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Weighted Estimates for the Maximal Commutator of Singular Integral Operator on Spaces of Homogeneous Type*

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Abstract: Weighted estimates with general weights are established for the maximal operator associated with the commutator generated by singular integral operator and BMO function on spaces of homogeneous type, where the associated kernel satisfies the Hölder condition on the first variable and some condition which is fairly weaker than the Hölder condition on the second variable.

Key words: spaces of homogeneous type, weighted estimates, singular integral operator, commutator, maximal operator

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1 Introduction

We work on a space of homogeneous type. Let $\mathscr{X}$ be a set endowed with a positive Borel regular measure μ and a symmetric quasi-metric d satisfying that there exists a constant $\kappa \geq 1$ such that for all $x, y, z \in \mathscr{X}$,

$$d(x, y) \leq \kappa[d(x, z) + d(z, y)].$$

The triple $(\mathscr{X}, d, \mu)$ is said to be a space of homogeneous type in the sense of Coifman and Weiss[1], if μ satisfies the following doubling condition: there exists a constant $C \geq 1$ such that for all $x \in \mathscr{X}$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Moreover, if C is the smallest constant for which the measure μ verifies the doubling condition, then $D = \log_2 C$ is called the doubling order of μ and we have

$$\frac{\mu(B_1)}{\mu(B_2)} \leq C_\mu \Big(\frac{r_{B_1}}{r_{B_2}}\Big)^D \qquad \text{for all balls } B_2 \subset B_1 \subset \mathscr{X},$$

where r_{B_i} denotes the radius of B_i, $i = 1, 2$.

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We remark that although all balls defined by d satisfy the axioms of complete system of neighborhood in $\mathscr{X}$, and therefore induce a topology in $\mathscr{X}$, the balls $B(x, r)$ for $x \in \mathscr{X}$ and $r > 0$ need not to be open with respect to this topology. However, by a remarkable result of Macías and Segovia in [2], we know that there exists another quasi-metric $\tilde{d}$ which is equivalent to d such that the balls corresponding to $\tilde{d}$ are open in the topology induced by $\tilde{d}$. Thus, throughout this paper, we assume that the balls $B(x, r)$ for $x \in \mathscr{X}$ and $r > 0$ are open.

Let T be a linear $L^2(\mathscr{X})$-bounded operator with kernel K in the sense that for all $f \in L^2(\mathscr{X})$ with bounded support and almost all $x \notin \operatorname{supp} f$,

$$Tf(x) = \int_{\mathscr{X}} K(x,y)f(y)\mathrm{d}\mu(y), \tag{1.1}$$

where K is a locally integrable function on $\mathscr{X} \times \mathscr{X} \backslash \{(x,y) : x = y\}$. For $b \in \mathrm{BMO}(\mathscr{X})$, define the commutator generated by T and b by

$$T_b f(x) = b(x)Tf(x) - T(bf)(x), \qquad f \in L_0^\infty(\mathscr{X}), \tag{1.2}$$

where and in the following, $L_0^\infty(\mathscr{X})$ denotes the set of bounded functions with bounded support. The maximal operator associated with the commutator T_b is defined by

$$T_b^* f(x) = \sup_{\epsilon>0} |T_{\epsilon;\, b} f(x)|, \tag{1.3}$$

where

$$T_{\epsilon;\ b} = b(x)T_\epsilon f(x) - T_\epsilon(bf)(x), \qquad f \in L_0^\infty(\mathscr{X}),$$

and T_ϵ ($\epsilon > 0$) is the truncated operator defined by

$$T_\epsilon f(x) = \int_{d(x,y)>\epsilon} K(x,\, y)f(y)\mathrm{d}\mu(y).$$

The operator T_b^* has been considered by many authors. When the associated kernel K satisfies the size condition

$$|K(x,y)| \le \frac{C}{\mu(x, d(x,y))}, \qquad x, y \in \mathscr{X},\ x \ne y \tag{1.4}$$

and the Hölder smoothness conditions

$$|K(x,y) - K(x,y')| \le C\frac{(d(y,y'))^\eta}{\mu(B(y, d(x,y)))(d(x,y))^\eta}, \qquad \text{if } d(x,y) \ge 2d(y,y') \tag{1.5}$$

and

$$|K(y,x) - K(y',x)| \le C\frac{(d(y,y'))^\eta}{\mu(B(y, d(x,y)))(d(x,y))^\eta}, \qquad \text{if } d(x,y) \ge 2d(y,y') \tag{1.6}$$

with some $\eta \in (0,1]$, for the operator T_b^*, Hu and Wang[3] proved L^p weighted estimates with general weight, Hu *et al.*[4] established weighted endpoint estimates with general weight. Whether one of the smoothness conditions can be replaced by the weaker one, it is of considerable interest. To state our result, we first give some notations.

Let E be a measurable set with $\mu(E) < \infty$. For any fixed $l > 0$ and a suitable function f, set

$$\|f\|_{L(\lg L)^l,\, E} = \inf\Big\{\lambda > 0 : \frac{1}{\mu(E)}\int_E \frac{|f(x)|}{\lambda} \lg^l\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big)\mathrm{d}\mu(x) \le 1\Big\}.$$

The maximal operator $\mathcal{M}_{L(\lg L)^l}$ is defined by

$$\mathcal{M}_{L(\lg L)^l} f(x) = \sup_{B \ni x} \|f\|_{L(\lg L)^l,\, B},$$

where the supremum is taken over all balls containing x. Our main result can be stated as follows.

Theorem 1.1 *Let T be a linear $L^2(\mathscr{X})$-bounded operator with kernel K in the sense of* (1.1) *and $b \in \mathrm{BMO}(\mathscr{X})$. Suppose that K satisfies* (1.4), (1.5) *and the following condition*

$$\sum_{k=1}^{\infty} k \int_{2^k R<d(x,\, y)\le 2^{k+1}R} |K(y,\, x) - K(y',\, x)|\mathrm{d}\mu(x) \le C \tag{1.7}$$

for any $R > 0$ and $y, y' \in \mathscr{X}$ with $d(y, y') < R$, where C is independent of x, y, y' and R. Then

(1) *for any $p \in (1, \infty)$ and $\delta > 0$, there is a constant $C > 0$ such that for any weight w and any $f \in L_0^\infty(\mathscr{X})$,*

$$\int_{\mathscr{X}} (T_b^* f(x))^p w(x)\mathrm{d}\mu(x) \le C \int_{\mathscr{X}} |f(x)|^p \mathcal{M}_{L(\lg L)^{2p+1+\delta}} w(x)\mathrm{d}\mu(x); \tag{1.8}$$

(2) *for any $\delta > 0$, there is a constant $C > 0$ such that for any $\lambda > 0$, any weight w and any $f \in L_0^\infty(\mathscr{X})$,*

$$w(\{x \in \mathscr{X} : |T_b^* f(x)| > \lambda\}) \le C \int_{\mathscr{X}} \frac{|f(x)|}{\lambda} \lg\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big)\mathcal{M}_{L(\lg L)^{3+\delta}} w(x)\mathrm{d}\mu(x). \tag{1.9}$$

Remark 1.1 The iteration of the Hardy-Littlewood maximal operators on the right-hand sides of (1.8) and (1.9) are bigger than that of Theorem 1 in [3] and that of Theorem 1.3 in [4] respectively. We guess that this is due to the weaker smoothness condition on the second variable of the kernel K here.

