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# Upper and Lower Semicontinuity of Solution Sets for Parametric Generalized Vector Quasi-equilibrium Problems*

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**Abstract:** In this paper, two kinds of parametric generalized vector quasi-equilibrium problems are introduced and the relations between them are studied. The upper and lower semicontinuity of their solution sets to parameters are investigated.
**Key words:** parametric generalized vector quasi-equilibrium problem, solution set, upper semicontinuity, lower semicontinuity, set-valued mapping
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## 1 Introduction and Preliminaries

Equilibrium theory, including optimization problems, variational inequality problems, saddle point problems and complementary problems as special cases, provides us a general framework for studying other fields. Up to now, main efforts for equilibrium problems have been made for the solution existence; see, e.g., [1]–[3], and the references therein. A few results have been obtained for properties of solution sets; see, e.g., [4]–[6], in which stability of solutions to parameters were studied. In most cases, stability can be viewed as the semicontinuity, continuity, Lipschitz continuity or some kinds of (generalized) differentiability of solutions to parameters. Although much efforts have been made for establishing the continuity, Lipschitz continuity and (generalized) differentiability of solutions to parameters, few works was concentrated on the semicontinuity of the solution sets.

Khanh and Luu[5] studied the lower and upper semicontinuity of the solution sets for multivalued quasivariational inequalities with a single parameter. Anh and Khanh[6] considered the semicontinuity of solution sets for parametric multivalued vector quasiequilibrium problems. Inspired and motivated by their works, in this paper, we introduce two kinds

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of parametric generalized vector quasi-equilibrium problems, which are more general than that in the literature, and study the upper and lower semicontinuity of the solution sets to parameters under some relaxed assumptions.

Throughout this paper, let $X$, $Y$, $Z$, $\Lambda$, $M$ be real Hausdorff topological vector spaces, $D$ be a nonempty compact subset of $X$ and $E$ a nonempty subset of $Y$. Let $2^E$ denote the family of all nonempty subsets of $E$, and $T : D\times \Lambda \to 2^E$, $G : D\times M \to 2^D$ and $C : D \to 2^Z$ be set-valued mappings such that $C(x)$ is a closed convex pointed cone in $Z$ and $\mathrm{int}C(x) \neq \emptyset$ for each $x \in D$. Let $f : D \times E \times D \to Z$ be a single-valued mapping.

A set-valued mapping $F : X \to 2^Y$ is said to be upper semicontinuous (shortly, u.s.c.) at $x_0 \in X$ if for any open set $V \supseteq F(x_0)$, there exists an open neighborhood $U$ of $x_0$ such that $F(U) \subseteq V$. $F$ is said to be u.s.c. on $X$ if it is u.s.c. at each point in $X$.

$F : X \to 2^Y$ is said to be lower semicontinuous (shortly, l.s.c.) at $x_0 \in X$ if for each $y \in F(x_0)$ and any open neighborhood $V$ of $y$, there exists an open neighborhood $U$ of $x_0$ such that $F(z) \cap V \neq \emptyset$ for each $z \in U$, which can be equivalently stated as: $F$ is said to be l.s.c. at $x_0$ if for any net $\{x_\alpha\}$ with $x_\alpha \to x_0$ and any $y \in F(x_0)$, there exists a net $\{y_\alpha\}$ with $y_\alpha \in F(x_\alpha)$ for all $\alpha$ such that $y_\alpha \to y$. $F$ is said to be l.s.c. on $X$ if it is l.s.c. at each point in $X$.

$F : X \to 2^Y$ is said to be a closed set-valued mapping if its graph, denoted by graph$(F)$, is a closed set in $X \times Y$, where $\mathrm{graph}(F) = \{(x, y) : x \in X,\ y \in F(x)\}$.

A single-valued mapping $f : D\times E\times D \to Z$ is said to be $Y\setminus -\mathrm{int}C(x)$-quasiconvex with respect to $T$ of type $\Pi$ if for any nonempty finite subset $\{z_1, \cdots, z_n\} \subseteq D$, any $x \in \mathrm{co}\{z_1, \cdots, z_n\}$ and any $\lambda \in \Lambda$, there exists some $i$ $(i = 1, \cdots, n)$ and $y \in T(x, \lambda)$ such that $f(x, y, z_i) \in Y\setminus(-\mathrm{int}C(x))$.

## 2 Parametric Generalized Vector Quasi-equilibrium Problems

In this section, we introduce two kinds of parametric generalized vector quasi-equilibrium problems (shortly, PGVQEP) and study the relations between them.

