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Asymptotic Distribution of a Kind of Dirichlet Distribution*

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Abstract: The Dirichlet distribution that we are concerned with in this paper is very special, in which all parameters are different from each other. We prove that the asymptotic distribution of this kind of Dirichlet distributions is a normal distribution by using the central limit theorem and Slutsky theorem.
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1 Introduction

The Dirichlet distribution is a common multivariate distribution. It not only plays a very important role in modern non-parameter statistics, but also serves as the conjugate priori distribution of the multinomial distribution which is wildly used in Bayes statistics. As we know, the limit distribution of the Beta distribution is a normal distribution (see [1]), and the Dirichlet distribution is the multivariate form of the Beta distribution (see [2]). So it is very important to investigate the asymptotic distribution of the Dirichlet distribution. For the Dirichlet distribution in which all parameters are the same, we have obtained an important conclusion (see [3]). But the Dirichlet distribution that we are concerned with in this paper is very special, in which all parameters are different from each other. We prove that the asymptotic distribution of this kind of Dirichlet distributions is still a normal distribution by using the central limit theorem and Slutsky theorem.

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2 Main Result

Lemma 2.1[4] *Suppose that* $\xi_1,\ \xi_2,\ \cdots,\ \xi_n,\ \cdots$ *are all i.i.d random vectors of dimension* $m(\in Z^+)$, $\bar{\xi}_n = \dfrac{1}{n}\sum\limits_{i=1}^{m}\xi_i$, *and* $N(x,\mu,\Sigma)$ *denotes the value on* x *of the multivariate normal distribution function with expectation* μ *and covariance matrix* Σ. *For* $E(\xi_j)=\mu$ *and* $Var(\xi_j)=\Sigma>0$, *there is (vector form* $\boldsymbol{a}<\boldsymbol{b}$ *expresses each of their corresponding components satisfies the same inequality relation)*

$$\lim_{n\to\infty}P\{\sqrt{n}(\bar{\xi}_n-\mu)<x\}\xrightarrow{W}N(x,0,\Sigma).$$

Lemma 2.2[5] *If* $\{\xi_n\}$ *and* $\{\eta_n\}$ *are all random variable sequences with* $(\xi_n,\ \eta_n)^T\xrightarrow{L}(\xi,\ \eta)^T$, $\zeta_n\xrightarrow{P}0$ *and* $\tau_n\xrightarrow{P}0$, *then* $(\xi_n+\zeta_n,\ \eta_n+\tau_n)^T\xrightarrow{L}(\xi,\ \eta)^T$.

Theorem 2.1 *Suppose that the random vector* $(Y_1,\ Y_2,\ \cdots,\ Y_n)^T$ *obeys the Dirichlet distribution with parameters* $(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_m,\ \alpha_{m+1})$ *satisfying*

$$Y_i\geq 0,\quad \sum_{i=1}^{m}Y_i\leq 1,\quad \alpha_i\geq 1\qquad (i=1,2,\cdots,m+1).$$

Particularly, when $\alpha_1=n_1,\ \alpha_2=n_2,\ \cdots,\ \alpha_m=n_m,\ \alpha_{m+1}=n_{m+1}\ (n_i\in Z^+)$, *we denote* $\sum\limits_{k=1}^{m+1}n_k=N$. *Given a matrix* $\boldsymbol{A}$ *and a vector* $\boldsymbol{C}$, *if* $\dfrac{n_k}{N}\longrightarrow\gamma_k$ *when* $n_k\to\infty$, *where* $\gamma_k\in(0,1)$, $k=1,2,\cdots,m+1$, *then*

$$\boldsymbol{A}[(Y_1,\ Y_2,\ \cdots,\ Y_m)^T-\boldsymbol{C}]\xrightarrow{L}\boldsymbol{Z}\sim\boldsymbol{N}_m(\boldsymbol{0},\ \Sigma),$$

where

$$\boldsymbol{A}=(\lambda_{ij})_{m\times m}$$

with

$$\lambda_{ij}=\begin{cases}\dfrac{N^2}{\sqrt{n_i(N-n_i)N}}, & i=j;\\ 0, & i\neq j,\end{cases}\qquad i,j=1,2,\cdots,m,$$

