Communications in Mathematical Research
26(1)(2010), 41–52

Star-shaped Differentiable Functions and Star-shaped Differentials*

Pan Shao-rong[1,2], Zhang Hong-wei[1] and Zhang Li-wei[1]
(*1. School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024*)
(*2. School of Mathematical Sciences, Harbin Normal University, Harbin, 150080*)

Communicated by Yin Jing-xue

Abstract: Based on the isomorphism between the space of star-shaped sets and the space of continuous positively homogeneous real-valued functions, the star-shaped differential of a directionally differentiable function is defined. Formulas for star-shaped differential of a pointwise maximum and a pointwise minimum of a finite number of directionally differentiable functions, and a composite of two directionally differentiable functions are derived. Furthermore, the mean-value theorem for a directionally differentiable function is demonstrated.

Key words: The space of star-shaped sets, gauge function, isometrical isomorphism, directionally differentiable function, star-shaped differential, mean-value theorem
2000 MR subject classification: 90C30, 90C90
Document code: A
Article ID: 1674-5647(2010)01-0041-12

1 Introduction

In 1950s and 1960s Fenchel and Rockafellar[1] investigated differential theory and optimization theory for nonsmooth convex functions. A convex function is directionally differentiable and its subdifferential is a convex set. In 1970s Clarke[2] made contributions to the differential and optimization theory for Lipschitz continuous functions. The generalized subdifferential for a Lipschitz function is a convex set, but the set of generalized directional derivatives for Lipschitz continuous functions is not a linear space. In 1969, Pshenichyi[3] suggested the concept of quasidifferentiability where the directional derivative is a sublinear function. But the class of quasidifferentiable functions in the sense of Pshenichnyi is insufficient to describe many important situations; for instance, it cannot include the so-called D. C. functions. In 1970s, Demyanov, Rubinov and Polyakova extended the concept of quasidifferentiable functions in the sense of Pshenichnyi by introducing a new definition, which

*Received date: June 17, 2008.
Foundation item: The NSF (10771026) of China.

ensures many good properties about arithmetic operations of quasidifferentiable functions in the new sense (see [4] and [5]). A quasidifferentiable function in the sense of Demyanov and Rubinov is directionally differentiable and its directional derivative is representable as a difference of two convex functions with the differential as a pair of convex compact sets. The class of quasidifferentiable functions, in the sense of Demyanov and Rubinov, have a very wide practical applied background, such as problems for storages and problems for optimum layout of circuits.

Recently, noticing that many functions appearing in bilevel programs are a new class of important directionally differentiable functions, Zhang *et al.*[6] proposed the concept of generalized quasidifferentiable functions, and Zhang *et al.*[7] explored the optimality conditions for nondifferentiable optimization problems of generalized quasidifferentiable functions. But for a general directionally differentiable function without any structures, there is no suitable defintion for its differential. Observing that if the directionally derivative is a positively homogeneous continuous function in direction, then it can be represented as a difference of two nonnegative positively homogeneous continuous functions, we can express the directional derivative as the difference of two gauge functions of star-shaped sets. A directionally differentiable function whose directional derivatives is continuous in direction is defined as a star-shaped differentiable function. A star-shaped differential of a star-shaped differentiable function is a pair of star-shaped sets, but not a pair of convex compact sets. We can verify that any quasidifferentiable function in the sense of Demyanov and Rubinov is star-shaped differentiable.

In this paper, we first present some preliminaries about the space of star-shaped sets. Secondly, we give the concept of star-shaped differential and verify arithmetic operations of star-shaped differentials, and derive formulas for star-shaped differentials of a pointwise maximum and of a pointwise minimum of a finite number of directionally differentiable functions, and a composite of two directionally differentiable functions. Finally, we demonstrate the mean-value theorem for a star-shaped differentiable function.

2 Preliminaries

In this section we recall some results about the space of star-shaped sets and the space of positively homogeneous continuous functions (see [8]).

Definition 2.1[9] *A closed subset A of $\mathbf{R}^n$ is called a star-shaped set if it contains the origin as an interior point and every ray*

$$L_x = \{\lambda x \mid \lambda \geq 0\}, \qquad \forall\, x \in \mathbf{R}^n \backslash \{0\}$$

does not intersect the boundary of A more than once.