Remark 1.2 We do not know if there is a Cotlar inequality linking the operators T_b^* and T_b when K satisfies (1.4), (1.5) and (1.7), so our argument in the proof of Theorem 1.1 is fairly different from which was used in [5]. On the other hand, we do not have the sharp estimate as Lemma 3 in [3], so the theorem of Lerner in [6] cannot be applied directly. To overcome these difficulties, we use the sharp function $\mathcal{M}^\sharp_{0,\, s}$ to estimate T_b^* via the Calderón-Zygmund decomposition.

We now make some conventions. Throughout this paper, we always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Constant with subscript such as C_2, does not change in different occurrences. For a measurable set E and a weight w, χ_E denotes the characteristic function of E, and

$$w(E) = \int_E w(x)\mathrm{d}\mu(x).$$

Given $\lambda > 0$ and a ball B, λB denotes the ball with the same center as B and whose radius is λ times that of B. For a fixed p with $p \in (1, \infty)$, p' denotes the dual exponent of p, namely, $p' = p/(p-1)$. For a locally integrable function f on $\mathscr{X}$ and a bounded measurable set E, $m_E(f)$ denotes the mean value of f over E, that is,

$$m_E(f) = [\mu(E)]^{-1} \int_E f(x)\mathrm{d}\mu(x).$$

For $0 < s < 1$, the operators $\mathcal{M}_{0,\, s}$ and $\mathcal{M}^\sharp_{0,\, s}$ are defined by

$$\mathcal{M}_{0,\, s} f(x) = \sup_{B\ni x} \inf\{t > 0 : \mu(\{y \in B : |f(y)| > t\}) < s\mu(B)\}$$

and

$$\mathcal{M}^{\sharp}_{0,\,s}f(x) = \sup_{B\ni x}\inf_{c\in\mathbb{C}}\inf\{t>0 : \mu(\{y\in B : |f(y)-c|>t\}) < s\mu(B)\}$$

for any locally integrable function f and $x\in\mathscr{X}$. Let $\mathcal{M}$ be the Hardy-Littlewood maximal operator and

$$\mathcal{M}_{\delta}f(x) = (\mathcal{M}(|f|^{\delta})(x))^{\frac{1}{\delta}}, \qquad \delta>0.$$

For a locally integrable function f, define the Fefferman-Stein sharp maximal function $\mathcal{M}^{\sharp}f$ as

$$\mathcal{M}^{\sharp}f(x) = \sup_{B\ni x}\frac{1}{\mu(B)}\int_B |f(y)-m_B(f)|\mathrm{d}\mu(y),$$

where the supremum is taken over all balls B containing x. For fixed $q\in(0,1)$, let the sharp maximal function be

$$\mathcal{M}^{\sharp}_{q}f(x) = (\mathcal{M}^{\sharp}(|f|^{q})(x))^{\frac{1}{q}}.$$

The commutator of the Hardy-Littlewood maximal operator is defined by

$$\mathcal{M}_b f(x) = \sup_{B\ni x}\frac{1}{\mu(B)}\int_B |b(x)-b(y)||f(y)|\mathrm{d}\mu(y),$$

where $b\in\mathrm{BMO}(\mathscr{X})$.

The following inequalities will be used in the proof of Theorem 1.1. Let

$$\mathcal{M} = \mathcal{M}_{L(\log L)^0}.$$

For $\alpha,\beta\in[0,\infty)$ and any weight w, we have

$$\mathcal{M}_{L(\log L)^{\alpha}}(\mathcal{M}_{L(\log L)^{\beta}}w)(x) \le C\mathcal{M}_{L(\log L)^{\alpha+\beta+1}}w(x) \tag{1.10}$$

(see [3]). For any suitable function f, set

$$\|f\|_{\exp\{L\},\,E} = \inf\Big\{\lambda>0 : \frac{1}{\mu(E)}\int_E \exp\Big\{\frac{|f(x)|}{\lambda}\Big\}\mathrm{d}\mu(x)\le 2\Big\}.$$

Then the following generalization of Hölder's inequality

$$\frac{1}{\mu(E)}\int_E |f(x)g(x)|\mathrm{d}\mu(x) \le C\|f\|_{L\lg L,\,E}\|g\|_{\exp\{L\},\,E}$$

holds for any suitable functions f and g; see [7] for details.

2 Some Lemmas

Lemma 2.1 *Suppose that $\mu(\mathscr{X})<\infty$, $0<\delta<1$, and S is an operator which satisfies the weak type estimate*

$$\mu(\{x\in\mathscr{X} : |Sf(x)|>\lambda\}) \le C_0\int_{\mathscr{X}}\frac{f(x)}{\lambda}\lg\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)\mathrm{d}\mu(x), \tag{2.1}$$

where C_0 is independent of f and λ. Then there exists a positive constant C such that for any $x\in\mathscr{X}$,

$$\Big(\frac{1}{\mu(\mathscr{X})}\int_{\mathscr{X}}|Sf(y)|^{\delta}\mathrm{d}\mu(y)\Big)^{\frac{1}{\delta}} \le C\mathcal{M}_{L\lg L}f(x). \tag{2.2}$$

Proof. Repeating the proof of the Kolmogorov inequality, we see that there exists a positive constant C such that for any $f \in L_0^\infty(\mathscr{X})$,

$$\left(\frac{1}{\mu(B)}\int_B |Sf(y)|^\delta \mathrm{d}\mu(y)\right)^{\frac{1}{\delta}} \le C\|f\|_{L\lg L,\,B},$$

where $\operatorname{supp} f \subset B$. It is easily known that for any $x \in \mathscr{X}$, there exists a ball B satisfying that $\operatorname{supp} f \subset B$ and $x \in B$, such that for any nonnegative integer k,

$$\left(\frac{1}{\mu(2^kB)}\int_{2^kB} |Sf(y)|^\delta \mathrm{d}\mu(y)\right)^{\frac{1}{\delta}} \le C\|f\|_{L\lg L,\,2^kB} \le C\mathcal{M}_{L\lg L}f(x)$$

with C independent of x, f and B. Taking the limit $k \to \infty$, we get (2.2). This completes the proof.