and

$$\boldsymbol{C}=\Big(\frac{n_1}{N},\ \frac{n_2}{N},\ \cdots,\ \frac{n_i}{N},\ \cdots,\ \frac{n_{m-1}}{N},\ \frac{n_m}{N}\Big)^T,$$

and $\boldsymbol{N}_m(\boldsymbol{0},\Sigma)$ *is the m-dimensional normal distribution whose covariance matrix is*

$$\Sigma=(\sigma_{ij})_{m\times m}=\begin{cases}1, & i=j;\\ -\sqrt{\dfrac{\gamma_i\gamma_j}{(1-\gamma_i)(1-\gamma_j)}}, & i\neq j,\end{cases}\qquad i,j=1,2,\cdots,m.$$

Proof. (1) Suppose that $X_{11},X_{12},\cdots,X_{1n_1},\cdots;X_{21},X_{22},\cdots,X_{2n_2},\cdots;\cdots;X_{m1},X_{m2},\cdots,X_{mn_m},\cdots;X_{m+1,1},\ X_{m+1,2},\ \cdots,\ X_{m+1,n_{m+1}},\ \cdots$ are all random variable sequences with independent and identical distribution and obey the exponential type distribution: $Exp(1)$. Because $Exp(1)$ is also $Ga(1,1)$ (see [6]), according to the countable additivity of Γ-distribution, we know that $n_i\bar{X}_i\sim\Gamma(n_i,1)$, where

$$\bar{X}_i=\frac{1}{n_i}\sum_{j=1}^{n_i}X_{ij}.$$

Let

$$\sum_{k=1}^{m+1} = N, \qquad Y_i = \frac{n_i \bar{X}_i}{\sum\limits_{k=1}^{m+1} n_k \bar{X}_k}.$$

Then

$$Y_i \sim \beta(n_i,\ N - n_i), \qquad i = 1, 2, \cdots, m+1.$$

Obviously,

$$\sum_{k=1}^{m+1} Y_k = 1,$$

and by [7] we know that

$$(Y_1,\ Y_2,\ \cdots,\ Y_m)^T \sim D(n_1,\ n_2,\ \cdots,\ n_m,\ n_{m+1}),$$

and its density function is

$$f(y_1,\ y_2,\ \cdots,\ y_m) = \begin{cases} \dfrac{\Gamma\Big[\sum\limits_{k=1}^{m+1} n_k\Big]}{\prod\limits_{k=1}^{m+1} \Gamma(n_k)} \prod\limits_{k=1}^{m} y_k^{n_k - 1} \Big(1 - \sum\limits_{k=1}^{m} y_k\Big)^{n_{m+1}-1}, & y_i \geq 0, \quad \sum\limits_{k=1}^{m+1} y_k = 1; \\ 0, & \text{elsewhere.} \end{cases}$$

Noting that

$$E(X_{ij}) = E(\bar{X}_i) = 1,$$

we have

$$Var(X_{ij}) = 1, \qquad Var(\bar{X}_i) = \frac{1}{n_i}.$$

Since every Y_i is the function of $(\bar{X}_1,\ \bar{X}_2,\ \cdots,\ \bar{X}_m,\ \bar{X}_{m+1})$, we denote

$$\begin{aligned} h(\bar{X}_1,\ \bar{X}_2,\ \cdots,\ \bar{X}_m,\ \bar{X}_{m+1}) &= \left(\frac{n_1 \bar{X}_1}{\sum\limits_{k=1}^{m+1} n_k \bar{X}_k},\ \frac{n_2 \bar{X}_2}{\sum\limits_{k=1}^{m+1} n_k \bar{X}_k},\ \cdots,\ \frac{n_m \bar{X}_m}{\sum\limits_{k=1}^{m+1} n_k \bar{X}_k} \right)^T \\ &= (Y_1,\ Y_2,\ \cdots,\ Y_m)^T. \end{aligned}$$