Define

$$K = \{A \subseteq \mathbf{R}^n \mid A \text{ is a star-shaped set}\}.$$

We introduce a partial order in K^2, denoted by $\succeq$, in the sense that for $(A_i, B_i) \in K^2$, $i = 1, 2$,

$$(A_1, B_1) \succeq (A_2, B_2)$$

if and only if

$$A_1 \oplus B_2 \subseteq A_2 \oplus B_1,$$

where $\oplus$ (called inverse sum) is defined by

$$A \oplus B \equiv \mathrm{cl} \bigcup_{0 \le \alpha \le 1} (\alpha A \cap (1-\alpha)B), \qquad A, B \in K,$$

and it is assumed that

$$0 \cdot A = \bigcap_{\alpha > 0} \alpha A.$$

An equivalence relation, denoted by $\sim$, is deduced from this partial order, i.e., for $(A_i, B_i) \in K^2$, $i = 1, 2$,

$$(A_1, B_1) \sim (A_2, B_2)$$

if and only if

$$A_1 \oplus B_2 = A_2 \oplus B_1.$$

Let $K_1 = K^2/\sim$ and define the inverse sum $\oplus$ and inverse scalar multiplication $\odot$ in K_1 as follows:

$$(A_1, B_1) \oplus (A_2, B_2) = (A_1 \oplus A_2,\ B_1 \oplus B_2),$$

$$\alpha \odot (A, B) = \begin{cases} (\alpha \odot A,\ \alpha \odot B), & \text{if } \alpha \ge 0; \\ (|\alpha| \odot B,\ |\alpha| \odot A), & \text{if } \alpha < 0, \end{cases}$$

where

$$\alpha \odot A = \begin{cases} \dfrac{1}{\alpha} A, & \text{if } \alpha > 0; \\ \mathbf{R}^n, & \text{if } \alpha = 0. \end{cases}$$

We introduce a norm in K_1, defined by:

$$\|(A, B)\|_{K_1} = \inf\{\lambda > 0 \mid B \oplus \frac{1}{\lambda} U \subseteq A,\ A \oplus \frac{1}{\lambda} U \subseteq B\},$$

where

$$U = \{x \in \mathbf{R}^n \mid \|x\| \le 1\}, \qquad (A, B) \in K_1.$$

Define

$$H = \{\sigma : \mathbf{R}^n \to \mathbf{R} \mid \sigma \text{ is a nonnegative positively homogeneous and continuous function}\}.$$

A partial order in H^2, denoted by $\succeq$, is defined by

$$(\sigma_1, \tau_1) \succeq (\sigma_2, \tau_2)$$

if and only if

$$\sigma_1 - \tau_1 \ge \sigma_2 - \tau_2,$$

where

$$(\sigma_i, \tau_i) \in H^2, \qquad i = 1, 2.$$

An equivalence relation, denoted by $\sim$, is deduced from this partial order, i.e.,

$$(\sigma_1, \tau_1) \sim (\sigma_2, \tau_2)$$

if and only if

$$\sigma_1 - \tau_1 = \sigma_2 - \tau_2.$$

Define $H_1 = H^2/\sim$ and introduce addition operation and scalar multiplication operation in H_1 as follows:

$$(\sigma_1, \tau_1) + (\sigma_2, \tau_2) = (\sigma_1 + \sigma_2, \tau_1 + \tau_2),$$

$$\alpha(\sigma, \tau) = \begin{cases} (\alpha\sigma, \alpha\tau), & \text{if } \alpha \geq 0; \\ (|\alpha|\tau, |\alpha|\sigma), & \text{if } \alpha < 0. \end{cases}$$

Defined a norm in H_1 by

$$\|(\sigma, \tau)\|_{H_1} = \sup_{u \in U} |\sigma(u) - \tau(u)|.$$

Let $A \subseteq \mathbf{R}^n$, $0 \in \text{int}A$. The function $\phi(A) : \mathbf{R}^n \to \mathbf{R}$ defined by

$$\phi(A)(y) = \inf\{\lambda > 0 \mid y \in \lambda A\}$$

is called the gauge (or the Minkowski gauge function) of a set A. If A is convex then the gauge coincides with the gauge function in the sense of convex analysis. The following lemma plays an important role in studying the isomorphism between the space of star-shaped sets and the space of positively homogeneous continuous functions.

Lemma 2.1[9] *Let $\psi : \mathbf{R}^n \to \mathbf{R}$. The following statements are equivalent*:

(i) *the functional ψ is positively homogeneous, nonnegative and continuous*;

(ii) *ψ coincides with the gauge of a star-shaped set Ω, where $\Omega = \{x \mid \psi(x) \leq 1\}$.*

Define

$$C_{p,h} = \{f : \mathbf{R}^n \to \mathbf{R} \mid f \text{ is positively homogeneous continuous function}\}.$$