Lemma 2.2 *For any s with $0 < s < 1$, there is a positive constant C such that for any weight w and any nonnegative function f, which satisfies that $\mu(\{x \in \mathscr{X} : f(x) > \lambda\}) < \infty$ for any $\lambda > 0$,*

(1) *if $\mu(\mathscr{X}) = \infty$, then*

$$\int_{\mathscr{X}} f(x)w(x)\mathrm{d}\mu(x) \le C\int_{\mathscr{X}} \mathcal{M}^\sharp_{0,s}f(x)\mathcal{M}w(x)\mathrm{d}\mu(x);$$

(2) *if $\mu(\mathscr{X}) < \infty$, then*

$$\int_{\mathscr{X}} f(x)w(x)\mathrm{d}\mu(x) \le C\int_{\mathscr{X}} \mathcal{M}^\sharp_{0,\,s}f(x)\mathcal{M}w(x)\mathrm{d}\mu(x) + Cw(\mathscr{X})m_{\mathscr{X}}(f).$$

The proof of Lemma 2.2 is similar to the proof of Theorem 2.5 in [5], and is omitted. For the setting of Euclidean space, this lemma was proved by Lerner in [6].

Lemma 2.3 *Under the hypothesis of Theorem* 1.1, *for any s with $0 < s < 1$, there exists a constant $C > 0$ such that for any $f \in L_0^\infty(\mathscr{X})$,*

$$\mathcal{M}^\sharp_{0,\,s}(T^*_bf)(x) \le C(\|b\|_{\mathrm{BMO}(\mathscr{X})}\mathcal{M}_s(T^*f)(x) + \|b\|_{\mathrm{BMO}(\mathscr{X})}\|f\|_{L^\infty(\mathscr{X})}).$$

Proof. Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}(\mathscr{X})} = 1$. By a trivial computation, we see that for any $s, q \in (0,1)$ with $q < s$ and any locally integrable function f,

$$\mathcal{M}^\sharp_{0,\,s}f(x) \le s^{-1/q}\mathcal{M}^\sharp_q f(x)$$

and

$$\mathcal{M}^\sharp_q(T^*_bf)(x) \le \sup_{B\ni x}\inf_{c\in\mathbb{C}}\left(\frac{1}{\mu(B)}\int_B |T^*_bf(y) - c|^q\mathrm{d}\mu(y)\right)^{\frac{1}{q}}.$$

Let $f \in L_0^\infty(\mathscr{X})$. Our goal is now to prove that

$$\sup_{B\ni x}\inf_{c\in\mathbb{C}}\left(\frac{1}{\mu(B)}\int_B |T^*_bf(y) - c|^q\mathrm{d}\mu(y)\right)^{\frac{1}{q}} \le C\mathcal{M}_s(T^*f)(x) + C\|f\|_{L^\infty(\mathscr{X})} \tag{2.3}$$

with C independent of f, x, and B. For each fixed $x \in \mathscr{X}$ and a ball B containing x, decompose f as

$$f(y) = f_1(y) + f_2(y) = f(y)\chi_{C_1B}(y) + f(y)\chi_{\mathscr{X}\setminus C_1B}(y), \qquad C_1 = \kappa(4\kappa^2+1).$$

Notice that $(m_B(b) - b)f_2 \in L^2(\mathscr{X})$. The $L^p(\mathscr{X})$ $(1 < p < \infty)$ boundedness of T^* (see [5]) states that

$$\sup_{\epsilon>0}|T_\epsilon((m_B(b) - b)f_2)(y)| < \infty, \qquad \text{a.e. } y \in \mathscr{X}.$$

Choose $y_0 \in B$ such that

$$\sup_{\epsilon>0} |T_\epsilon((m_B(b)-b)f_2)(y_0)| < \infty$$

and set

$$C_B = T^*((m_B(b)-b)f_2)(y_0).$$

Write

$$T_{\epsilon;\,b}f(y) = T_\epsilon((m_B(b)-b)f_1)(y) + T_\epsilon((m_B(b)-b)f_2)(y) - (m_B(b)-b(y))T_\epsilon f(y).$$

Then for any $y \in B$,

$$\begin{aligned}
|T_b^* f(y) - C_B| &= \Big|\sup_{\epsilon>0}|T_{\epsilon;\,b}f(y)| - \sup_{\epsilon>0}|T_\epsilon((m_B(b)-b)f_2)(y_0)|\Big| \\
&\le \sup_{\epsilon>0}\Big|T_{\epsilon;\,b}f(y) - T_\epsilon((m_B(b)-b)f_2)(y_0)\Big| \\
&\le T^*((m_B(b)-b)f_1\big)(y) + |m_B(b)-b(y)|T^*f(y) \\
&\quad + \sup_{\epsilon>0}\Big|T_\epsilon((m_B(b)-b)f_2)(y) - T_\epsilon((m_B(b)-b)f_2)(y_0)\Big| \\
&= \mathrm{I}(y) + \mathrm{II}(y) + \mathrm{III}(y).
\end{aligned}$$

The Kolmogorov inequality, via the fact that T^* is bounded from $L^1(\mathscr{X})$ to $L^{1,\infty}(\mathscr{X})$ (see [5]), tells us that

$$\begin{aligned}
\left(\frac{1}{\mu(B)}\int_B |I(y)|^q \mathrm{d}\mu(y)\right)^{\frac{1}{q}} &= \left(\frac{1}{\mu(B)}\int_B |T^*((m_B(b)-b)f_1\big)(y)|^q \mathrm{d}\mu(y)\right)^{\frac{1}{q}} \\
&\le \frac{C}{\mu(C_1B)}\int_{C_1B} |m_B(b)-b(y)||f(y)|\mathrm{d}\mu(y) \\
&\le C\|f\|_{L^\infty(\mathscr{X})}.
\end{aligned}$$

