+\frac{1}{2}}}^{t_{j+1}} (t_{j+1} - t) u_{tt} \mathrm{d}t)(u^{j+\frac{1}{2}} + u^{j+1}) \\ &\quad + \Big(\frac{\triangle t}{2} u_t^{j+\frac{1}{2}} - \int_{t_j}^{t_{j+\frac{1}{2}}} (t - t_j) u_{tt} \mathrm{d}t)(u^{j+\frac{1}{2}} + u^j \Big) \Big\| \\ &\le C \triangle t^{\frac{3}{2}} \Big(\int_{t_j}^{t_{j+1}} \|u_{tt}\|^2 \mathrm{d}t \Big)^{\frac{1}{2}} + \frac{\triangle t^{\frac{3}{2}}}{12} \|u_t^{j+\frac{1}{2}}\|_\infty \cdot \Big(\int_{t_j}^{t_{j+1}} \|u_t\|^2 \mathrm{d}t \Big)^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned}
\|\rho_1^j\| &= \frac{1}{12}\Big\|((2u^{j+\frac{1}{2}})^3-(u^{j+1}+u^j)^3)+\frac{1}{6}\Big[2(u^{j+\frac{1}{2}})^3-(u^{j+1})^3-(u^j)^3\Big]\Big\| \\
&\le \frac{1}{12}\Big\|\Big[2u^{j+\frac{1}{2}}-(u^{j+1}+u^j)\Big]\cdot\Big[4(u^{j+\frac{1}{2}})^2+2(u^{j+1}+u^j)u^{j+\frac{1}{2}}+(u^{j+1}+u^j)^2\Big]\Big\| \\
&\quad +\frac{1}{6}\Big\|(u^{j+\frac{1}{2}}-u^{j+1})((u^{j+\frac{1}{2}})^2+u^{j+1}u^{j+\frac{1}{2}}+(u^{j+1})^2) \\
&\quad +(u^{j+\frac{1}{2}}-u^j)((u^{j+\frac{1}{2}})^2+u^ju^{j+\frac{1}{2}}+(u^j)^2)\Big\| \\
&\le C\triangle t^{\frac{3}{2}}\Big\{\Big(\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2\mathrm{d}t\Big)^{\frac{1}{2}}+\|u_t^{j+\frac{1}{2}}\|_\infty\cdot\Big(\int_{t_j}^{t_{j+1}}\|u_t\|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\Big(A(u^{j+\frac{1}{2}})-P_N\tilde{A}(U_N^{j+1},U_N^j),D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\Big) \\
\le{}& \frac{3}{4\varepsilon}\|A(u^{j+\frac{1}{2}})-P_NA(u^{j+\frac{1}{2}})\|^2+\frac{\varepsilon}{3}\|D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\|^2 \\
&+\frac{3}{4\varepsilon}\|A(u^{j+\frac{1}{2}})-\tilde{A}(u^{j+1},u^j)\|^2+\frac{\varepsilon}{3}\|D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\|^2 \\
&+\frac{3}{4\varepsilon}\|\tilde{A}(u^{j+1},u^j)-\tilde{A}(U_N^{j+1},U_N^j)\|^2+\frac{\varepsilon}{3}\|D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\|^2 \\
\le{}& \varepsilon\|D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\|^2+\frac{3}{4\varepsilon}\|A(u^{j+\frac{1}{2}})-P_NA(u^{j+\frac{1}{2}})\|^2+\frac{3}{4\varepsilon}\|A(u^{j+\frac{1}{2}})-\tilde{A}(u^{j+1},u^j)\|^2 \\
&+\frac{3}{4\varepsilon}\|\tilde{A}(u^{j+1},u^j)-\tilde{A}(U_N^{j+1},U_N^j)\|^2. \qquad (3.8)
\end{aligned}$$

Directly computation yields

$$\begin{aligned}
\|D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\|^2 &\le (M_0^2+M_1^2)\|D^2\bar{e}^{j+\frac{1}{2}}\|^2+C\|\bar{e}^{j+\frac{1}{2}}\|^2, \\
\|A(u^{j+\frac{1}{2}})-P_NA(u^{j+\frac{1}{2}})\|^2 &\le CN^{-8}, \\
\|A(u^{j+\frac{1}{2}})-\tilde{A}(u^{j+1},u^j)\|^2 &\le C(\|\rho_1^j\|^2+\|\rho_2^j\|^2+\|\rho_3^j\|^2), \\
\|\tilde{A}(u^{j+1},u^j)-\tilde{A}(U_N^{j+1},U_N^j)\|^2 &\le (\|G_1^j\|_\infty^2+\|G_2^j\|_\infty^2)(\|e^{j+1}\|^2+\|e^j\|^2+CN^{-8}),
\end{aligned}$$