Noting that

$$f(\cdot) = \max\{f(\cdot),\ 0\} - \max\{-f(\cdot),\ 0\},$$

one has

$$C_{p,h} = H - H.$$

Define $\Phi : K_1 \to H_1$ by

$$\Phi(A, B)(\cdot) = (\Phi(A)(\cdot),\ \Phi(B)(\cdot)), \qquad (A, B) \in K_1$$

and define $T : H_1 \to C_{p,h}$ by

$$T(\sigma, \tau) = \sigma - \tau, \qquad \sigma, \tau \in H_1.$$

Theorem 2.1 *The mapping $T\Phi : K_1 \to C_{p,h}$ is a linear isometrically Riesz isomorphism.*

3 Star-shaped Differentials

Definition 3.1 *Let $f:\mathbf{R}^n\to\mathbf{R}$ be directionally differentiable at $x\in\mathbf{R}^n$. If there exists a pair of star-shaped sets $(\underline{Q}f(x),\overline{Q}f(x))\in K_1$ such that*

$$f'(x;\,\cdot\,)=T\Phi((\underline{Q}f(x),\ \overline{Q}f(x)))(\,\cdot\,),\tag{3.1}$$

then we say that f is star-shaped differentiable at x. The pair $Qf(x)=(\underline{Q}f(x),\ \overline{Q}f(x))$ is called a star-shaped differential of f at x; $\underline{Q}f(x)$ and $\overline{Q}f(x)$ are called a star-shaped subdifferential and a star-shaped superdifferential, respectively, of f at x. If f is star-shaped differentiable at every point of the space $\mathbf{R}^n$, we say that f is a star-shaped differentiable function.

For a subset V of the space $\mathbf{R}^n$, we use V° to denote its polar:

$$V^\circ=\{x\mid\langle v,x\rangle\le 1,\forall\ v\in V\}.$$

From (3.1) and definitions of the mapping T and Φ, respectively, we have that

$$f'(x;d)=\inf\{\lambda>0\mid d\in\lambda\underline{Q}f(x)\}-\inf\{\lambda>0\mid d\in\lambda\overline{Q}f(x)\},\qquad\forall d\in\mathbf{R}^n.$$

If $\underline{Q}f(x)$ and $\overline{Q}f(x)$ are star-shaped convex sets, then

$$f'(x;d)=\max_{v\in\underline{\partial}f(x)}\langle v,d\rangle+\min_{w\in\overline{\partial}f(x)}\langle w,d\rangle,$$

where

$$\underline{\partial}f(x)=(\underline{Q}f(x))^o,\qquad\overline{\partial}f(x)=-(\overline{Q}f(x))^o$$

are convex compact sets. Hence, f is quasidifferentiable at x in the sense of Demyanov and Rubinov. On the other hand, if f is quasidifferentiable at x in the sense of Demyanov and Rubinov, then f is star-shaped differentiable at x, and

$$\underline{Q}f(x)=(\underline{\partial}f(x))^o,\qquad\overline{Q}f(x)=-(\overline{\partial}f(x))^o$$

are star-shaped convex sets. Thus all quasidifferentiable functions are star-shaped differentiable.

Example 3.1 [10] Consider the function

$$f(x_1,x_2)=2\sqrt{|x_1x_2|},\qquad\bar{x}=(0,0)\in\mathbf{R^2}.$$

The function f is directionally differentiable at $\bar{x}=(0,0)$, and for every $d\in\mathbf{R}^2$, we have

$$f'(\bar{x};d)=2\sqrt{|d_1d_2|}.$$

Let

$$\phi_1(d)=(\sqrt{|d_1|}+\sqrt{|d_2|})^2,\qquad\phi_2(d)=|d_1|+|d_2|.$$

Then

$$f'(\bar{x};d)=\phi_1(d)-\phi_2(d).$$

It is clear that $\phi_1(d)$ and $\phi_2(d)$ are nonnegative positively homogeneous continuous functions in d. From Lemma 2.1, we obtain

$$\phi_1(d)=\phi(\Omega_1)(d),\qquad\phi_2(d)=\phi(\Omega_2)(d).$$

The star-shaped sets

$$\Omega_1=\{d\in R^2|(\sqrt{|d_1|}+\sqrt{d_2|})^2\le 1\},\qquad\Omega_2=\{d\in R^2||d_1|+|d_2|\le 1\}$$

are shown in Fig. 3.1.