On the other hand, a straightforward computation involving the Hölder inequality leads to that

$$\begin{aligned}
\left(\frac{1}{\mu(B)}\int_B |\mathrm{II}(y)|^q \mathrm{d}\mu(y)\right)^{\frac{1}{q}} &\le C\left(\frac{1}{\mu(B)}\int_B |m_B(b)-b(y)|^q |T^*f(y)|^q \mathrm{d}\mu(y)\right)^{\frac{1}{q}} \\
&\le C\left(\frac{1}{\mu(B)}\int_B |T^*f(y)|^s \mathrm{d}\mu(y)\right)^{\frac{1}{s}} \\
&\quad \times \left(\frac{1}{\mu(B)}\int_B |m_B(b)-b(y)|^{\frac{qs}{s-q}} \mathrm{d}\mu(y)\right)^{\frac{s-q}{qs}} \\
&\le C\mathcal{M}_s(T^*f)(x).
\end{aligned}$$

It remains to estimate $\mathrm{III}(y)$. For each fixed point $y \in B$, we can write

$$\begin{aligned}
\mathrm{III}(y) &= \sup_{\epsilon>0}\Big|T_\epsilon((m_B(b)-b)f_2)(y) - T_\epsilon((m_B(b)-b)f_2)(y_0)\Big| \\
&\le \int_{\mathscr{X}\setminus C_1B} |K(y,z)-K(y_0,z)||m_B(b)-b(z)||f(z)|\mathrm{d}\mu(z) \\
&\quad + 2\sup_{\epsilon>0}\int_{d(y,\,z)\le\epsilon,\,d(y_0,\,z)>\epsilon} |K(y_0,z)||m_B(b)-b(z)||f_2(z)|\mathrm{d}\mu(z) \\
&= D_1 + D_2.
\end{aligned}$$

The smoothness condition (1.7) states that

$$\begin{aligned}
D_1 &\le \int_{\mathscr{X}\setminus C_1B} |K(y,z)-K(y_0,z)||b(z)-m_{C_1B}(b)||f(z)|d\mu(z)\\
&\le \|f\|_{L^\infty(\mathscr{X})}\sum_{k=1}^{\infty}|m_{C_1B}(b)-m_{2^kC_1B}(b)|\int_{2^kC_1B\setminus 2^{k-1}C_1B}|K(y,z)-K(y_0,z)|\mathrm{d}\mu(z)\\
&\quad+\|f\|_{L^\infty(\mathscr{X})}\sum_{k=1}^{\infty}\int_{2^kC_1B\setminus 2^{k-1}C_1B}|K(y,z)-K(y_0,z)||b(z)-m_{2^kC_1B}(b)|\mathrm{d}\mu(z)\\
&\le C\|f\|_{L^\infty(\mathscr{X})}\sum_{k=1}^{\infty}\int_{2^kC_1B\setminus 2^{k-1}C_1B}|K(y,z)-K(y_0,z)||b(z)-m_{2^kC_1B}(b)|\mathrm{d}\mu(z)\\
&\quad+C\|f\|_{L^\infty(\mathscr{X})}.
\end{aligned}$$

Recall that there is a positive constant C such that for any $t_1, t_2 > 0$,

$$t_1t_2 \le C(t_1\lg(2+t_1)+\exp\{t_2\})$$

(see [4]). Thus by the John-Nirenberg inequality

$$\|b-m_B(b)\|_{\exp\{L\},B} \le C\|b\|_{\mathrm{BMO}(\mathscr{X})},$$

we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\int_{2^kC_1B\setminus 2^{k-1}C_1B}|K(y,z)-K(y_0,z)||b(z)-m_{2^kC_1B}(b)|\mathrm{d}\mu(z)\\
\le& C\sum_{k=1}^{\infty}\frac{1}{2^k\mu(2^{k-1}C_1B)}\int_{2^kC_1B}\exp\{|b(z)-m_{2^kC_1B}(b)|\}\mathrm{d}\mu(z)\\
&+C\sum_{k=1}^{\infty}\int_{2^kC_1B\setminus 2^{k-1}C_1B}|K(y,z)-K(y_0,z)|\\
&\qquad\times\lg(2+2^k\mu(2^{k-1}C_1B)|K(y,z)-K(y_0,z)|)\mathrm{d}\mu(z)\\
\le& C,
\end{aligned}$$

and so

$$D_1 \le C\|f\|_{L^\infty(\mathscr{X})}.$$

As for D_2, it is easy to verify that

$$\begin{aligned}
D_2 &\le C\sup_{\epsilon>0}\int_{\epsilon<d(y_0,\,z)\le C\epsilon}|K(y_0,z)f(z)||m_B(b)-b(z)|\mathrm{d}\mu(z)\\
&\le C\sup_{\epsilon>0}\frac{1}{\mu(B(y_0,\,\epsilon))}\int_{d(y_0,\,z)\le C\epsilon}|m_B(b)-b(z)||f(z)|\mathrm{d}\mu(z)\\
&\le C\|f\|_{L^\infty(\mathscr{X})}.
\end{aligned}$$

Combining the estimates for the terms I, II and III yields the inequality (2.3), and then completes the proof of Lemma 2.3.

3 Proof of Theorem 1.1

We first establish a weighted inequality for the composite operator $\mathcal{M}^{\sharp}_{0,\,s}T^*_b$.

Theorem 3.1 *Let T be a linear $L^2(\mathscr{X})$-bounded operator with kernel K in the sense of* (1.1) *and $b \in \mathrm{BMO}(\mathscr{X})$. Suppose that K satisfies the conditions* (1.4), (1.5) *and* (1.7). *Then for any $\delta > 0$, $s \in (0,1)$ and $p \in (1,\infty)$, there is a constant $C > 0$ such that for any weight w and $f \in L_0^\infty(\mathscr{X})$,*

$$\int_{\mathscr{X}} (\mathcal{M}^\sharp_{0,\,s}(T_b^* f)(x))^p w(x)\mathrm{d}\mu(x) \leq C \int_{\mathscr{X}} |f(x)|^p \mathcal{M}_{L(\lg L)^{p+1+\delta}} w(x)\mathrm{d}\mu(x). \tag{3.1}$$

Proof. Notice that for any $p \in (1,\infty)$, δ is an arbitrary positive number. By the Marcinkiewicz interpolation theorem, we see that the proof of Theorem 3.1 can be reduced to prove that for any $\delta > 0$, $s \in (0,1)$ and $p \in (1,\infty)$, there is a constant $C > 0$ such that for any weight w and any $f \in L_0^\infty(\mathscr{X})$,

$$w(\{x \in \mathscr{X} : \mathcal{M}^\sharp_{0,\,s}(T_b^* f)(x) > C\lambda\}) \leq C\lambda^{-p} \int_{\mathscr{X}} |f(x)|^p \mathcal{M}_{L(\lg L)^{p+1+\delta}} w(x)\mathrm{d}\mu(x). \tag{3.2}$$