For any given parameters $\lambda \in \Lambda$ and $\mu \in M$, we consider the following two parametric generalized vector quasi-equilibrium problems:

(PGVQEP1) Find $x \in D$ such that there exists $y \in T(x, \lambda)$ satisfying
$$f(x, y, z) \notin -\mathrm{int}C(x), \qquad z \in G(x, \mu).$$

(PGVQEP2) Find $x \in D$ such that for any $z \in G(x, \mu)$ there exists $y \in T(x, \lambda)$ satisfying
$$f(x, y, z) \notin -\mathrm{int}C(x).$$

We denote their solution sets by $S_1(\lambda, \mu)$ and $S_2(\lambda, \mu)$, respectively. Firstly, we study the nonempty of the solution set $S_1(\lambda, \mu)$.

**Theorem 2.1** *For* $(PGVQEP1)$*, let*

(i) *the set* $\{x \in D : z \in G(x, \mu)\}$ *be open for any* $z \in D$ *and* $\mu \in M$*;*

(ii) $f$ *be* $Y\setminus -\mathrm{int}C(x)$*-quasiconvex with respect to* $T$ *of type* $\Pi$*;*

(iii) *the set* $\{x \in D: \ y \in T(x,\lambda), \ f(x,y,z) \notin -\text{int}C(x)\}$ *be closed for any* $z \in D$ *and* $\lambda \in \Lambda$.
*Then* $S_1(\lambda,\mu) \neq \emptyset$.

*Proof.* By using the well-known F-KKM Theorem (see [3]) and some similar arguments in [5], we can easily get the conclusion.

The following theorem states the relations between (PGVQEP1) and (PGVQEP2).

**Theorem 2.2** *For any given* $(\lambda,\mu) \in \Lambda \times M$, *one has* $S_1(\lambda,\mu) \subseteq S_2(\lambda,\mu)$. *But the converse inclusion is not necessarily true.*

*Proof.* For each given $x_0 \in S_1(\lambda,\mu)$, there exists $y_0 \in T(x_0,\lambda)$ satisfying

$$f(x_0,y_0,z) \notin -\text{int}C(x_0), \qquad z \in G(x_0,\mu). \tag{2.1}$$

Assume to the contrary that $x_0 \notin S_2(\lambda,\mu)$. Then there exists $z_0 \in G(x_0,\mu)$ such that

$$f(x_0,y,z_0) \in -\text{int}C(x_0), \qquad y \in T(x_0,\lambda), \tag{2.2}$$

which together with (2.1) indicates that

$$f(x_0,y_0,z_0) \notin -\text{int}C(x_0) \quad \text{and} \quad f(x_0,y_0,z_0) \in -\text{int}C(x_0).$$

This is a contradiction. Therefore, $S_1(\lambda,\mu) \subseteq S_2(\lambda,\mu)$, which indicates that $S_2(\lambda,\mu) \neq \emptyset$ under the same assumption of Theorem 2.1.

The following example shows that the converse inclusion is not true.

**Example 2.1** Let $X$, $Y$, $Z = R$, $D = E = \Lambda = M = [0,1]$, and

$$T(x,\lambda) = \{1,-1\}, \qquad x \in D, \ \lambda \in \Lambda,$$
$$G(x,\mu) = \left\{\mu, \ \frac{\mu}{2}, \ \frac{\mu}{3}\right\}, \qquad x \in D, \ \mu \in M,$$
$$C(x) = [0, \ +\infty), \qquad x \in D,$$
$$f(x,y,z) = y(z-x), \qquad x, z \in D, \ y \in E.$$

We can easily deduce that $S_1(\lambda,\mu) = \left[0, \ \frac{\mu}{3}\right] \cup [\mu, 1]$ and $S_2(\lambda,\mu) = [0,1]$.

# 3 Upper and Lower Semicontinuity

In this section, we assume that $S_1(\lambda,\mu)$ and $S_2(\lambda,\mu)$ are nonempty sets for all $\lambda \in \Lambda$ and $\mu \in M$. In order to study the upper and lower semicontinuity of solution sets to parameters, we first recall two basic results.

**Lemma 3.1**[8] *Let* $X$ *and* $Y$ *be real topological vector spaces and* $F: X \to 2^Y$ *be a set-valued mapping. If* $F$ *has compact values, then* $F$ *is u.s.c. at* $x$ *if and only if for any nets* $\{x_\alpha\} \subseteq X : x_\alpha \to x$ *and* $\{y_\alpha\} : y_\alpha \in F(x_\alpha)$ *for all* $\alpha$, *there exist* $y \in F(x)$ *and a subnet* $\{y_\beta\}$ *of* $\{y_\alpha\}$ *such that* $y_\beta \to y$.