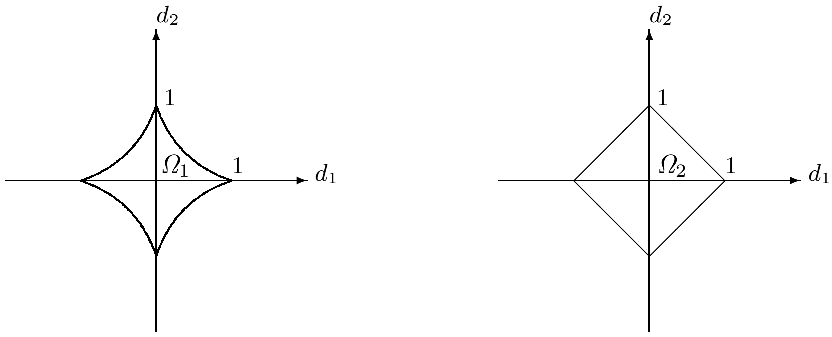

Fig. 3.1

Thus,

$$f'(\bar{x};d) = T\Phi((\Omega_1, \Omega_2))(d),$$

where $(\Omega_1, \Omega_2) \in K_1$. Therefore, f is star-shaped differentiable at $\bar{x} = (0,0)$. But f is not Lipschitz continuous around $\bar{x}$, so, f is not quasidifferentiable at $\bar{x}$ in the sense of Demyanov and Rubinov.

The following conclusion is obvious.

Theorem 3.1 *Let $f : \mathbf{R}^n \to \mathbf{R}$ be directional differentiable at $x \in \mathbf{R}^n$. The function f is star-shaped differentiable at x if and only if $f'(x;d)$ is positively homogeneous continuous in $d \in \mathbf{R}^n$.*

Lemma 3.1 (Arithmetic operations of star-shaped differentials)

(i) *Let $f_i : \mathbf{R}^n \to \mathbf{R}$ $(i = 1, 2, \cdots, N)$ be star-shaped differentiable at $x \in \mathbf{R}^n$, and $c_i \in \mathbf{R}$ $(i = 1, 2, \cdots, N)$. Then the function $f \equiv \sum_{i=1}^{N} c_i f_i$ is star-shaped differentiable at $x \in \mathbf{R}^n$, and*

$$Qf(x) = \Big(\sum_{i=1}^{N} \oplus\Big)(c_i \odot Qf_i(x)).$$

(ii) *Let $f_i : \mathbf{R}^n \to \mathbf{R}$ $(i = 1, 2)$ be star-shaped differentiable at $x \in \mathbf{R}^n$. Then the function $f \equiv f_1 f_2$ is star-shaped differentiable at $x \in \mathbf{R}^n$, and*

$$Qf(x) = f_1(x) \odot Qf_2(x) \oplus f_2(x) \odot Qf_1(x).$$

(iii) *Let $f : \mathbf{R}^n \to \mathbf{R}$ be star-shaped differentiable at $x \in \mathbf{R}^n$, and $f(x) \neq 0$. Then the function $\varphi \equiv \dfrac{1}{f}$ is star-shaped differentiable at $x \in \mathbf{R}^n$, and*

$$Q\varphi(x) = -\frac{1}{f^2(x)} \odot Qf(x).$$

(iv) *Let $f_i : \mathbf{R}^n \to \mathbf{R}$ $(i = 1, 2)$ be star-shaped differentiable at $x \in \mathbf{R}^n$, and $f_2(x) \neq 0$. Then the function $f \equiv \dfrac{f_1}{f_2}$ is star-shaped differentiable at $x \in \mathbf{R}^n$, and*

$$Qf(x) = \frac{1}{f_2^2(x)}[f_2(x) \odot Qf_1(x) \ominus f_1(x) \odot Qf_2(x)].$$

Proof. Since

$$f'(x;\,\cdot\,) = \Big(\sum_{i=1}^{N} c_i f_i\Big)'(x;\,\cdot\,) = \sum_{i=1}^{N} c_i f_i'(x;\,\cdot\,), \qquad f_i'(x;\,\cdot\,) = T\,\Phi(Qf_i(x))(\,\cdot\,),$$

we have

$$f'(x;\,\cdot\,) = \sum_{i=1}^{N} c_i T\,\Phi(Qf_i(x))(\cdot\,),$$

which from the linearity of T and Φ yields that

$$f'(x;\,\cdot\,) = T\,\Phi\Big(\Big(\sum_{i=1}^{N}\oplus\Big)(c_i \odot Qf_i(x))\Big)(\cdot\,).$$

From the definition of a directionally differentiable function and the inclusion

$$\Big(\sum_{i=1}^{N}\oplus\Big)(c_i \odot Qf_i(x)) \in K_1,$$

we obtain that f is star-shaped differentiable at $x \in \mathbf{R}^n$, and

$$Qf(x) = \Big(\sum_{i=1}^{N}\oplus\Big)(c_i \odot Qf_i(x)).$$

We can prove (ii), (iii) and (iv) in a similar way, and we omit the details here. The proof is completed.