If $\mu(\mathscr{X}) < \infty$ and $\lambda^p \leq \|f\|^p_{L^p(\mathscr{X})}[\mu(\mathscr{X})]^{-1}$, the inequality (3.2) is trivial. So it remains to consider the case that $\lambda^p > \|f\|^p_{L^p(\mathscr{X})}[\mu(\mathscr{X})]^{-1}$. For each fixed $f \in L_0^\infty(\mathscr{X})$ and $\lambda^p > \|f\|^p_{L^p(\mathscr{X})}[\mu(\mathscr{X})]^{-1}$, applying the Calderón-Zygmund decomposition (see [8]) to $|f|^p$ at level λ^p, we can obtain a sequence of pairwise disjoint balls $\{B_j\}_{j=1}^\infty$ and a constant $C_2 \geq 1$ such that

$$m_{C_2 B_j}(|f|^p) \leq \lambda^p < m_{B_j}(|f|^p) \tag{3.3}$$

and

$$m_B(|f|^p) \leq \lambda^p$$

for every ball B centered at $x \in \mathscr{X} \setminus \cup_j C_2 B_j$. As in the proof of Lemma 2.10 in [8], set

$$V_1 = C_2 B_1 - \bigcup_{n=2}^\infty B_n, \qquad V_j = C_2 B_j - \left[\bigcup_{n=1}^{j-1} V_n \cup \bigcup_{n=j+1}^\infty B_n\right].$$

Then it follows that

$$B_j \subset V_j \subset C_2 B_j \quad \text{and} \quad \bigcup_j V_j = \bigcup_j C_2 B_j.$$

Decompose f as

$$f(x) = g(x) + h(x) = g(x) + \sum_j h_j(x),$$

where

$$g(x) = f(x)\chi_{\mathscr{X}\setminus \bigcup_j V_j}(x) + \sum_j m_{V_j}(f)\chi_{V_j}(x)$$

and

$$h_j(x) = (f(x) - m_{V_j}(f))\chi_{V_j}(x).$$

Although $\mathcal{M}^\sharp_{0,\,s}$ is not sublinear, we can prove that for any locally integrable functions f_1 and f_2,

$$\mathcal{M}^\sharp_{0,\,s}(f_1 + f_2)(x) \leq \mathcal{M}^\sharp_{0,\,s/2} f_1(x) + \mathcal{M}_{0,\,s/2} f_2(x).$$

So we have that

$$\begin{aligned} &w(\{x \in \mathscr{X} : \mathcal{M}^\sharp_{0,\,s}(T_b^* f)(x) > C\lambda\}) \\ &\leq w(\{x \in \mathscr{X} : \mathcal{M}^\sharp_{0,\,s/2}(T_b^* g)(x) > C\lambda\}) + w(\{x \in \mathscr{X} : \mathcal{M}_{0,\,s/2}(T_b^* h)(x) > C\lambda\}). \end{aligned} \tag{3.4}$$

Now we deal with the first term on the right of (3.4). Let

$$C_3 = \kappa(4\kappa^2+1)C_2, \qquad \Omega = \bigcup_j C_3 B_j$$

and

$$w^* = w\chi_{\mathscr{X}\backslash\Omega}.$$

Following an argument similar to the case of Euclidean spaces (see [9], p.159), we can verify that there exists a constant $C > 0$ depending only on the space $\mathscr{X}$ such that for any $x \in C_2 B_j$,

$$\mathcal{M}w^*(x) \le C \inf_{y\in C_2 B_j} \mathcal{M}w^*(y).$$

Lemma 2.3 and the fact

$$T^*g(x) \le C(\mathcal{M}(Tg)(x) + \|g\|_{L^\infty(\mathscr{X})})$$

(see [10]) tell us that

$$\begin{aligned}
&w(\{x\in\mathscr{X}\backslash\Omega : \mathcal{M}^{\sharp}_{0,\,s/2}(T^*_b g)(x) > C\lambda\})\\
&\le Cw(\{x\in\mathscr{X}\backslash\Omega : \mathcal{M}(T^*g)(x) + \|g\|_{L^\infty(\mathscr{X})} > C\lambda\})\\
&\le Cw(\{x\in\mathscr{X}\backslash\Omega : \mathcal{M}_{L\lg L}(Tg)(x) > C\lambda\})\\
&\le C\lambda^{-p}\int_{\mathscr{X}} |Tg(x)|^p \mathcal{M}w^*(x)\mathrm{d}\mu(x),\\
&\le C\lambda^{-p}\int_{\mathscr{X}} |g(x)|^p \mathcal{M}_{L(\lg L)^{p-1+\delta}}(\mathcal{M}w^*)(x)\mathrm{d}\mu(x).\\
&\le C\lambda^{-p}\int_{\mathscr{X}} |f(x)|^p \mathcal{M}_{L(\lg L)^{p+\delta}}w^*(x)\mathrm{d}\mu(x).\\
&\quad + C\lambda^{-p}\sum_j\int_{V_j} |m_{V_j}(|f|)|^p \mathcal{M}_{L(\lg L)^{p+\delta}}w^*(x)\mathrm{d}\mu(x),
\end{aligned}$$

where in the third to the last inequality we have invoked (1.10) and the inequalities

$$\int_{\mathscr{X}} (\mathcal{M}_{L\lg L}g'(x))^p u(x)\mathrm{d}\mu(x) \le C\int_{\mathscr{X}} |g'(x)|^p \mathcal{M}u(x)\mathrm{d}\mu(x)$$

(this conclusion is an easy consequence of Theorem 1.4 in [11]) and

$$\int_{\mathscr{X}} (Tg'(x))^p u(x)\mathrm{d}\mu(x) \le C\int_{\mathscr{X}} |g'(x)|^p \mathcal{M}_{L(\lg L)^{p-1+\delta}}u(x)\mathrm{d}\mu(x)$$