where

$$G_1^j=\frac{\gamma_2}{8}((u^{j+1}+U_N^{j+1})^2+(U_N^{j+1}+U_N^j)^2+(u^{j+1}+U_N^j)^2)+\frac{\gamma_1}{3}(U_N^{j+1}+u^{j+1}+U_N^j)-\frac{1}{2},$$

$$G_2^j=\frac{\gamma_2}{8}((u^j+U_N^j)^2+(u^j+u^{j+1})^2+(u^{j+1}+U_N^j)^2)+\frac{\gamma_1}{3}(u^{j+1}+u^j+U_N^j)-\frac{1}{2}.$$

Applying Theorem 2.1 and Lemma 2.1, we obtain that

$$\|G_1\|_\infty\le C(m,u_0,\gamma_1,\gamma_2)$$

and

$$\|G_2\|_\infty\le C(m,u_0,\gamma_1,\gamma_2).$$

Taking $\varepsilon=\dfrac{m_0}{4(M_0^2+M_1^2)}$ in (3.8), we have

$$\begin{aligned}
&\Big(A(u^{j+\frac{1}{2}})-P_N\tilde{A}(U_N^{j+1},U_N^j),D(m^{j+\frac{1}{2}}D\bar{e}^{j+\frac{1}{2}})\Big) \\
\le{}& \frac{m_0}{4}\|D^2\bar{e}^{j+\frac{1}{2}}\|^2+C_2\Big\{\|e^{j+1}\|^2+\|e^j\|^2+N^{-8} \\
&+\triangle t^3\Big(\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2\mathrm{d}t+\|u_t^{j+\frac{1}{2}}\|_\infty^2\int_{t_j}^{t_{j+1}}\|u_t\|^2\mathrm{d}t\Big)\Big\},
\end{aligned}$$

where $C_2 = C_2(u_0, m, \gamma_1, \gamma_2) > 0$ is a constant.

Thus, we obtain the following theorem.

Theorem 3.1 *Assume that $u(x,t)$ is the solution of the C-H equations (1.1)–(1.3) satisfying*

$$u \in L^\infty(0,T;W^{4,\infty}), \qquad u_t \in L^2(0,T;L^2(0,1)) \cap L^\infty(0,T;L^2(0,1)),$$
$$u_{tt} \in L^2(0,T;H^2(0,1)), \qquad u_{ttt} \in L^2(0,T;L^2(0,1)).$$

Also $U_N^j \in S_N$ $(j = 1,2,\cdots,k)$ is a solution of the full-discretization equations (2.6), (2.7). If $\triangle t$ is sufficiently small, then there exists a positive constant $C = C(m,u_0,\gamma_1,\gamma_2) > 0$ such that, for $j = 0,1,2,\cdots,k$,

$$\|e^{j+1}\| = \|P_N u(t_{j+1}) - U_N^{j+1}\| \le C(N^{-2} + \|e^0\| + \triangle t^2). \tag{3.9}$$

Proof. By (3.1), (3.5) and (3.7), we obtain

$$\|e^{j+1}\|^2 \le \|e^j\|^2 + \triangle t C_1(N^{-4} + \|e^{j+1}\|^2 + \|e^j\|^2)$$
$$+ C_2\triangle t^4 \int_{t_j}^{t_{j+1}} (\|D^2u_{tt}\|^2 + \|u_{tt}\|^2 + \|u_{ttt}\|^2 + \|u_t^{j+\frac{1}{2}}\|_\infty^2 \|u_t\|^2)\mathrm{d}t,$$

where $C_1 = C_1(m,u_0,\gamma_1,\gamma_2) > 0$ and $C_2 = C_2(m,u_0,\gamma_1,\gamma_2) > 0$ are constants. For $\triangle t$ being sufficiently small such that $C_1\triangle t \le \frac{1}{2}$, denoted by $\tilde{C} = 2(C_1 + C_2)$, we have

$$\|e^{j+1}\|^2 \le (1 + \tilde{C}\triangle t)\|e^j\|^2 + \tilde{C}(\triangle t N^{-4} + \triangle t^4 B^j),$$

where

$$B^j = \int_{t_j}^{t_{j+1}} (\|D^2u_{tt}\|^2 + \|u_{tt}\|^2 + \|u_{ttt}\|^2 + \|u_t^{j+\frac{1}{2}}\|_\infty^2 \|u_t\|^2)\mathrm{d}t.$$