In the following, formulas for star-shaped differential of a pointwise maximum, a pointwise minimum of a finite number of directionally differentiable functions, and a composite of two directionally differentiable functions are derived. Define

$$\bigvee_{i=1}^{N} f_i(x) \equiv \max\{f_i(x) \mid i = 1, 2, \cdots, N\}, \qquad x \in \mathbf{R}^n,$$
$$\bigwedge_{i=1}^{N} f_i(x) \equiv \min\{f_i(x) \mid i = 1, 2, \cdots, N\}, \qquad x \in \mathbf{R}^n.$$

$\bigvee_{i=1}^{N} f_i$ and $\bigwedge_{i=1}^{N} f_i$ are called pointwise maximum function and pointwise minimum function, of $f_i, i = 1, 2, \cdots, N$, respectively.

Lemma 3.2 *Let $f_i : \mathbf{R}^n \to \mathbf{R}$ $(i = 1, 2, \cdots, N)$ be star-shaped differentiable at $x \in \mathbf{R}^n$. Then $\bigvee_{i=1}^{N} f_i$ and $\bigwedge_{i=1}^{N} f_i$ are star-shaped differentiable at $x \in \mathbf{R}^n$, and*

$$Q\bigvee_{i=1}^{N} f_i(x) = \bigvee_{i\in R(x)} Qf_i(x), \qquad Q\bigwedge_{i=1}^{N} f_i(x) = \bigwedge_{i\in J(x)} Qf_i(x), \tag{3.2}$$

where

$$R(x) = \{i \mid f_i(x) = \bigvee_{i=1}^{N} f_i(x), i = 1, 2, \cdots, N\},$$

$$J(x) = \{i \mid f_i(x) = \bigwedge_{i=1}^{N} f_i(x), i = 1, 2, \cdots, N\}.$$

Proof. By the Corollary 3.2 of [11], we can easily check that $\bigvee_{i=1}^{N} f_i$ and $\bigwedge_{i=1}^{N} f_i$ are directionally differentiable at $x \in \mathbf{R}^n$, and for any $d \in \mathbf{R}^n$,

$$\Big(\bigvee_{i=1}^{N} f_i\Big)'(x;d) = \bigvee_{i\in R(x)} f_i'(x;d), \qquad \Big(\bigwedge_{i=1}^{N} f_i\Big)'(x;d) = \bigwedge_{i\in J(x)} f_i'(x;d).$$

Since f_i $(i = 1, 2, \cdots, N)$ are star-shaped differentiable at $x \in \mathbf{R}^n$, we have

$$f_i'(x;d) = T\Phi(Qf_i(x))(d), \qquad i = 1, 2, \cdots, N,$$

and so

$$\Big(\bigvee_{i=1}^{N} f_i\Big)'(x;d) = \bigvee_{i\in R(x)} T\Phi(Qf_i(x))(d) = T\Phi\Big(\bigvee_{i\in R(x)} Qf_i(x)\Big)(d), \qquad \bigvee_{i\in R(x)} Qf_i(x) \in K_1$$

and

$$\Big(\bigwedge_{i=1}^{N} f_i\Big)'(x;d) = \bigwedge_{i\in J(x)} T\Phi(Qf_i(x))(d) = T\Phi(\bigwedge_{i\in J(x)} Qf_i(x))(d), \qquad \bigwedge_{i\in J(x)} Qf_i(x) \in K_1.$$

Hence $\bigvee_{i=1}^{N} f_i$ and $\bigwedge_{i=1}^{N} f_i$ are star-shaped differentiable at $x \in \mathbf{R}^n$ with (3.2) being true. The proof is completed.

Let us consider the following composite function:

$$F(x) = \psi(f_1(x), f_2(x), \cdots, f_m(x)),$$

where

$$f_i : \mathbf{R}^n \to R \ (i = 1, 2, \cdots, m), \qquad \psi : \mathbf{R}^m \to \mathbf{R}.$$

Lemma 3.3 *Let $f_i : \mathbf{R}^n \to \mathbf{R}$ $(i = 1, 2, \cdots, m)$ be star-shaped differentiable at $x_0 \in \mathbf{R}^n$ and ψ be continuously differentiable at*

$$y_0 = (y_0^{(1)}, y_0^{(2)}, \cdots, y_0^{(m)}) \equiv (f_1(x_0), f_2(x_0), \cdots, f_m(x_0)).$$

Then F is star-shaped differentiable at $x_0 \in \mathbf{R}^n$ and

$$QF(x_0) = \Big(\sum_{i=1}^{m} \oplus\Big)\Big(\frac{\partial\psi(y_0)}{\partial y^{(i)}} \odot Qf_i(x_0)\Big).$$

