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Quadratic Lyapunov Function and Exponential Dichotomy on Time Scales*

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Abstract: In this paper, we study the relationship between exponential dichotomy and quadratic Lyapunov function for the linear equation $x^{\Delta} = A(t)x$ on time scales. Moreover, for the nonlinear perturbed equation $x^{\Delta} = A(t)x + f(t, x)$ we give the instability of the zero solution when f is sufficiently small.

Key words: quadratic Lyapunov function, exponential dichotomy, time scale, instability

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1 Introduction

Exponential dichotomy plays an important role in the theory of nonautonomous dynamical systems. When people deal with the nonlinear problems which are the perturbation of the linear ones, exponential dichotomy is very useful. Copple[1] studied in detail the exponential dichotomy for ordinary differential equations. Coff and Schäffer[2] studied exponential dichotomy for difference equations. In 1990, Hilger[3] introduced the theory of time scales, and then there are numerous works using this notion to unify and generalize theories of continuous and discrete dynamical systems (see [3]–[5]).

Recently, Barreia and Valls[6] studied the relationship between nonuniform exponential dichotomy and quadratic Lyapunov function. In this paper, we firstly introduce strict quadratic Lyapunov function on time scales. Then we study the relationship between exponential dichotomy and strict quadratic Lyapunov function on time scales. We obtain that the linear equation $x^{\Delta} = A(t)x$ has a strict quadratic Lyapunov function if it admits strong exponential dichotomy; conversely, the linear equation admits exponential dichotomy if it has a strict quadratic Lyapunov function with some property. And by quadratic Lyapunov function, we investigate the instability of the zero solution of the nonlinear perturbed equa-

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tion. The stability and instability of solutions to the nonlinear perturbed equation on time scales have been studied in [7] and [8] by considering appropriate eigenvalue conditions, and in [9] by assuming the existence of Lyapunov functions.

This paper is organized as follows. In next section, we review some useful notions and basic properties on time scales. In Section 3, the main concepts, exponential dichotomy and strict quadratic Lyapunov function on time scales, are given. Furthermore, our main results are stated and proved. Finally, we study instability of the zero solution to the nonlinear perturbed equation on time scales in Section 4.

2 Preliminaries on Time Scales

For the convenience of readers, we review some preliminary definitions and theories on time scales. The reader may refer to [5] for details. Let $\mathbf{T}$ be a time scale which is an arbitrary nonempty closed subset of the real numbers. Such as the sets of real numbers and integers are the special time scales and the union of arbitrary nonempty closed intervals is also a time scale.

Definition 2.1 *Let $\mathbf{T}$ be a time scale. The forward jump operator is defined by*

$$\sigma(t) := \inf\{s \in \mathbf{T} : s > t\}$$

for every $t \in \mathbf{T}$. Let $\mu(t) := \sigma(t) - t$ be the graininess function.

It is clear that the graininess function $\mu(t)$ is nonnegative. In this paper, we always suppose that the graininess function $\mu(t)$ is bounded.

Definition 2.2 *A function $f : \mathbf{T} \to \mathbf{R}^n$ is called rd-continuous if it is continuous at right dense points in $\mathbf{T}$ and left-sided limits exist at left dense points in $\mathbf{T}$.*

We denote the set of rd-continuous functions by $C_{rd}(\mathbf{T}, \mathbf{R}^n)$.

Definition 2.3 *A function $f : \mathbf{T} \to \mathbf{R}^n$ is called differentiable at $t \in \mathbf{T}$, if for any $\varepsilon > 0$, there exists a $\mathbf{T}$-neighborhood U of t and $f^{\Delta}(t) \in \mathbf{R}^n$ such that for any $s \in U$ we have*

$$\|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\| \le \varepsilon(\sigma(t) - s),$$

and $f^{\Delta}(t)$ is called the derivative of f at t.