By the Gronwall's inequality of the discrete form, we obtain

$$\|e^{j+1}\|^2 \le \exp\Big\{\tilde{C}(j+1)\triangle t\Big\}\Big\{\|e^0\|^2 + \tilde{C}(j\triangle t N^{-4} + \triangle t^4 \sum_{l=0}^{j} B^l)\Big\}.$$

Directly computation gives

$$\sum_{l=0}^{j} B^l \le \int_0^{t_{j+1}} \Big(\|D^2u_{tt}\|^2 + \|u_{tt}\|^2 + \|u_{ttt}\|^2 + \max_{0\le l\le j}\{\|u_t^{l+\frac{1}{2}}\|_\infty^2\} \cdot \|u_t\|^2\Big)\mathrm{d}t.$$

Then we get the conclusion (3.9).

Furthermore, we get the following theorem.

Theorem 3.2 *Assume that $\triangle t$ is sufficiently small. The solution $u(x,t)$ of the C-H equations (1.1)–(1.3) satisfies*

$$u \in L^\infty(0,T;W^{4,\infty}), \qquad u_t \in L^2(0,T;L^2(0,1)) \cap L^\infty(0,T;L^2(0,1)),$$
$$u_{tt} \in L^2(0,T;H^2(0,1)), \qquad u_{ttt} \in L^2(0,T;L^2(0,1)),$$

$U_N^j \in S_N$ $(j = 1,2,\cdots,k)$ is the solution of the full-discretization equations (2.6), (2.7), and the initial value U^0 satisfies

$$\|e^0\| = \|P_N u^0 - U^0\| \le CN^{-4}.$$

Then there exists a constant $C = C(m,u_0,\gamma_1,\gamma_2)$, independent of N and $\triangle t$, such that,

$$\|u(x,t_j) - U_N^j\| \le C(N^{-2} + \triangle t^2), \qquad j = 1,2,\cdots,k. \tag{3.10}$$

4 Numerical Experiments

In this section, using the spectral method described in (2.6), (2.7), we carry out some numerical computations to illustrate out results in previous section. Take

$$m(x,t) = 1 + x\sin t, \qquad A(s) = s^3 - s.$$

Then the full-discretization spectral method is read as: find

$$U_N^j = \sum_{l=0}^{N} \alpha_{jl} \cos l\pi x, \qquad j = 1, 2, \cdots, k$$

such that (2.6), (2.7) hold. As an example we choose $u_0 = x^4(1-x)^4$, $\triangle t = 5 \times 10^{-6}$, $N = 32$, and get the solution which evolves from $t = 0$ to $t = 5 \times 10^{-4}$ (cf. Fig 4.1).

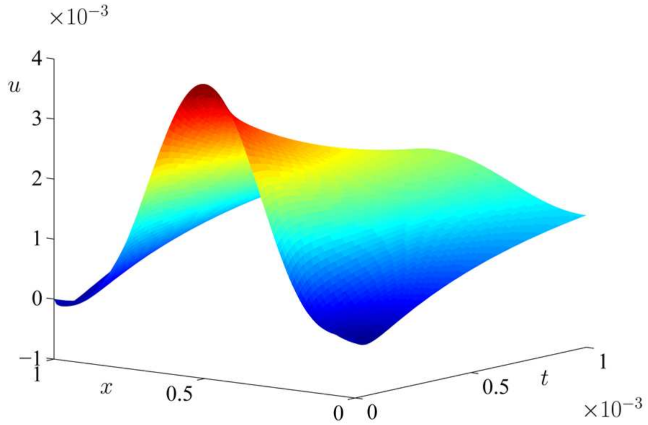

Fig. 4.1 The expanded property of the solution when $N = 32$

Since no exact solution to (1.1)–(1.3) is known, we make a comparison between the solution of (2.6), (2.7) on a coarse mesh and on a fine mesh.

We choose $\triangle t = 0.01, 0.005, 0.001, 0.0001$ respectively to solve (2.6), (2.7). Denote by

$$err(t, \triangle t) = \Big(\int_0^1 (U_N^j(x, 5\times 10^{-6}) - U_N^j(x, \triangle t))^2 \mathrm{d}x\Big)^{\frac{1}{2}}.$$

Then the error is showed in the following table at $T = 0.1$.

$\triangle t$	$err(0.1, \triangle t)$
0.01	4.5635×10^{-4}
0.002	2.6309×10^{-5}
0.001	4.6806×10^{-6}
0.0002	6.3412×10^{-8}
0.0001	9.1733×10^{-9}

We see that the order of error estimates is $O(\triangle t^2)$ proved in Theorem 3.2.

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