Proof. For any $d \in \mathbf{R}^n$, we have

$$\begin{aligned} F'(x_0;d) &= \frac{\partial\psi(y_0)}{\partial y^{(1)}} f_1'(x_0;d) + \frac{\partial\psi(y_0)}{\partial y^{(2)}} f_2'(x_0;d) + \cdots + \frac{\partial\psi(y_0)}{\partial y^{(m)}} f_m'(x_0;d) \\ &= \frac{\partial\psi(y_0)}{\partial y^{(1)}} T\Phi(Qf_1(x_0))(d) + \frac{\partial\psi(y_0)}{\partial y^{(2)}} T\Phi(Qf_2(x_0))(d) + \cdots \\ &\quad + \frac{\partial\psi(y_0)}{\partial y^{(m)}} T\Phi(Qf_m(x_0))(d) \\ &= T\Phi\Big(\Big(\sum_{i=1}^{m} \oplus\Big)\Big(\frac{\partial\psi(y_0)}{\partial y^{(i)}} \odot Qf_i(x_0)\Big)\Big)(d), \end{aligned}$$

which implies that F is star-shaped differentiable at x_0 and

$$QF(x_0) = \Big(\sum_{i=1}^{m} \oplus\Big)\Big(\frac{\partial\psi(y_0)}{\partial y^{(i)}} \odot Qf_i(x_0)\Big) \in K_1.$$

Definition 3.2 [12] *Let $f : \mathbf{R}^n \to \mathbf{R}$ be directionally differentiable at $x \in \mathbf{R}^n$. If*

$$\lim_{\lambda \downarrow 0, d' \to d} \frac{f(x + \lambda d') - f(x)}{\lambda} = f'(x; d)$$

for any vector $d \in \mathbf{R}^n$, we say that f is directionally differentiable at $x \in \mathbf{R}^n$ in the Hadamard sense.

It follows from Proposition 2.46 of [12] that if f is directionally differentiable at $x \in \mathbf{R}^n$ in the Hadamard sense, then $f'(x; \cdot)$ is continuous, so f is star-shaped differentiable at x.

Theorem 3.2 *Let $f_i : \mathbf{R}^n \to \mathbf{R}$ $(i = 1, 2, \cdots, m)$ be star-shaped differentiable at $x_0 \in \mathbf{R}^n$ and ψ be directionally differentiable at*

$$y_0 = (y_0^{(1)}, y_0^{(2)}, \cdots, y_0^{(m)}) \equiv (f_1(x_0), f_2(x_0), \cdots, f_m(x_0))$$

in the Hadamard sense. Then F is star-shaped differentiable at x_0.

Proof. By Proposition 2.47 of [12], we have that F is directionally differentiable at x_0 and for any $d \in \mathbf{R}^n$,

$$F'(x_0; d) = \psi'(y_0; (f_1'(x_0; d), f_2'(x_0; d), \cdots, f_m'(x_0; d))). \tag{3.3}$$

Define

$$\phi_l(d) = (\phi_l^1(d), \phi_l^2(d), \cdots, \phi_l^m(d)), \qquad \phi_r(d) = (\phi_r^1(d), \phi_r^2(d), \cdots, \phi_r^m(d)),$$

where

$$\phi_l^i(d) \equiv \phi(\underline{Q} f_i(x_0))(d), \quad \phi_r^i(d) \equiv \phi(\overline{Q} f_i(x_0))(d), \qquad i = 1, 2, \cdots, m.$$

Then

$$\phi_l(d) - \phi_r(d) = (f_1'(x_0; d), f_2'(x_0; d), \cdots, f_m'(x_0; d)). \tag{3.4}$$

Again, since ψ is star-shaped differentiable at y_0, combining (3.3) with (3.4), we obtain

$$F'(x_0; d) = \phi(\underline{Q}\psi(y_0))(\phi_l(d) - \phi_r(d)) - \phi(\overline{Q}\psi(y_0))(\phi_l(d) - \phi_r(d)).$$

Since $\phi(\underline{Q}\psi(y_0))(\phi_l(d) - \phi_r(d))$ and $\phi(\overline{Q}\psi(y_0))(\phi_l(d) - \phi_r(d))$ are nonnegative positively homogeneous and continuous functions in d, by Lemma 2.1, we have

$$\phi(\underline{Q}\psi(y_0))(\phi_l(d) - \phi_r(d)) = \phi(\Omega_1)(d), \qquad \phi(\overline{Q}\psi(y_0))(\phi_l(d) - \phi_r(d)) = \phi(\Omega_2)(d),$$