(this inequality can be obtained by the same lines to the proof of Lemma 4.3 in [5] with obvious changes) for any weight u and any nonnegative functions g'. Moreover, an application of Hölder inequality via (3.3) gives that

$$\begin{aligned}
&\int_{V_j} |m_{V_j}(|f|)|^p \mathcal{M}_{L(\log L)^{p+\delta}}w^*(x)\mathrm{d}\mu(x)\\
&\le C\lambda^p\mu(V_j)\inf_{y\in C_2B_j}\mathcal{M}_{L(\lg L)^{p+\delta}}w^*(y)\\
&\le C\lambda^p\mu(B_j)\inf_{y\in C_2B_j}\mathcal{M}_{L(\lg L)^{p+\delta}}w^*(y)\\
&\le C\int_{B_j} |f(x)|^p \mathcal{M}_{L(\lg L)^{p+\delta}}w(x)\mathrm{d}\mu(x).
\end{aligned}$$

On the other hand, a simple computation states that

$$\begin{aligned}w(\Omega) &\le C\sum_j \frac{w(C_3B_j)}{\mu(C_3B_j)}\mu(B_j)\\ &\le C\sum_j \inf_{y\in B_j}\mathcal{M}w(y)\lambda^{-p}\int_{B_j}|f(x)|^p\mathrm{d}\mu(x)\\ &\le C\lambda^{-p}\int_{\mathscr{X}}|f(x)|^p\mathcal{M}w(x)\mathrm{d}\mu(x).\end{aligned}$$

So we obtain that

$$w(\{x\in\mathscr{X}:\mathcal{M}^{\sharp}_{0,\,s/2}(T_b^*g)(x)>C\lambda\})\le C\lambda^{-p}\int_{\mathscr{X}}|f(x)|^p\mathcal{M}_{L(\lg L)^{p+\delta}}w(x)\mathrm{d}\mu(x).$$

Then we deal with the second term on the right of (3.4). Noting that for any $s\in(0,1)$,

$$\{x\in\mathscr{X}:\mathcal{M}_{0,\,s/2}(T_b^*h)(x)>C\lambda\}\subset\{x\in\mathscr{X}:\mathcal{M}(\chi_{\{y\in\mathscr{X}:T_b^*h(x)>C\lambda\}})(x)\ge s/2\},$$

we can obtain that

$$\begin{aligned}&w(\{x\in\mathscr{X}:\mathcal{M}_{0,\,s/2}(T_b^*h)(x)>C\lambda\})\\ \le\ & w(\{x\in\mathscr{X}:\mathcal{M}(\chi_{\{y\in\mathscr{X}:T_b^*h(x)>C\lambda\}})(x)\ge s/2\})\\ \le\ & C2^{1/s}s^{-1/s}\sup_{\tau>C2^{1/s}s^{-1/s}}\tau\mathcal{M}w(\{x\in\mathscr{X}:\chi_{\{y\in\mathscr{X}:T_b^*h(x)>C\lambda\}}(x)\ge\tau\})\\ \le\ & C2^{1/s}s^{-1/s}\int_{\mathscr{X}}\chi_{\{y\in\mathscr{X}:T_b^*h(x)>C\lambda\}}\mathcal{M}w(x)\mathrm{d}\mu(x)\\ \le\ & C2^{1/s}s^{-1/s}\mathcal{M}w(\{x\in\mathscr{X}\backslash\Omega:|T_b^*h(x)|>C\lambda\})+C2^{1/s}s^{-1/s}\mathcal{M}w(\Omega),\end{aligned}$$

where in the second inequality we have invoked the fact that for $\gamma>0$ and any weight w,

$$w(\{x\in\mathscr{X}:\mathcal{M}_sf(x)\ge\gamma\})\le C\gamma^{-1}\sup_{\tau>C\gamma}\tau\mathcal{M}w(\{x\in\mathscr{X}:|f(x)|\ge\tau\})$$

(this inequality follows from a similar argument as in the case of Euclidean spaces; see [12], P.651). It suffices to prove that

$$w(\{x\in\mathscr{X}\backslash\Omega:|T_b^*h(x)|>C\lambda\})\le C\lambda^{-p}\int_{\mathscr{X}}|f(x)|^p\mathcal{M}_{L(\lg L)^{p+\delta}}w(x)\mathrm{d}\mu(x),\tag{3.5}$$

which implies that

$$w(\{x\in\mathscr{X}:\mathcal{M}_{0,\,s/2}(T_b^*h)(x)>C\lambda\})\le C\lambda^{-p}\int_{\mathscr{X}}|f(x)|^p\mathcal{M}_{L(\lg L)^{p+1+\delta}}w(x)\mathrm{d}\mu(x).$$

For each fixed $x\in\mathscr{X}\backslash\Omega$, we define

$$\begin{aligned}J_1(x,\epsilon)&=\{j:\text{for all }y\in C_2B_j,\ d(x,\ y)\le\epsilon\},\\ J_2(x,\epsilon)&=\{j:\text{for all }y\in C_2B_j,\ d(x,\ y)>\epsilon\},\\ J_3(x,\epsilon)&=\{j:C_2B_j\cap\{y\in\mathscr{X}:d(x,\ y)>\epsilon\}\neq\emptyset\\ &\qquad\text{and }C_2B_j\cap\{y\in\mathscr{X}:d(x,\ y)\le\epsilon\}\neq\emptyset\}.\end{aligned}$$

It then follows that

$$\begin{aligned}|T_{\epsilon;\,b}h(x)| &\le \Big|T_{\epsilon;b}\Big(\sum_{j\in J_2(x,\epsilon)} h_j\Big)(x)\Big| + \Big|T_{\epsilon;b}\Big(\sum_{j\in J_3(x,\epsilon)} h_j\Big)(x)\Big| \\ &\le \Big|\sum_{j\in J_2(x,\epsilon)} (b(x)-m_{B_j}(b))T_\epsilon h_j(x)\Big| + \Big|T_\epsilon\Big(\sum_j (b-m_{B_j}(b))h_j\Big)(x)\Big| \\ &\quad + \Big|T_\epsilon\Big(\sum_{j\in J_3(x,\epsilon)} (b-m_{B_j}(b))h_j\Big)(x)\Big| + \Big|T_{\epsilon;b}\Big(\sum_{j\in J_3(x,\epsilon)} h_j\Big)(x)\Big| \\ &= U_\epsilon(x)+V_\epsilon(x)+X_\epsilon(x)+Y_\epsilon(x).\end{aligned}$$