If f and g are differentiable at t, then the following equalities hold:

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t),$$
$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t).$$

The integral of a mapping f is always understood in Lebesgue's sense and written as $\int_s^t f(\tau)\Delta\tau$, for $s, t \in \mathbf{T}$. If $f \in C_{rd}(\mathbf{T}, \mathbf{R}^n)$ and $t \in \mathbf{T}$, then

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t). \tag{2.1}$$

Definition 2.4 *A function $p : \mathbf{T} \to \mathbf{R}$ is called regressive if $1+\mu(t)p(t) \neq 0$ for all $t \in \mathbf{T}$, and the $n \times n$ matrix valued function $A(t)$ on time scale $\mathbf{T}$ is called regressive if $Id+\mu(t)A(t)$ is invertible for all $t \in \mathbf{T}$.*

Let a, b be regressive. Define

$$(a \oplus b)(t) = a(t) + b(t) + \mu(t)a(t)b(t),$$

$$(\ominus a)(t) = -\frac{a(t)}{1+\mu(t)a(t)}$$

for all $t \in \mathbf{T}$. Then the regressive set is an Abelian group. It is not hard to verify that the following properties hold.

Lemma 2.1 *Suppose that a, b are regressive. Then we have*

(1) $a \ominus a = 0$;

(2) $\ominus(\ominus a) = a$;

(3) $a \ominus b = \dfrac{a-b}{1+\mu(t)b}$;

(4) $\ominus(a \oplus b) = (\ominus a) \oplus (\ominus b)$.

Now we give the definition of exponential function on time scales. Moreover, we state some basic properties of the exponential function which are to be used in our paper.

Definition 2.5 *Let p be regressive. We define the exponential function by*

$$e_p(t, s) = \exp\left\{\int_s^t \frac{1}{\mu(\tau)} \ln(1 + \mu(\tau)p)\Delta\tau\right\}$$

for all $s, t \in \mathbf{T}$. When $\mu(\tau) = 0$, we define $e_p(t, s) = e^{p(t-s)}$.

Lemma 2.2 *Suppose that a, b are regressive. Then we have*

(1) $e_0(t, s) = 1$ *and* $e_a(t, t) = 1$;

(2) $e_a(t, s) = e_{\ominus a}(s, t)$;

(3) $e_a(t, s)e_a(s, r) = e_a(t, r)$;

(4) $e_a(t, s)e_b(t, s) = e_{a\oplus b}(t, s)$.

Lemma 2.3 *Suppose that a is regressive and $c \in \mathbf{T}$. Then we have*

$$[e_a(c, t)]_t^\Delta = -a(t)e_a(c, \sigma(t))$$

and

$$\int_s^t a(\tau)e_a(c, \sigma(\tau))\Delta\tau = e_a(c, s) - e_a(c, t),$$

where $[e_a(c, t)]_t^\Delta$ stands for the delta derivative of $e_a(c, t)$ with respect to t.

3 Exponential Dichotomy and Strict Quadratic Lyapunov Function

Let M be a bound of $\mu(t)$, i.e., $|\mu(t)| < M$ for any $t \in \mathbf{T}$. Consider the equation

$$x^\Delta = A(t)x, \tag{3.1}$$

where $A(t)$ is an $n \times n$ matrix valued function on time scale $\mathbf{T}$ satisfying that $A(t)$ is rd-continuous and regressive. The equation (3.1) is said to admit an exponential dichotomy on time scale $\mathbf{T}$ if for each $t \in \mathbf{T}$ there exists a projection $P(t) : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$T(t,s)P(s) = P(t)T(t,s), \qquad t,s \in \mathbf{T} \tag{3.2}$$

and

$$\|T(t,s)P(s)\| \le De_{\ominus a}(t,s), \qquad t \ge s, \ \ t,s \in \mathbf{T}, \tag{3.3}$$

$$\|T(t,s)Q(s)\| \le De_{a}(t,s), \qquad t \le s, \ \ t,s \in \mathbf{T} \tag{3.4}$$

for some constants $a > 0$ and $D > 1$. Here, $T(t,s)$ is the linear evolution operator associated to the equation (3.1), and $Q(s) = Id - P(s)$ for $s \in \mathbf{T}$. We say that the equation (3.1) admits a strong exponential dichotomy on time scale $\mathbf{T}$ if there exists a projection $P(t) : \mathbf{R}^n \to \mathbf{R}^n$ for each $t \in \mathbf{T}$ and constants $a > 0$, $D > 1$ satisfying (3.2), (3.3), (3.4), as well as a constant b with $b \ge a > 0$ such that

$$\|T(t,s)P(s)\| \le De_{b}(s,t), \qquad t \le s, \ t,s \in \mathbf{T}, \tag{3.5}$$

$$\|T(t,s)Q(s)\| \le De_{b}(t,s), \qquad t \ge s, \ t,s \in \mathbf{T}. \tag{3.6}$$