Using the Taylor formula to expand every component of $h(\bar{X}_1,\ \bar{X}_2,\ \cdots,\ \bar{X}_m,\ \bar{X}_{m+1})$ at $(1, 1, \cdots, 1)_{1\times(m+1)}$, we get

$$h(\bar{X}_1,\ \bar{X}_2,\ \cdots,\ \bar{X}_m,\ \bar{X}_{m+1}) = h(1, 1, \cdots, 1, 1) + \boldsymbol{Q} + \boldsymbol{R},$$

where

$$h(1, 1, \cdots, 1, 1) = \left(\frac{n_1}{N},\ \frac{n_2}{N},\ \cdots,\ \frac{n_i}{N},\ \cdots,\ \frac{n_{m-1}}{N},\ \frac{n_m}{N} \right)^T$$

which is denoted by $\boldsymbol{C}$; moreover,

$$(Y_1,\ Y_2,\ \cdots,\ Y_m)^T - \boldsymbol{C} = \boldsymbol{Q} + \boldsymbol{R},$$

where

$$\boldsymbol{Q} = (Q_1,\ Q_2,\ \cdots,\ Q_m)^T, \qquad \boldsymbol{R} = (R_1,\ R_2,\ \cdots,\ R_m)^T.$$

We know that

$$Q_i = \sum_{k=1}^{m+1} \frac{\partial Y_i}{\partial \overline{X}_k}\Big|_{\bar{X}_k=1,k=1,2,\cdots,m+1} (\bar{X}_k - 1), \qquad i = 1, 2, \cdots, m,$$

so

$$\begin{aligned} Q_i = {} & \frac{n_i \sum\limits_{k=1,k\neq i}^{m+1} n_k \bar{X}_k}{\Big(\sum\limits_{k=1}^{m+1} n_k \bar{X}_k\Big)^2}\Bigg|_{\bar{X}_k=1;k=1,2,\cdots,m+1} (\bar{X}_i - 1) \\ & - \frac{n_i}{\Big(\sum\limits_{k=1}^{m+1} n_k \bar{X}_k\Big)^2}\Bigg|_{\bar{X}_k=1;k=1,2,\cdots,m+1} \cdot \sum_{k=1,k\neq i}^{m+1} n_k(\bar{X}_k - 1) \\ = {} & \frac{n_i \sum\limits_{k=1,k\neq i}^{m+1} n_k}{\Big(\sum\limits_{k=1}^{m+1} n_k\Big)^2} (\bar{X}_i - 1) - \frac{n_i}{\Big(\sum\limits_{k=1}^{m+1} n_k\Big)^2} \sum_{k=1,k\neq i}^{m+1} n_k(\bar{X}_k - 1) \\ = {} & \frac{n_i(N - n_i)}{N^2}(\bar{X}_i - 1) - \frac{n_i}{N^2} \sum_{k=1,k\neq i}^{m+1} n_k(\bar{X}_k - 1). \end{aligned}$$

Set

$$1 + \theta(\bar{X}_k - 1) = T_k$$

with

$$0 < \theta < 1, \qquad k = 1, 2, \cdots, m + 1.$$

Meanwhile,

$$\begin{aligned} R_i = {} & \frac{1}{2} \sum_{k=1}^{m+1} \frac{\partial^2 Y_i}{(\partial \overline{X}_k)^2}\Bigg|_{\bar{X}_k=T_k;k=1,2,\cdots,m+1} (\bar{X}_k - 1)^2 \\ & + \frac{1}{2} \sum_{k=1,k\neq l}^{m+1} \sum_{l=1}^{m+1} \frac{\partial^2 Y_i}{\partial \overline{X}_k \partial \bar{X}_l}\Bigg|_{\bar{X}_k=T_k,\bar{X}_l=T_l;k,l=1,2,\cdots,m+1} (\bar{X}_k - 1)(\bar{X}_l - 1), \\ & \hfill i = 1, 2, \cdots, m. \end{aligned}$$