where

$$\begin{aligned} \Omega_1 &= \{d \in \mathbf{R}^n \mid \phi(\underline{Q}\psi(y_0))(\phi_l(d) - \phi_r(d)) \le 1\} \\ &= \{d \in \mathbf{R}^n \mid \phi_l(d) - \phi_r(d) \in \underline{Q}\psi(y_0)\} \\ &= \{d \in \mathbf{R}^n \mid (f_1'(x_0; d), f_2'(x_0; d), \cdots, f_m'(x_0; d)) \in \underline{Q}\psi(y_0)\} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \Omega_2 &= \{d \in \mathbf{R}^n \mid \phi(\overline{Q}\psi(y_0))(\phi_l(d) - \phi_r(d)) \le 1\} \\ &= \{d \in \mathbf{R}^n \mid \phi_l(d) - \phi_r(d) \in \overline{Q}\psi(y_0)\} \\ &= \{d \in \mathbf{R}^n \mid (f_1'(x_0; d), f_2'(x_0; d), \cdots, f_m'(x_0; d)) \in \overline{Q}\psi(y_0)\} \end{aligned} \tag{3.6}$$

are star-shaped sets. Hence,

$$F'(x_0; d) = \phi(\Omega_1)(d) - \phi(\Omega_2)(d) = T\Phi(\Omega_1, \Omega_2)(d)$$

and F is star-shaped differentiable at x_0 with $QF(x_0) = (\Omega_1, \Omega_2) \in K_1$, where Ω_1, Ω_2 are given by (3.5) and (3.6).

4 An Mean-value Theorem

Consider a univalent function $\varphi : \mathbf{R} \to \mathbf{R}$ and let φ be a star-shaped differentiable function on R. Since a star-shaped set in the univariate space R is a convex set, the mean-value theorem for a univalent quasidifferentiable function, in the sense of Demyanov and Rubinov, applies for a univalent directionally differentiable function.

Lemma 4.1[11] *Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a star-shaped differentiable function on $\mathbf{R}$. Then for any $\xi_1, \xi_2 \in \mathbf{R}$, there exist ζ in the midst of ξ_1, ξ_2, $u \in (\underline{Q}\varphi(\zeta))^\circ$ and $v \in (\overline{Q}\varphi(\zeta))^\circ$, such that*

$$\varphi(\xi_1) - \varphi(\xi_2) = (u - v)(\xi_1 - \xi_2).$$

Theorem 4.1 *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a star-shaped differentiable function on $\mathbf{R}^n$. Then for arbitrary $x, y \in \mathbf{R}^n$, there exist $z \in (x, y)$ and*

$$u \in \{\eta \in R \mid \eta(y - x) \in \underline{Q}f(z)\}^\circ, \qquad v \in \{\eta \in R | \eta(y - x) \in \overline{Q}f(z)\}^\circ$$

such that

$$f(y) - f(x) = u - v.$$

Proof. For arbitrary $x, y \in \mathbf{R}^n$, define the auxiliary function

$$\varphi(t) = f(x + t(y - x)),$$

then for any $\eta \in \mathbf{R}$, by the star-shaped differentiability of f on $\mathbf{R}^n$, we have

$$\begin{aligned}\varphi'(t; \eta) &= \lim_{\lambda \downarrow 0} \frac{\varphi(t + \lambda\eta) - \varphi(t)}{\lambda} \\ &= f'(x + t(y - x); \eta(y - x)) \\ &= T\Phi(Qf(x + t(y - x)))(\eta(y - x)) \\ &= \phi(\underline{Q}f(x + t(y - x)))(\eta(y - x)) - \phi(\overline{Q}f(x + t(y - x)))(\eta(y - x)).\end{aligned}$$

From the definition of a gauge function, one has that $\phi(\underline{Q}f(x + t(y - x)))(\eta(y - x))$ and $\phi(\overline{Q}f(x + t(y - x)))(\eta(y - x))$ are positively homogeneous, nonnegative and continuous in η. By Lemma 2.1, we have

$$\begin{aligned}&\phi(\underline{Q}f(x + t(y - x)))(\eta(y - x)) \\ &= \phi(\Omega_1(t))(\eta), \phi(\overline{Q}f(x + t(y - x)))(\eta(y - x)) \\ &= \phi(\Omega_2(t))(\eta),\end{aligned}$$

where

$$\Omega_1(t) = \{\eta \in R | \eta(y - x) \in \underline{Q}f(x + t(y - x))\} \tag{4.1}$$

and

$$\Omega_2(t) = \{\eta \in R | \eta(y - x) \in \overline{Q}f(x + t(y - x))\} \tag{4.2}$$

are star-shaped sets. Therefore we obtain

$$\begin{aligned}\varphi'(t; \eta) &= \phi(\Omega_1(t))(\eta) - \phi(\Omega_2(t))(\eta) \\ &= T\Phi(\Omega_1(t), \Omega_2(t))(\eta).\end{aligned}$$