Next we give the definition of Lyapunov functions. Consider a continuous function $V : \mathbf{T} \times \mathbf{R}^n \to \mathbf{R}^n$. The function V is called a Lyapunov function, if the following two conditions are satisfied:

(H1) For each $\tau \in \mathbf{T}$, set $V_\tau := V(\tau, \cdot)$ and

$$C^s(V_\tau) := \{0\} \cup V_\tau^{-1}(-\infty, 0), \qquad C^u(V_\tau) := \{0\} \cup V_\tau^{-1}(0, +\infty).$$

Let r_s and r_u be respectively the maximal dimensions of linear subspaces inside $C^s(V_\tau)$ and $C^u(V_\tau)$ satisfying

$$r_s + r_u = n;$$

(H2) For every $t \ge \tau$, $t, \tau \in \mathbf{T}$ and $x \in \mathbf{R}^n$, we have

$$V(t, T(t,\tau)x) \ge V(\tau, x).$$

Set

$$E_\tau^s := \bigcap_{t \in \mathbf{T}} T(\tau,t)\overline{C^s(V_t)}, \qquad E_\tau^u := \bigcap_{t \in \mathbf{T}} T(\tau,t)\overline{C^u(V_t)}.$$

Then

$$T(t,\tau)E_\tau^s = E_t^s, \quad T(t,\tau)E_\tau^u = E_t^u, \qquad t,\tau \in \mathbf{T}.$$

Suppose that V is a Lyapunov function for the equation (3.1). V is called a strict Lyapunov function, if there exist $r > 0$ and $K > 0$ such that the following conditions are satisfied:

(H3) If $x \in E_\tau^s$, then

$$V^2(t, T(t,\tau)x) \le e_{\ominus r}(t,\tau)V^2(\tau,x), \qquad t \ge \tau, \ \ t,\tau \in \mathbf{T};$$

(H4) If $x \in E_\tau^u$, then

$$V^2(t, T(t,\tau)x) \ge e_{r}(t,\tau)V^2(\tau,x), \qquad t \ge \tau, \ \ t,\tau \in \mathbf{T};$$

(H5) If $x \in E_\tau^s \cup E_\tau^u$, then

$$|V(\tau,x)| \ge \frac{1}{K}\|x\|, \qquad \tau \in \mathbf{T}.$$

Furthermore, let $S(t)$ for each $t \in \mathbf{T}$ be a symmetric invertible $n \times n$ matrix. Let

$$H(t,x) = \langle S(t)x, x\rangle, \qquad V(t,x) = -\mathrm{sign} H(t,x)\sqrt{|H(t,x)|} \tag{3.7}$$

for $t \in \mathbf{T},\ x \in \mathbf{R}^n$. If $V(t,x)$ in (3.7) is a strict Lyapunov function, then we say that $V(t,x)$ is a strict quadratic Lyapunov function.

Now we state and prove our main results.

The following Theorems 3.1 and 3.2 indicate that, for the equation (3.1), the strong exponential dichotomy property and the existence of a strict quadratic Lyapunov function are equivalent in some sense.

Theorem 3.1 *If the equation* (3.1) *admits a strong exponential dichotomy, then it has a strict quadratic Lyapunov function.*

Proof.

$$\begin{aligned} S(t) := &\int_t^{+\infty} (T(\sigma(v),t)P(t))^* T(\sigma(v),t)P(t) e_{c\ominus a}(t,\sigma(v))\Delta v \\ &- \int_{-\infty}^t (T(\sigma(v),t)Q(t))^* T(\sigma(v),t)Q(t) e_{a\ominus c}(t,\sigma(v))\Delta v, \end{aligned} \tag{3.8}$$

where $0 < c < a$. Let $H(t,x) = \langle S(t)x, x\rangle$.