So

$$\begin{aligned} R_i = {} & -\frac{n_i^2 \sum\limits_{k=1,k\neq i}^{m+1} n_k \bar{X}_k}{\Big[\sum\limits_{k=1}^{m+1} n_k \bar{X}_k\Big]^3}\Bigg|_{\bar{X}_k=T_k;k=1,2,\cdots,m+1} (\bar{X}_i - 1)^2 \\ & + \frac{n_i \bar{X}_i}{\Big[\sum\limits_{k=1}^{m+1} n_k \bar{X}_k\Big]^3}\Bigg|_{\bar{X}_k=T_k;k=1,2,\cdots,m+1} \sum_{k=1,k\neq i}^{m+1} n_k^2(\bar{X}_k - 1)^2 \\ & + \frac{n_i(2n_i\bar{X}_i - \sum\limits_{k=1}^{m+1} n_k \bar{X}_k)}{\Big[\sum\limits_{k=1}^{m+1} n_k \bar{X}_k\Big]^3}\Bigg|_{\bar{X}_k=T_k;k=1,2,\cdots,m+1} (\bar{X}_i - 1) \sum_{k=1,k\neq i}^{m+1} n_k(\bar{X}_k - 1) \end{aligned}$$

$$+\frac{2n_i\bar{X}_i}{\Big[\sum\limits_{k=1}^{m+1} n_k\bar{X}_k\Big]^3}\Bigg|_{\bar{X}_k=T_k;k=1,2,\cdots,m+1}\sum_{k=1,k\neq i,k\neq l}^{m+1}\ \sum_{l=1,l\neq i}^{m+1} n_k(\bar{X}_k-1)n_l(\bar{X}_l-1)$$

$$= -\frac{n_i^2\sum\limits_{k=1,k\neq i}^{m+1} n_kT_k}{\Big[\sum\limits_{k=1}^{m+1} n_kT_k\Big]^3}(\bar{X}_i-1)^2+\frac{n_iT_i}{\Big[\sum\limits_{k=1}^{m+1} n_kT_k\Big]^3}\sum_{k=1,k\neq i}^{m+1} n_k^2(\bar{X}_k-1)^2$$

$$+\frac{n_i\Big(2n_iT_i-\sum_{k=1}^{m+1} n_kT_k\Big)}{\Big[\sum\limits_{k=1}^{m+1} n_k\bar{X}_k\Big]^3}(\bar{X}_i-1)\sum_{k=1,k\neq i}^{m+1} n_k(\bar{X}_k-1)$$

$$+\frac{2n_iT_i}{\Big[\sum\limits_{k=1}^{m+1} n_kT_k\Big]^3}\sum_{k=1,k\neq i,k\neq l}^{m+1}\ \sum_{l=1,l\neq i}^{m+1} n_k(\bar{X}_k-1)n_l(\bar{X}_l-1).$$

(2) We are to prove that

$$\boldsymbol{AQ}\xrightarrow{L}\boldsymbol{Z}\sim\boldsymbol{N}_m(\boldsymbol{0},\ \Sigma)\ \text{as}\ n_k\longrightarrow\infty,\qquad k=1,2,\cdots,m+1.$$

Noting that the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{Q}$ are shown before, by means of an easy calculation we get

$$\boldsymbol{AQ}=(\eta_1,\eta_2,\cdots,\eta_i,\cdots,\eta_{m-1},\eta_m)^T,$$

in which the i-th component of $\boldsymbol{AQ}$ is

$$\begin{aligned}\eta_i&=\frac{N^2}{\sqrt{n_i(N-n_i)N}}Q_i\\&=\sqrt{\frac{(N-n_i)}{N}}\sqrt{n_i}(\bar{X}_i-1)-\sum_{k=1,k\neq i}^{m+1}\sqrt{\frac{n_in_k}{(N-n_i)N}}\sqrt{n_k}(\bar{X}_k-1).\end{aligned}$$