For each fixed j, let y_j and r_j be the center and radius of B_j. Noticing that for $x\in\mathscr{X}\backslash\Omega$ and $j\in J_2(x,\epsilon)$, we have

$$T_\epsilon h_j(x) = Th_j(x).$$

By the vanishing moment of h_j we have

$$\sup_{\epsilon>0} U_\epsilon(x) \le \sum_j |b(x)-m_{B_j}(b)| \int_{\mathscr{X}} \frac{(d(y,y_j))^\eta}{\mu(B(y,d(x,y)))(d(x,y))^\eta}|h_j(y)|\mathrm{d}\mu(y),$$

and so

$$\begin{aligned}&w(\{x\in\mathscr{X}\backslash\Omega : \sup_{\epsilon>0} U_\epsilon(x) > C\lambda\}) \\ &\le C\lambda^{-1}\sum_j \int_{\mathscr{X}} |h_j(x)|(d(y,y_j))^\eta \int_{\mathscr{X}\backslash C_3B_j} \frac{|b(x)-m_{B_j}(b)|w(x)}{\mu(B(y,d(x,y)))(d(x,y))^\eta}\mathrm{d}\mu(x)\mathrm{d}\mu(y) \\ &\le C\lambda^{-1}\sum_j \int_{\mathscr{X}} |h_j(x)|(d(y,y_j))^\eta \\ &\quad\times \sum_{k=1}^\infty \int_{2^kC_3B_j\backslash 2^{k-1}C_3B_j} \frac{|b(x)-m_{B_j}(b)|w(x)}{\mu(B(y,d(x,y)))(d(x,y))^\eta}\mathrm{d}\mu(x)\mathrm{d}\mu(y) \\ &\le C\lambda^{-1}\sum_j \int_{\mathscr{X}} |h_j(x)|\mathrm{d}\mu(y) \inf_{y\in C_3B_j} \mathcal{M}_{L\lg L}w(y) \\ &\le C\lambda^{-p}\int_{\mathscr{X}} |f(x)|^p \mathcal{M}_{L\lg L}w(x)\mathrm{d}\mu(x),\end{aligned}$$

where in the third inequality, we have used the fact that for each fixed j, $y\in V_j$ and any positive integer k, a standard argument involving the Hölder inequality and John-Nirenberg inequality yields

$$\int_{2^kC_3B_j\backslash 2^{k-1}C_3B_j} \frac{|b(x)-m_{B_j}(b)|w(x)}{\mu(B(y,d(x,y)))(d(x,y))^\eta}\mathrm{d}\mu(x) \le Ck(2^kr_j)^{-\eta} \inf_{z\in C_3B_j} \mathcal{M}_{L\lg L}w(z).$$

On the other hand, because T^* is bounded from $L^{(p+1)/2}(\mathscr{X}, \mathcal{M}_{L(\lg L)^{(p+1)/2+\delta}}w)$ to $L^{(p+1)/2}(\mathscr{X}, w)$ (this conclusion can be obtained from [5] with some obvious change in the proof there), we have

$$\begin{aligned}&w(\{x\in\mathscr{X}\backslash\Omega : \sup_{\epsilon>0} V_\epsilon(x) > C\lambda\}) \\ &\le C\lambda^{-\frac{p+1}{2}}\sum_j \int_{\mathscr{X}} |b(x)-m_{B_j}(b)|^{\frac{p+1}{2}}|h_j(x)|^{\frac{p+1}{2}}\mathrm{d}\mu(x) \inf_{y\in C_3B_j} \mathcal{M}_{L(\lg L)^{(p+1)/2+\delta}}w(y)\end{aligned}$$

$$\le C\lambda^{-\frac{p+1}{2}}\sum_j \mu(V_j)^{\frac{p-1}{2p}}\Big(\int_{V_j}|f(x)|^p\mathrm{d}\mu(x)\Big)^{\frac{p+1}{2p}}\inf_{y\in C_3B_j}\mathcal{M}_{L(\lg L)^{(p+1)/2+\delta}}w(y)$$

$$\le C\lambda^{-p}\int_{\mathscr{X}}|f(x)|^p\mathcal{M}_{L(\lg L)^{(p+1)/2+\delta}}w(x)\mathrm{d}\mu(x).$$

Notice that for $x\in\mathscr{X}\backslash\Omega$ and $j\in J_3(x,\epsilon)$, we have $C_2B_j\subset B(x,C_4\epsilon)\backslash B(x,C_5\epsilon)$, where C_4 and C_5 are two positive constants satisfying $C_4>C_5$. Therefore, for all $x\in\mathscr{X}\backslash\Omega$,

$$\sup_{\epsilon>0}(X_\epsilon(x)+Y_\epsilon(x))\le C\mathcal{M}\Big(\sum_j|b-m_{B_j}(b)||h_j|\Big)(x)+C\mathcal{M}_b\Big(\sum_j|h_j|\Big)(x),$$

which leads to

$$w(\{x\in\mathscr{X}\backslash\Omega:\sup_{\epsilon>0}(X_\epsilon(x)+Y_\epsilon(x))>C\lambda\})$$

$$\le C\lambda^{-p}\sum_j\int_{V_j}|h_j(x)|^p\mathrm{d}\mu(x)\inf_{y\in C_3B_j}\mathcal{M}_{L(\lg L)^{p+\delta}}w(y)$$

$$+C\lambda^{-\frac{p+1}{2}}\sum_j\int_{V_j}|b(x)-m_{B_j}(b)|^{\frac{p+1}{2}}|h_j(x)|^{\frac{p+1}{2}}\mathrm{d}\mu(x)\inf_{y\in C_3B_j}\mathcal{M}w(y)$$

$$\le C\lambda^{-p}\int_{\mathscr{X}}|f(x)|^p\mathcal{M}_{L(\lg L)^{p+\delta}}w(x)\mathrm{d}\mu(x),$$

where in the first inequality we have used the boundedness of $\mathcal{M}$ (see [11]) and invoked a consequence from [4] that for any weight u and $g'\in L_0^\infty(\mathscr{X})$,

$$\int_{\mathscr{X}}(\mathcal{M}_bg'(x))^pu(x)\mathrm{d}\mu(x)\le C\int_{\mathscr{X}}|g'(x)|^p\mathcal{M}_{L(\lg L)^{p+\delta}}u(x)\mathrm{d}\mu(x).$$

Combining the estimates for the terms $\sup_{\epsilon>0}U_\epsilon$, $\sup_{\epsilon>0}V_\epsilon$ and $\sup_{\epsilon>0}(X_\epsilon+Y_\epsilon)$ yields (3.5), and then completes the proof of Theorem 3.1.