Therefore, φ is star-shaped differentiable on R, and

$$Q\varphi(t) = (\Omega_1(t), \Omega_2(t)),$$

where $\Omega_1(t)$ and $\Omega_2(t)$ are given by (4.1) and (4.2). From Lemma 4.1, there exist $\bar{t} \in (0,1)$, $u \in (\underline{Q}\varphi(\bar{t}))^\circ \equiv (\Omega_1(\bar{t}))^\circ$ and $v \in (\overline{Q}\varphi(\bar{t}))^\circ \equiv (\Omega_2(\bar{t}))^\circ$, such that

$$\varphi(1) - \varphi(0) = u - v.$$

From the definition of φ, we have that there exist

$$z \equiv x + \bar{t}(y-x) \in (x,y)$$

and

$$u \in \{\eta \in R \mid \eta(y-x) \in \underline{Q}f(z)\}^\circ, \qquad v \in \{\eta \in R \mid \eta(y-x) \in \overline{Q}f(z)\}^\circ$$

such that

$$f(y) - f(x) = u - v.$$

Remark 4.1 Let $w \in (\underline{Q}f(x+t(y-x)))^\circ$. For arbitrary $\eta \in \Omega_1(t)$, one has $\eta(y-x) \in \underline{Q}f(x+t(y-x))$. From the definition of a polar, it is clear that

$$\eta\langle w, y-x\rangle \le 1, \qquad \forall\, \eta \in \Omega_1(t).$$

Hence,

$$\langle w, y-x\rangle \in (\Omega_1(t))^\circ, \qquad \forall\, w \in (\underline{Q}f(x+t(y-x)))^\circ,$$

i.e.,

$$\{\langle w, y-x\rangle \mid w \in (\underline{Q}f(x+t(y-x)))^\circ\} \subseteq (\Omega_1(t))^\circ.$$

In a similar way we can prove that

$$\{\langle w, y-x\rangle \mid w \in (\overline{Q}f(x+t(y-x)))^\circ\} \subseteq (\Omega_2(t))^\circ.$$

Whether the opposite inclusion holds is still unknown.

If $\bar{t} \in (0,1)$ satisfies

$$\{\langle w, y-x\rangle \mid w \in (\underline{Q}f(x+\bar{t}(y-x)))^\circ\} = (\Omega_1(\bar{t}))^\circ$$

and

$$\{\langle w, y-x\rangle \mid w \in (\overline{Q}f(x+\bar{t}(y-x)))^\circ\} = (\Omega_2(\bar{t}))^\circ,$$

then that conclusion of Theorem 4.1 can be rewritten as: there exist $z \in (x,y)$, $w_1 \in (\underline{Q}f(z))^\circ$ and $w_2 \in (\overline{Q}f(z))^\circ$, such that

$$f(y) - f(x) = \langle w_1 - w_2,\ y-x\rangle,$$

which is similar to the classical mean-value theorem.

References

[1] Rockfellar, R. T., Convex Analysis, Princeton University Press, Princeton, New Jersey, 1970.

[2] Clarke, F. H., Genaralized gradients and applications, *Trans. Amer. Math. Soc.*, **205**(1975), 247–262.

[3] Pschenichnyi, B. N., Necessary Conditions for an Extremum, Marcel Dekker, New York, 1971.

[4] Demyanov, V. F., Polyakova, L. N. and Rubinov, A. M., On one generalization of the concept of subdifferential, Abstracts of Reports in All Union Conference on Dynamic Control, Sverdlovsk, 1979, pp.79–84.

[5] Demyanov, V. F. and Rubinov, A. M., On quasidifferentiable functionals, *Dokl. Akad. Nauk UzSSR*, **250**(1980), 21–25.

[6] Zhang, H. W., Zhang, L. W. and Xia, Z. Q., Calculus of generalized quasi-differentiable functions I: some results on the space of pairs of convex-set collections, *Northeast. Math. J.*, **19**(2003), 75–85.

[7] Zhang, H. W., Zhang, L. W., Xia, Z. Q. and Song, C. L., Optimality conditions for generalized quasidifferentiable optimization problems with inequality constraints, *J. Dalian Univ. Tech.*, **46**(2006), 299–301.

[8] Zhang, H. W., Pan, S. R. and Zhang, L. W., The space of star-shaped sets and its applications, submitted.

[9] Rubinov, A. M. and Yagubov, A. A., The space of star-shaped sets and its applications in nonsmooth optimization, *Math. Program. Study*, **29**(1986), 176–202.

[10] Demyanov, V. F. and Rubinov, A. M., Quasidifferentiablity and Related Topics, Kluwer Acsdemic Publishers, Dordrecht, 2000.

[11] Demyanov, V. F. and Rubinov, A. M., Constructive Nonsmooth Analysis, Peter Lang, Frankfurt am Main, New York, 1995.

[12] Bonnans, J. F. and Shapiro, A., Perturbation Analysis of Optimization Problems, Springer, New York, 2000.