If we set

$$\boldsymbol{B}_t=(a_{ij})_{m\times m}=\begin{cases}\sqrt{\dfrac{N-n_i}{N}}, & i=j=t;\\ -\sqrt{\dfrac{n_tn_i}{(N-n_i)N}}, & i=j,\ i\neq t,\ j\neq t;\\ 0, & i\neq j\end{cases}\qquad \text{for } t=1,2,\cdots,m,$$

$$\boldsymbol{B}_{m+1}=(b_{ij})_{m\times m}=\begin{cases}\sqrt{-\dfrac{n_in_{m+1}}{(N-n_i)N}}, & i=j;\\ 0, & i\neq j,\end{cases}$$

$$\bar{X}_t=[\sqrt{n_t}(\bar{X}_t-1),\ \sqrt{n_t}(\bar{X}_t-1),\ \cdots,\ \sqrt{n_t}(\bar{X}_t-1)]^T_{1\times m},$$

$$\boldsymbol{W}_{tj}=(X_{tj}-1,\ X_{tj}-1,\ \cdots,\ X_{tj}-1)^T_{1\times m},$$

$$\boldsymbol{0}=(0,0,0,\cdots,0)^T_{1\times m},$$

then we have

$$\boldsymbol{A}\boldsymbol{Q} = \sum_{t=1}^{m+1} \boldsymbol{B}_t \bar{X}_t.$$

For any real number t and $j = 1, 2, \cdots, n_t$, $\boldsymbol{B}_t \boldsymbol{W}_{tj}$ is of $i.i.d.$ By Lemma 2.1 (the multivariate central limit theorem) we obtain that

$$\sum_{j=1}^{n_t} \boldsymbol{B}_t \boldsymbol{W}_{tj} = \sum_{j=1}^{n_t} \boldsymbol{B}_t \cdot (X_{tj} - 1, X_{tj} - 1, \cdots, X_{tj} - 1)^T = \sqrt{n_t} \boldsymbol{B}_t \bar{X}_t,$$

$$\begin{aligned}\sum_{j=1}^{n_t} \boldsymbol{B}_t \boldsymbol{W}_{tj} - \boldsymbol{0} = &\sum_{j=1}^{n_t} \boldsymbol{B}_t \cdot (X_{tj} - 1, X_{tj} - 1, \cdots, X_{tj} - 1)^T - (0, 0, \cdots, 0)^T \\ = &\sqrt{n_t} \boldsymbol{B}_t \bar{X}_t \xrightarrow{L} \boldsymbol{N}_m(\boldsymbol{0}, \boldsymbol{E}_t'),\end{aligned}$$

where

$$\boldsymbol{E}_t' = Var\Big(\sum_{j=1}^{n_t} \boldsymbol{B}_t \boldsymbol{W}_{tj}\Big) = n_t Var(\boldsymbol{B}_t \boldsymbol{W}_{tj}\Big) = n_t \boldsymbol{B}_t Var(\boldsymbol{W}_{tj}) \boldsymbol{B}_t'$$

and

$$Var(\boldsymbol{W}_{tj}) = (\boldsymbol{1})_{m\times m}.$$

Let

$$\boldsymbol{E}_t = \boldsymbol{B}_t Var(\boldsymbol{W}_{tj}) \boldsymbol{B}_t'.$$

Then

$$\boldsymbol{B}_t \bar{X}_t \xrightarrow{L} \boldsymbol{N}_m(\boldsymbol{0}, \boldsymbol{E}_t).$$

Let

$$\boldsymbol{Y} = \sum_{t=1}^{m+1} \boldsymbol{E}_t.$$

By means of a series of calculations we can abtain

$$\boldsymbol{Y} = (\pi_{ij})_{m\times m} = \begin{cases} 1, & i = j; \\ -\sqrt{\dfrac{n_i n_j}{(N - n_i)(N - n_j)}}, & i \neq j, \end{cases} \qquad i, j = 1, 2, \cdots, m.$$