When $t \geq \tau,\ t, \tau \in \mathbf{T}$, we have

$$\begin{aligned} H(t, T(t,\tau)x) = &\int_t^{+\infty} \|T(\sigma(v),\tau)P(\tau)x\|^2 e_{c\ominus a}(t,\sigma(v))\Delta v \\ &- \int_{-\infty}^t \|T(\sigma(v),\tau)Q(\tau)x\|^2 e_{a\ominus c}(t,\sigma(v))\Delta v \\ \leq &\int_\tau^{+\infty} \|T(\sigma(v),\tau)P(\tau)x\|^2 e_{c\ominus a}(\tau,\sigma(v))\Delta v e_{c\ominus a}(t,\tau) \\ &- \int_{-\infty}^\tau \|T(\sigma(v),\tau)Q(\tau)x\|^2 e_{a\ominus c}(\tau,\sigma(v))\Delta v e_{a\ominus c}(t,\tau). \end{aligned}$$

Since $c \ominus a = \dfrac{c-a}{1+\mu(t)a} < 0$, $a \ominus c = \dfrac{a-c}{1+\mu(t)c} > 0$ and $t \geq \tau$, we obtain

$$e_{c\ominus a}(t,\tau) \leq 1, \qquad e_{a\ominus c}(t,\tau) \geq 1.$$

Thus,

$$\begin{aligned} H(t, T(t,\tau)x) \leq &\int_\tau^{+\infty} \|T(\sigma(v),\tau)P(\tau)x\|^2 e_{c\ominus a}(\tau,\sigma(v))\Delta v \\ &- \int_{-\infty}^\tau \|T(\sigma(v),\tau)Q(\tau)x\|^2 e_{a\ominus c}(\tau,\sigma(v))\Delta v = H(\tau, x). \end{aligned} \tag{3.9}$$

Set

$$V(t,x) = -\mathrm{sign} H(t,x)\sqrt{|H(t,x)|}.$$

We define subspaces

$$E_t^s = P(t)(\mathbf{R}^n), \quad E_t^u = Q(t)(\mathbf{R}^n), \qquad t \in \mathbf{T}.$$

Thus

$$E_t^s \oplus E_t^u = \mathbf{R}^n, \qquad t \in \mathbf{T}.$$

When $x \in E_t^s/\{0\}$, we have $H(t,x) > 0$, and hence $V(t,x) < 0$; when $x \in E_t^u/\{0\}$, we have $H(t,x) < 0$, and hence $V(t,x) > 0$. We also get that $H(t,x) = 0$ if and only if $x = 0$ which implies $S(t)$ is invertible for each $t \in \mathbf{T}$. Note that $S(t)$ is symmetric for each $t \in \mathbf{T}$.

When $t \geq \tau$, from (3.9) we have

$$V(t, T(t,\tau)x) = -\sqrt{H(t, T(t,\tau)x)} \geq -\sqrt{H(\tau, x)} = V(\tau, x), \qquad x \in E_t^s$$

and

$$V(t, T(t,\tau)x) = \sqrt{|H(t, T(t,\tau)x)|} \geq \sqrt{|H(\tau, x)|} = V(\tau, x), \qquad x \in E_t^u.$$

Therefore, when $t \geq \tau$, we get $V(t, T(t,\tau)x) \geq V(\tau, x)$. It follows that $V(t,x)$ is a Lyapunov function.

From (3.3), (3.4) and Lemmas 2.1–2.3, we have

$$\begin{aligned}
|H(t,x)| \leq & \int_t^{+\infty} \|T(\sigma(v),t)P(t)x\|^2 e_{c\ominus a}(t,\sigma(v))\Delta v \\
& + \int_{-\infty}^t \|T(\sigma(v),t)Q(t)x\|^2 e_{a\ominus c}(t,\sigma(v))\Delta v \\
\leq & \int_t^{+\infty} D^2 e_{\ominus a}(\sigma(v),t) e_{\ominus a}(\sigma(v),t) e_{c\ominus a}(t,\sigma(v))\Delta v \|x\|^2 \\
& + \int_{-\infty}^t D^2 e_a(\sigma(v),t) e_a(\sigma(v),t) e_{a\ominus c}(t,\sigma(v))\Delta v \|x\|^2 \\
= & \int_t^{+\infty} \frac{-D^2}{a\oplus c}[e_{a\oplus c}(t,v)]_v^\Delta \Delta v \|x\|^2 \\
& + \int_{-\infty}^t \frac{-D^2}{\ominus(a\oplus c)}[e_{\ominus(a\oplus c)}(t,v)]_v^\Delta \Delta v \|x\|^2.
\end{aligned}$$