**Lemma 3.2**[9] *Let* $X$ *and* $Y$ *be real topological vector spaces and* $F: X \to 2^Y$ *be a set-valued mapping. If* $F$ *is u.s.c. and has closed values, then graph*$(F)$ *is a closed set.*

The following theorem provides a sufficient condition for the upper semicontinuity of the solution set $S_1(\lambda, \mu)$ to parameters $(\lambda, \mu)$.

**Theorem 3.1** *For any given $x_0 \in X$, $\lambda_0 \in \Lambda$ and $\mu_0 \in M$, let $(PGVQEP1)$ satisfy the following assumptions*:

(i) $T(\cdot, \lambda_0)$ *is u.s.c. and has compact values in $D$*;

(ii) $G(\cdot, \mu_0)$ *is l.s.c. in $D$*;

(iii) $W(\cdot)$ *is u.s.c. at $x_0$, where $W(x) := Z \setminus (-\text{int}C(x))$*;

(iv) $f(\cdot, \cdot, \cdot)$ *is continuous in $D \times E \times D$.*

*Then $S_1(\cdot, \cdot)$ is u.s.c. and closed at $(\lambda_0, \mu_0)$.*

*Proof.* Suppose to the contrary that $S_1(\cdot, \cdot)$ is not u.s.c. at $(\lambda_0, \mu_0)$. Then there exists an open set $V$ of $S_1(\lambda_0, \mu_0)$ such that for any $(\lambda_\alpha, \mu_\alpha) \to (\lambda_0, \mu_0)$ there exists $x_\alpha \in S_1(\lambda_\alpha, \mu_\alpha) \backslash V$. By the compactness of $D$, without loss of generality, we can assume that $x_\alpha \to x_0 \in D \setminus V$. Consequently, we can deduce that $x_0 \in D \setminus S_1(\lambda_0, \mu_0)$, which implies that for any $y \in T(x_0, \lambda_0)$ there exists $z_0 \in G(x_0, \mu_0)$ such that

$$f(x_0, y, z_0) \in -\text{int}C(x_0). \tag{3.1}$$

On the other hand, $x_\alpha \in S_1(\lambda_\alpha, \mu_\alpha)$ implies that there exists $y_\alpha \in T(x_\alpha, \lambda_\alpha)$ such that

$$f(x_\alpha, y_\alpha, z_\alpha) \notin -\text{int}C(x_\alpha), \qquad z_\alpha \in G(x_\alpha, \mu_\alpha). \tag{3.2}$$

It follows from Lemma 3.1 that there exist a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ and $y_0 \in T(x_0, \lambda_0)$ such that $y_\beta \to y_0$. By (ii), there exists $\bar{z}_\alpha \in G(x_\alpha, \mu_\alpha)$ such that $\bar{z}_\alpha \to z_0$, which together with (3.2) shows that

$$f(x_\alpha, y_\beta, \bar{z}_\alpha) \notin -\text{int}C(x_\alpha), \tag{3.3}$$

that is, $f(x_\alpha, y_\beta, \bar{z}_\alpha) \in W(x_\alpha)$. By Lemma 3.2, we get $f(x_0, y_0, z_0) \in W(x_0)$, which contradicts (3.1). Therefore, $S_1(\cdot, \cdot)$ is u.s.c. at $(\lambda_0, \mu_0)$.

Next, we show that $S_1(\cdot, \cdot)$ is closed at $(\lambda_0, \mu_0)$.

Suppose to the contrary that $S_1(\cdot, \cdot)$ is not closed at $(\lambda_0, \mu_0)$. Then there exists a net $(\lambda_\alpha, \mu_\alpha, x_\alpha) \to (\lambda_0, \mu_0, x_0)$ with $x_\alpha \in S_1(\lambda_\alpha, \mu_\alpha)$ and $x_0 \notin S_1(\lambda_0, \mu_0)$. Using a similar way above, we can finish the rest part of the proof.

In a similar proof way of Theorem 3.1, we can obtain the following result.

**Theorem 3.2** *For $(PGVQEP2)$, let hypotheses (i)–(iv) in Theorem 3.1 hold. Then $S_2(\cdot, \cdot)$ is u.s.c. and closed at $(\lambda_0, \mu_0)$.*

Now, we consider the lower semicontinuity of $S_1(\cdot, \cdot)$ and $S_2(\cdot, \cdot)$ to parameters $(\lambda, \mu)$.