By means of the condition in the theorem:

$$\text{“}\ \frac{n_k}{N} \longrightarrow \gamma_k \text{ when } n_k \longrightarrow \infty, \qquad \gamma_k \in (0, 1),\ k = 1, 2, \cdots, m+1\text{”},$$

we get

$$\frac{n_k}{N - n_k} \longrightarrow \frac{\gamma_k}{1 - \gamma_k},$$

and thus

$$\boldsymbol{Y} \longrightarrow \Sigma, \qquad n_k \longrightarrow \infty,$$

where

$$\Sigma = (\sigma_{ij})_{m\times m} = \begin{cases} 1, & i = j; \\ -\sqrt{\dfrac{\gamma_i \gamma_j}{(1 - \gamma_i)(1 - \gamma_j)}}, & i \neq j, \end{cases} \qquad i, j = 1, 2, \cdots, m.$$

Finally we have

$$\boldsymbol{AQ} = \sum_{t=1}^{m+1} \boldsymbol{B}_t \bar{X}_t \xrightarrow{L} \boldsymbol{N}_m\Big(\boldsymbol{0},\ \sum_{t=1}^{m+1} \boldsymbol{E}_t\Big) = \boldsymbol{N}_m(\boldsymbol{0},\ \boldsymbol{Y}) \longrightarrow \boldsymbol{N}_m(\boldsymbol{0},\ \varSigma).$$

Namely,

$$\boldsymbol{AQ} \xrightarrow{L} \boldsymbol{Z} \sim \boldsymbol{N}_m(\boldsymbol{0},\ \varSigma), \qquad n_k \longrightarrow \infty \ (k = 1, 2, \cdots, m+1).$$

(3) We are to prove

$$\boldsymbol{AR} \xrightarrow{P} 0, \qquad n_k \longrightarrow \infty \ (k = 1, 2, \cdots, m+1).$$

Noting that the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{R}$ are shown before, by means of an easy calculation we get

$$\boldsymbol{AR} = (\delta_1, \delta_2, \cdots, \delta_i, \cdots, \delta_{m-1}, \delta_m)^T,$$

in which the i-th component of $\boldsymbol{AR}$ is

$$\begin{aligned}
\delta_i &= \frac{N^2}{\sqrt{n_i(N-n_i)N}} R_i \\
&= -\frac{N^2}{\sqrt{n_i(N-n_i)N}} \frac{n_i^2 \sum\limits_{k=1,k\neq i}^{m+1} n_k T_k}{\Big[\sum\limits_{k=1}^{m+1} n_k T_k\Big]^3} (\bar{X}_i - 1)^2 \\
&\quad + \frac{N^2}{\sqrt{n_i(N-n_i)N}} \frac{n_i T_i}{\Big[\sum\limits_{k=1}^{m+1} n_k T_k\Big]^3} \sum_{k=1,k\neq i}^{m+1} n_k^2 (\bar{X}_k - 1)^2 \\
&\quad + \frac{N^2}{\sqrt{n_i(N-n_i)N}} \frac{n_i\Big[2n_i T_i - \sum\limits_{k=1}^{m+1} n_k T_k\Big]}{\Big[\sum\limits_{k=1}^{m+1} n_k T_k\Big]^3} (\bar{X}_i - 1) \sum_{k=1,k\neq i}^{m+1} n_k(\bar{X}_k - 1) \\
&\quad + \frac{N^2}{\sqrt{n_i(N-n_i)N}} \frac{2n_i T_i}{\Big[\sum\limits_{k=1}^{m+1} n_k T_k\Big]^3} \sum_{k=1,k\neq i,k\neq l}^{m+1} \sum_{l=1,l\neq i}^{m+1} n_k(\bar{X}_k - 1) n_l (\bar{X}_l - 1) \\
&= S_1 + S_2 + S_3 + S_4.
\end{aligned}$$

Then by repeated use of the Slutsky theorem we are to prove that each of S_1, S_2, S_3, S_4 converges to 0 in probability.