Since

$$a \oplus c = a + c + \mu(t)ac \geq a + c$$

and

$$\frac{-1}{\ominus(a\oplus c)} = \frac{1+\mu(t)(a+c+\mu(t)ac)}{a+c+\mu(t)ac} \leq \frac{1}{a+c} + M,$$

we obtain

$$|H(t,x)| \leq \Big(\frac{D^2}{a+c} + \frac{D^2}{a+c} + MD^2\Big)\|x\|^2 = \Big(\frac{2}{a+c} + M\Big)D^2\|x\|^2. \tag{3.10}$$

Thus

$$|V(t,x)| = \sqrt{|H(t,x)|} \leq D\sqrt{\frac{2}{a+c} + M}\|x\|.$$

Next, we show that the function $V(t,x)$ is actually a strict Lyapunov function. To do this, we only need to verify that (H3)–(H5) hold.

If $x \in E_\tau^s$ for $\tau \in \mathbf{T}$, there exists $y \in \mathbf{R}^n$ such that $x = P(\tau)y$. Then for $t \geq \tau,\ t, \tau \in \mathbf{T}$,

$$\begin{aligned}
H(t, T(t,\tau)x) = & \int_t^{+\infty} \|T(\sigma(v),\tau)P(\tau)x\|^2 e_{c\ominus a}(t,\sigma(v))\Delta v \\
\leq & \int_\tau^{+\infty} \|T(\sigma(v),\tau)P(\tau)x\|^2 e_{c\ominus a}(\tau,\sigma(v))\Delta v e_{c\ominus a}(t,\tau) \\
= & \, e_{c\ominus a}(t,\tau)H(\tau,x).
\end{aligned}$$

Therefore,

$$V^2(t,T(t,\tau)x)\le e_{c\ominus a}(t,\tau)V^2(\tau,x),\qquad x\in E^s_\tau,$$

i.e., (H3) is satisfied.

If $x\in E^u_\tau$ for $\tau\in\mathbf{T}$, there exists $z\in\mathbf{R}^n$ such that $x=Q(\tau)z$. Then for $t\ge\tau,\ t,\tau\in\mathbf{T}$,

$$\begin{aligned}|H(t,T(t,\tau)x)|&=\int_{-\infty}^{t}\|T(\sigma(v),\tau)Q(\tau)x\|^2e_{a\ominus c}(t,\sigma(v))\Delta v\\&\ge\int_{-\infty}^{\tau}\|T(\sigma(v),\tau)Q(\tau)x\|^2e_{a\ominus c}(\tau,\sigma(v))\Delta v e_{a\ominus c}(t,\tau)\\&=e_{a\ominus c}(t,\tau)|H(\tau,x)|.\end{aligned}$$

Therefore,

$$V^2(t,T(t,\tau)x)\ge e_{a\ominus c}(t,\tau)V^2(\tau,x),\qquad x\in E^u_\tau,$$

i.e., (H4) is satisfied.

Now we establish the condition (H5). From (3.3)–(3.6), when $t\ge\tau,\ t,\tau\in\mathbf{T}$, we have

$$\begin{aligned}\|T(t,\tau)\|&\le\|T(t,\tau)P(\tau)\|+\|T(t,\tau)Q(\tau)\|\\&\le De_{\ominus a}(t,\tau)+De_b(t,\tau),\end{aligned}\tag{3.11}$$

and when $t\le\tau,\ t,\tau\in\mathbf{T}$, we have

$$\begin{aligned}\|T(t,\tau)\|&\le\|T(t,\tau)P(\tau)\|+\|T(t,\tau)Q(\tau)\|\\&\le De_b(\tau,t)+De_a(t,\tau).\end{aligned}\tag{3.12}$$

If $x\in E^s_\tau,\ \tau\in\mathbf{T}$, then $P(\tau)x=x$ and

$$\begin{aligned}H(\tau,x)&=\int_{\tau}^{+\infty}\|T(\sigma(v),\tau)P(\tau)x\|^2e_{c\ominus a}(\tau,\sigma(v))\Delta v\\&\ge\int_{\tau}^{+\infty}\frac{\|x\|^2}{\|T(\tau,\sigma(v))\|^2}e_{c\ominus a}(\tau,\sigma(v))\Delta v\\&\ge\int_{\tau}^{+\infty}\frac{\|x\|^2e_{c\ominus a}(\tau,\sigma(v))}{[De_b(\sigma(v),\tau)+De_a(\tau,\sigma(v))]^2}\Delta v\\&\ge\frac{\|x\|^2}{4D^2}\int_{\tau}^{+\infty}e_{c\ominus a}(\tau,\sigma(v))e_{b\oplus b}(\tau,\sigma(v))\Delta v\\&=\frac{\|x\|^2}{4D^2}\int_{\tau}^{+\infty}e_{(b\oplus b)\oplus(c\ominus a)}(\tau,\sigma(v))\Delta v.\end{aligned}$$