**Theorem 3.3** *For any given $x_0 \in X$, $\lambda_0 \in \Lambda$ and $\mu_0 \in M$, let $(PGVQEP1)$ satisfy the following assumptions*:

(i) $G(\cdot, \mu_0)$ *is u.s.c. and has compact values in $D$*;

(ii) *$graphT(\cdot)$ is l.s.c. at $\lambda_0$, where $graphT(\lambda) := \{(x, y) \mid y \in T(x, \lambda)\}$*;

(iii) $C(\cdot)$ *is u.s.c. at $x_0$*;

(iv) $f(\cdot, \cdot, \cdot)$ *is continuous in $D \times E \times D$*;

(v) *$f(x,y,z) \notin -\partial C(x)$ for all $x \in S_1(\lambda,\mu)$, $y \in T(x,\lambda)$ and $z \in G(x,\mu)$, where $\partial C(x)$ denotes the boundary of the set $C(x)$.*
*Then $S_1(\cdot,\cdot)$ is l.s.c. at $(\lambda_0,\mu_0)$.*

*Proof.* Suppose to the contrary that $S_1(\cdot,\cdot)$ is not l.s.c. at $(\lambda_0,\mu_0)$. Then there exist $x_0 \in S_1(\lambda_0,\mu_0)$ and $(\lambda_\alpha,\mu_\alpha) \to (\lambda_0,\mu_0)$ such that, for all $x_\alpha \in S_1(\lambda_\alpha,\mu_\alpha)$, $x_\alpha$ does not converge to $x_0$.

$x_0 \in S_1(\lambda_0,\mu_0)$ implies that $x_0 \in D$ and there exists $y_0 \in T(x_0,\lambda_0)$ such that

$$f(x_0,y_0,z) \notin -\mathrm{int}C(x_0), \qquad z \in G(x_0,\mu_0). \tag{3.4}$$

Similarly, $x_\alpha \in S_1(\lambda_\alpha,\mu_\alpha)$ implies that $x_\alpha \in D$, and there exists $y_\alpha \in T(x_\alpha,\lambda_\alpha)$ such that

$$f(x_\alpha,y_\alpha,z_\alpha) \notin -\mathrm{int}C(x_\alpha), \qquad z_\alpha \in G(x_\alpha,\mu_\alpha).$$

By $(x_0,y_0) \in \mathrm{graph}T(\lambda_0)$ and (ii), we know that there exists a net $(\bar{x}_\alpha,\bar{y}_\alpha) \in \mathrm{graph}T(\lambda_\alpha)$ such that $(\bar{x}_\alpha,\bar{y}_\alpha) \to (x_0,y_0)$. Consequently, there exists a subnet $\{\bar{x}_\beta\}$ of $\{\bar{x}_\alpha\}$ such that $\bar{x}_\beta \notin S_1(\lambda_\beta,\mu_\beta)$ for each $\beta$, which indicates that for any $y_\beta \in T(\bar{x}_\beta,\lambda_\beta)$ there exists $z_\beta \in G(\bar{x}_\beta,\mu_\beta)$ such that $f(\bar{x}_\beta,y_\beta,z_\beta) \in -\mathrm{int}C(\bar{x}_\beta)$. Specially, for $(\bar{x}_\beta,\bar{y}_\beta) \in \mathrm{graph}T(\lambda_\beta)$, there exists $\bar{z}_\beta \in G(\bar{x}_\beta,\mu_\beta)$ such that

$$f(\bar{x}_\beta,\bar{y}_\beta,\bar{z}_\beta) \in -\mathrm{int}C(\bar{x}_\beta) \subseteq -C(\bar{x}_\beta).$$

By Lemma 3.1, there exists a subnet of $\{\bar{z}_\beta\}$, denoted still by $\{\bar{z}_\beta\}$, and $z_0 \in G(x_0,\mu_0)$ such that $\bar{z}_\beta \to z_0$. By Lemma 3.2 and (iii)–(iv), we can deduce that $f(x_0,y_0,z_0) \in -C(x_0)$. By (v), we get $f(x_0,y_0,z_0) \in -\mathrm{int}C(x_0)$, which contradicts (3.4).

In a similar proof way of Theorem 3.3, we can obtain the following result.

**Theorem 3.4** *For $(PGVQEP2)$, let hypotheses* (i)–(v) *in the Theorem* 3.3 *hold. Then $S_2(\cdot,\cdot)$ is l.s.c. at $(\lambda_0,\mu_0)$.*

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