For any integers i $(i = 1, 2, \cdots, m)$ and k $(k = 1, 2, \cdots, m+1)$, we suppose that when $n_k \longrightarrow \infty$ there is $\dfrac{n_k}{N} \longrightarrow \gamma_k$ $(\gamma_k \in (0, 1))$. Obviously,

$$\sum_{k=1}^{m+1} \gamma_k = 1$$

and

$$\gamma_i = 1 - \sum_{k=1,k\neq i}^{m+1} \gamma_k.$$

So we have

a) $\dfrac{N-n_i}{N} \longrightarrow 1-\gamma_i:\ \sqrt{\dfrac{N-n_i}{N}} \longrightarrow \sqrt{1-\gamma_i};$

b) $\dfrac{n_i n_k}{(N-n_i)N} = \dfrac{n_k}{N}\dfrac{\frac{n_i}{N}}{\frac{N-n_i}{N}} \longrightarrow \dfrac{\gamma_i\gamma_k}{1-\gamma_i}:\ \sqrt{\dfrac{n_i n_k}{(N-n_i)N}} \longrightarrow \sqrt{\dfrac{\gamma_i\gamma_k}{1-\gamma_i}}.$

Based on the central limit theorem and the Slutsky theorem, we have

c) $\sqrt{n_k}(\bar{X}_k-1) \xrightarrow{L} N(0,1);$

d) $\bar{X}_k - 1 \xrightarrow{P} 0;$

e) $T_k \xrightarrow{P} 1;\ 2T_k \xrightarrow{P} 2;$

f) $\sum\limits_{k=1,k\neq i}^{m+1} \dfrac{n_i n_k}{(N-n_i)N}T_k \xrightarrow{P} \sum\limits_{k=1,k\neq i}^{m+1} \dfrac{\gamma_i\gamma_k}{(1-\gamma_i)} = \gamma_i;$

g) $\sum\limits_{k=1,k\neq i}^{m+1} \dfrac{n_k}{N}T_k \xrightarrow{P} \sum\limits_{k=1}^{m+1}\gamma_k = 1;\ \Big(\sum\limits_{k=1}^{m+1}\dfrac{n_k}{N}T_k\Big)^t \xrightarrow{P} 1\ (t\in Z^+);$

h) $\dfrac{\sum\limits_{k=1,k\neq i}^{m+1}\dfrac{n_i n_k}{N(N-n_i)}T_k}{\Big[\sum\limits_{k=1}^{m+1}\dfrac{n_k}{N}T_k\Big]^3} \xrightarrow{P} \gamma_i;$

i) $\dfrac{T_i}{\Big[\sum\limits_{k=1}^{m+1}\dfrac{n_k}{N}T_k\Big]^3} \xrightarrow{P} 1;\ \dfrac{2T_i}{\Big[\sum\limits_{k=1}^{m+1}\dfrac{n_k}{N}T_k\Big]^3} \xrightarrow{P} 2;$

j) $\dfrac{\bar{X}_i - 1}{\Big[\sum\limits_{k=1}^{m+1}\dfrac{n_k}{N}T_k\Big]^3} \xrightarrow{P} 0.$

For

$$\begin{aligned} S_1 =& -\frac{N^2}{n_i(N-n_i)N}\frac{n_i^2\sum\limits_{k=1,k\neq i}^{m+1} n_k T_k}{\Big[\sum\limits_{k=1}^{m+1} n_k T_k\Big]^3}(\bar{X}_i-1)^2 \\ =& -\sqrt{\frac{N-n_i}{N}}\frac{\sum\limits_{k=1,k\neq i}^{m+1}\frac{n_i n_k}{N(N-n_i)}T_k}{\Big[\sum\limits_{k=1}^{m+1}\frac{n_k}{N}T_k\Big]^3}\sqrt{n_i}(\bar{X}_i-1)(\bar{X}_i-1), \end{aligned}$$

by a), c), d), h) and the Slutsky theorem, we have

$$S_1 \xrightarrow{P} 0.$$

For

$$\begin{aligned} S_2 =& \frac{N^2}{n_i(N-n_i)N}\frac{n_i T_i}{\Big[\sum\limits_{k=1}^{m+1} n_k T_k\Big]^3}\sum_{k=1,k\neq i}^{m+1} n_k^2(\bar{X}_k-1)^2 \\ =& \frac{T_i}{\Big[\sum\limits_{k=1}^{m+1}\frac{n_k}{N}T_k\Big]^3}\sum_{k=1,k\neq i}^{m+1}\frac{n_k}{N}\sqrt{\frac{n_i n_k}{N(N-n_i)}}\sqrt{n_k}(\bar{X}_k-1)(\bar{X}_k-1), \end{aligned}$$