Since

$$\begin{aligned}(b\oplus b)\oplus(c\ominus a)&=2b+\mu(t)b^2+\frac{c-a}{1+\mu(t)a}+\mu(t)(2b+\mu(t)b^2)\frac{c-a}{1+\mu(t)a}\\&\le 2b+c-a+Mb^2+2Mbc+M^2b^2c,\end{aligned}$$

and obviously $(b\oplus b)\oplus(c\ominus a)>0$, we have

$$\frac{1}{(b\oplus b)\oplus(c\ominus a)}\ge\frac{1}{2b+c-a+Mb^2+2Mbc+M^2b^2c}.$$

Therefore,

$$\begin{aligned}H(\tau,x)&\ge\frac{\|x\|^2}{4D^2}\int_{\tau}^{+\infty}\frac{-[e_{(b\oplus b)\oplus(c\ominus a)}(\tau,v)]^{\Delta}_{v}}{(b\oplus b)\oplus(c\ominus a)}\Delta v\\&\ge\frac{\|x\|^2}{4D^2(2b+c-a+Mb^2+2Mbc+M^2b^2c)}.\end{aligned}\tag{3.13}$$

If $x \in E^u_\tau,\ \tau \in \mathbf{T}$, then $Q(\tau)x = x$ and

$$\begin{aligned}
|H(\tau, x)| &= \int_{-\infty}^{\tau} \|T(\sigma(v), \tau)Q(\tau)x\|^2 e_{a\ominus c}(\tau, \sigma(v))\Delta v \\
&\geq \int_{-\infty}^{\tau} \frac{\|x\|^2}{\|T(\tau, \sigma(v))\|^2} e_{a\ominus c}(\tau, \sigma(v))\Delta v \\
&\geq \int_{-\infty}^{\tau} \frac{\|x\|^2 e_{a\ominus c}(\tau, \sigma(v))}{[De_b(\tau, \sigma(v)) + De_{\ominus a}(\tau, \sigma(v))]^2}\Delta v \\
&\geq \frac{\|x\|^2}{4D^2}\int_{-\infty}^{\tau} e_{a\ominus c}(\tau, \sigma(v)) e_{b\oplus b}(\sigma(v), \tau)\Delta v \\
&= \frac{\|x\|^2}{4D^2}\int_{-\infty}^{\tau} e_{(a\ominus c)\ominus(b\oplus b)}(\tau, \sigma(v))\Delta v.
\end{aligned}$$

Since

$$\begin{aligned}
-(a\ominus c)\ominus(b\oplus b) &= -\frac{\dfrac{a-c}{1+\mu(t)c} - (2b+\mu(t)b^2)}{1+\mu(t)(2b+\mu(t)b^2)} \\
&\leq 2b - a + c + Mb^2 + 2Mbc + M^2b^2c,
\end{aligned}$$

and obviously $-(a\ominus c)\ominus(b\oplus b) > 0$, we have

$$\frac{1}{-(a\ominus c)\ominus(b\oplus b)} \geq \frac{1}{2b+c-a+Mb^2+2Mbc+M^2b^2c}.$$

Therefore,

$$\begin{aligned}
|H(\tau, x)| &\geq \frac{\|x\|^2}{4D^2}\int_{-\infty}^{\tau} \frac{-[e_{(a\ominus c)\ominus(b\oplus b)}(\tau, v)]^{\Delta}_v}{(a\ominus c)\ominus(b\oplus b)}\Delta v \\
&\geq \frac{\|x\|^2}{4D^2(2b+c-a+Mb^2+2Mbc+M^2b^2c)}. \qquad (3.14)
\end{aligned}$$

From (3.13) and (3.14), we obtain

$$|H(t, x)| \geq \frac{\|x\|^2}{K^2}, \qquad (3.15)$$

where

$$K^2 = 4D^2(2b+c-a+Mb^2+2Mbc+M^2b^2c).$$

Thus, we get

$$|V(\tau, x)| = \sqrt{|H(\tau, x)|} \geq \frac{\|x\|}{K},$$

i.e., (H5) holds. The proof is completed.