Proof of Theorem 1.1 We may assume that $\mathcal{M}_{L(\lg L)^{2p+1+\delta}}w(x)$ is finite almost everywhere; otherwise there is nothing to prove. Fixed s with $0<s<1/2$, Theorem 3.1 tells us that $\mathcal{M}^\sharp_{0,\,s}T^*_b$ is bounded from $L^p(\mathscr{X},\mathcal{M}_{L(\lg L)^{p+1+\delta}}w)$ to $L^p(\mathscr{X},w)$. For each fixed $p\in(1,\infty)$ and $\delta>0$, choose $q=3p/(3p+\delta)$ and $\gamma=p/q$. It follows from the duality that

$$\Big(\int_{\mathscr{X}}(T^*_bf(x))^pw(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{\gamma}}=\sup_{h\ge0,\|h\|_{L^{\gamma'}(\mathscr{X},w^{1-\gamma'})}\le1}\int_{\mathscr{X}}(T^*_bf(x))^qh(x)\mathrm{d}\mu(x).$$

Lemma 2.2 states that if $\mu(\mathscr{X})=\infty$,

$$\begin{aligned}\int_{\mathscr{X}}(T^*_bf(x))^qh(x)\mathrm{d}\mu(x)&\le C\int_{\mathscr{X}}(\mathcal{M}^\sharp_{0,s}(T^*_bf)(x))^q\mathcal{M}h(x)\mathrm{d}\mu(x)\\&\le C\Big(\int_{\mathscr{X}}(\mathcal{M}^\sharp_{0,s}(T^*_bf)(x))^p\mathcal{M}_{L(\lg L)^{\gamma-1+\delta/3}}w(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{\gamma}}\\&\quad\times\Big(\int_{\mathscr{X}}(\mathcal{M}h(x))^{\gamma'}(\mathcal{M}_{L(\lg L)^{\gamma-1+\delta/3}}w(x))^{1-\gamma'}\mathrm{d}\mu(x)\Big)^{\frac{1}{\gamma'}}\\&\le C\Big(\int_{\mathscr{X}}|f(x)|^p\mathcal{M}_{L(\lg L)^{2p+1+\delta}}w(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{\gamma}}\\&\quad\times\Big(\int_{\mathscr{X}}|h(x)|^{\gamma'}w(x)^{1-\gamma'}\mathrm{d}\mu(x)\Big)^{\frac{1}{\gamma'}}\end{aligned}$$

$$\leq C\bigg(\int_{\mathscr{X}}|f(x)|^p\mathcal{M}_{L(\lg L)^{2p+1+\delta}}w(x)\mathrm{d}\mu(x)\bigg)^{\frac{1}{\gamma}},$$

where in the third inequality, we have invoked (1.10) and the following inequality that for any weight u and $g\in L^{\gamma'}(\mathscr{X},u^{1-\gamma'})$,

$$\int_{\mathscr{X}}(\mathcal{M}g(x))^{\gamma'}(\mathcal{M}_{L(\lg L)^{\gamma-1+\delta}}u(x))^{1-\gamma'}\mathrm{d}\mu(x)\leq C\int_{\mathscr{X}}|g(x)|^{\gamma'}u(x)^{1-\gamma'}\mathrm{d}\mu(x)$$

(see [3]); if $\mu(\mathscr{X})<\infty$, we firstly prove that T_b^* is an operator satisfying (2.1). For $s\in(0,1/2)$, we have

$$\mathcal{M}_s^\sharp f\leq C\mathcal{M}\mathcal{M}_{0,\,s}^\sharp f$$

and

$$\|\mathcal{M}f\|_{L^p(\mathscr{X})}\leq C\|\mathcal{M}^\sharp f\|_{L^p(\mathscr{X})}$$

(these two inequalities can be proven as in the case of Euclidean spaces; see the proofs of Lemma 3.7 in [13] and Theorem 5 in [14]), and so

$$\begin{aligned}\int_{\mathscr{X}}|T_b^*f(x)|^p\mathrm{d}\mu(x)&\leq\int_{\mathscr{X}}(\mathcal{M}(|T_b^*f|^s)(x))^{p/s}\mathrm{d}\mu(x)\\&\leq C\int_{\mathscr{X}}(\mathcal{M}^\sharp(|T_b^*f|^s)(x))^{p/s}\mathrm{d}\mu(x)\\&\leq C\|\mathcal{M}\mathcal{M}_{0,s}^\sharp(T_b^*f)\|_{L^p(\mathscr{X})}^p\\&\leq C\|\mathcal{M}_{0,s}^\sharp(T_b^*f)\|_{L^p(\mathscr{X})}^p\\&\leq C\|f\|_{L^p(\mathscr{X})}^p,\end{aligned}$$

where in the last inequality we have used Theorem 3.1. Using the same argument as in the proof of Theorem 1.3 in [4], we know that T_b^* satisfies the weak type estimate (2.1). With Lemma 2.1 we can obtain that

$$\begin{aligned}h(\mathscr{X})m_{\mathscr{X}}((T_b^*f)^q)&\leq C\int_{\mathscr{X}}(\mathcal{M}_{L\lg L}f(x))^qh(x)\mathrm{d}\mu(x)\\&\leq C\bigg(\int_{\mathscr{X}}(\mathcal{M}_{L\lg L}f(x))^pw(x)\mathrm{d}\mu(x)\bigg)^{1/\gamma}\\&\quad\times\bigg(\int_{\mathscr{X}}h(x)^{\gamma'}w(x)^{1-\gamma'}\mathrm{d}\mu(x)\bigg)^{1/\gamma'}\\&\leq C\bigg(\int_{\mathscr{X}}|f(x)|^p\mathcal{M}w(x)\mathrm{d}\mu(x)\bigg)^{1/\gamma}.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{\mathscr{X}}(T_b^*f(x))^qh(x)\mathrm{d}\mu(x)&\leq C\bigg(\int_{\mathscr{X}}|f(x)|^p\mathcal{M}_{L(\lg L)^{2p+1+\delta}}w(x)\mathrm{d}\mu(x)\bigg)^{\frac{1}{\gamma}}\\&\quad+Ch(\mathscr{X})m_{\mathscr{X}}((T_b^*f)^q)\\&\leq C\bigg(\int_{\mathscr{X}}|f(x)|^p\mathcal{M}_{L(\lg L)^{2p+1+\delta}}w(x)\mathrm{d}\mu(x)\bigg)^{\frac{1}{\gamma}},\end{aligned}$$

which implies that (1.8) is true. Repeating the argument used in the proof of Theorem 1.3 in [4] we can get (1.9), and then we complete the proof of Theorem 1.1.

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