by the assumption and b), c), d), i) as well as the Slutsky theorem, we have

$$S_2 \xrightarrow{P} 0.$$

For

$$\begin{aligned} S_3 &= \frac{N^2}{\sqrt{n_i(N-n_i)N}} \frac{n_i\Big[2n_iT_i - \sum\limits_{k=1}^{m+1} n_kT_k\Big]}{\Big[\sum\limits_{k=1}^{m+1} n_kT_k\Big]^3}(\bar{X}_i - 1)\sum_{k=1,k\neq i}^{m+1}(\bar{X}_k - 1) \\ &= \sqrt{\frac{N-n_i}{N}} \frac{2T_i}{\Big[\sum\limits_{k=1}^{m+1} \frac{n_k}{N}T_k\Big]^3}\sqrt{n_i}(\bar{X}_i - 1)\sum_{k=1,k\neq i}^{m+1}\frac{n_in_k}{N(N-n_i)}(\bar{X}_k - 1) \\ &\quad - \frac{(\bar{X}_i - 1)}{\Big[\sum\limits_{k=1}^{m+1} \frac{n_k}{N}T_k\Big]^2}\sum_{k=1,k\neq i}^{m+1}\sqrt{\frac{n_in_k}{N(N-n_i)}}\sqrt{n_k}(\bar{X}_k - 1), \end{aligned}$$

by a), b), c), d), i), j) and the Slutsky theorem, we have

$$S_3 \xrightarrow{P} 0.$$

For

$$\begin{aligned} S_4 &= \frac{N^2}{\sqrt{n_i(N-n_i)N}} \frac{2n_iT_i}{\Big[\sum\limits_{k=1}^{m+1} n_kT_k\Big]^3}\sum_{k=1,k\neq i,k\neq l}^{m+1}\ \sum_{l=1,l\neq i}^{m+1} n_k(\bar{X}_k - 1)n_l(\bar{X}_l - 1) \\ &= \frac{2T_i}{\Big[\sum\limits_{k=1}^{m+1} \frac{n_k}{N}T_k\Big]^3}\sum_{k=1,k\neq i,k\neq l}^{m+1}\ \sum_{l=1,l\neq i}^{m+1}\sqrt{\frac{n_in_k}{N(N-n_i)}}\sqrt{n_k}(\bar{X}_k - 1)\frac{n_l}{N}(\bar{X}_l - 1), \end{aligned}$$

by the assumption and b), c), d), i) as well as the Slutsky theorem, we have

$$S_4 \xrightarrow{P} 0.$$

Finally, by use of the Slutsky theorem again, we have

$$\frac{N^2}{\sqrt{n_i(N-n_i)N}}R_i = S_1 + S_2 + S_3 + S_4 \xrightarrow{P} 0 \qquad (i = 1, 2, \cdots, m).$$

Namely,

$$\boldsymbol{AR} \xrightarrow{P} 0.$$

(4) Based on the conclusion of (2) and making use of Lemma 2.2, we have

$$\boldsymbol{A}(\boldsymbol{Q} + \boldsymbol{R}) \xrightarrow{L} \boldsymbol{Z} \sim \boldsymbol{N}_m(\boldsymbol{0},\ \varSigma).$$

Namely,

$$\boldsymbol{A}[(Y_1,\ Y_2,\ \cdots,\ Y_k)^T - \boldsymbol{C}] = \boldsymbol{A}(\boldsymbol{Q} + \boldsymbol{R}) \xrightarrow{L} \boldsymbol{Z} \sim \boldsymbol{N}_m(\boldsymbol{0},\ \varSigma),$$

where, the covariance matix is

$$\varSigma = (\sigma_{ij})_{m\times m} = \begin{cases} 1, & i = j; \\ -\sqrt{\dfrac{\gamma_i\gamma_j}{(1-\gamma_i)(1-\gamma_j)}}, & i \neq j, \end{cases} \qquad i, j = 1, 2, \cdots, m.$$

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