Theorem 3.2 *If the equation* (3.1) *has a strict quadratic Lyapunov function and satisfies*

$$\sup_{t\in\mathbf{T}} \|S(t)\| < \infty,$$

then it admits an exponential dichotomy.

Proof. Let V be a strict quadratic Lyapunov function for the equation (3.1). Then V satisfies the conditions (H1)–(H5). For each $\tau \in \mathbf{T}$, set

$$E^s_\tau := \bigcap_{t\in\mathbf{T}} T(\tau, t)\overline{C^s(V_t)}, \qquad E^u_\tau := \bigcap_{t\in\mathbf{T}} T(\tau, t)\overline{C^u(V_t)}.$$

Then we have

$$T(t, \tau)E^s_\tau = E^s_t, \qquad T(t, \tau)E^u_\tau = E^u_t.$$

By condition (H1), there exist subspaces $F_t^s \subset E_t^s$ and $F_t^u \subset E_t^u$ satisfying $F_t^s \oplus F_t^u = \mathbf{R}^n$. Consider the projections

$$P(t) : \mathbf{R}^n \to F_t^s, \qquad Q(t) : \mathbf{R}^n \to F_t^u.$$

Obviously, $Q(t) = Id - P(t)$ for each $t \in \mathbf{T}$.

Since $\sup\limits_{t\in\mathbf{T}} \|S(t)\| < \infty$, there exists a constant $L > 0$ such that $\sup\limits_{t\in\mathbf{T}} \|S(t)\| \le L^2 < \infty$. Then

$$|V(\tau,x)| = \sqrt{|H(\tau,x)|} = \sqrt{|\langle S(t)x,x\rangle|} \le \sqrt{\sup_{t\in\mathbf{T}} \|S(t)\|\|x\|^2} \le L\|x\|. \tag{3.16}$$

If $x \in F_\tau^s$ for $\tau \in \mathbf{T}$, then, by (3.16) and (H3), we obtain

$$\|T(t,\tau)x\|^2 \le K^2|V(t,T(t,\tau)x)|^2 \le K^2 e_{\ominus r}(t,\tau)V^2(\tau,x) \le K^2L^2 e_{\ominus r}(t,\tau)\|x\|^2$$

when $t \ge \tau,\ t,\tau \in \mathbf{T}$. Therefore,

$$\|T(t,\tau)|_{F_\tau^s}\| \le KL[e_{\ominus r}(t,\tau)]^{\frac{1}{2}}. \tag{3.17}$$

Now we need to find some constat $a > 0$ such that

$$[e_{\ominus r}(t,\tau)]^{\frac{1}{2}} \le e_{\ominus a}(t,\tau), \qquad t \ge \tau,\ t,\ \tau \in \mathbf{T}. \tag{3.18}$$

By the notion of exponential function on time scales, we have

$$e_{\ominus r}(t,\tau) = \exp\left\{\int_\tau^t \frac{1}{\mu(v)} \ln\Big(\frac{1}{1+\mu(v)r}\Big)\Delta v\right\}.$$

It follows that we only need to find $a > 0$ such that

$$\exp\left\{\int_\tau^t \frac{1}{\mu(v)} \ln\Big(\frac{1}{1+\mu(v)r}\Big)^{\frac{1}{2}}\Delta v\right\} \le \exp\left\{\int_\tau^t \frac{1}{\mu(v)} \ln\Big(\frac{1}{1+\mu(v)a}\Big)\Delta v\right\}. \tag{3.19}$$

Note that if

$$\left(\frac{1}{1+\mu(v)r}\right)^{\frac{1}{2}} \le \frac{1}{1+\mu(v)a}, \tag{3.20}$$

then (3.19) holds. The inequality (3.20) is equivalent to

$$1 + \mu(v)r \ge 1 + 2\mu(v)a + \mu(v)^2a^2,$$

and hence

$$\mu(v)a^2 + 2a \le r.$$

Since $\mu(v)a^2 + 2a \le Ma^2 + 2a$, we only need find $a > 0$ such that

$$Ma^2 + 2a \le r, \tag{3.21}$$

i.e., $0 < a < \dfrac{\sqrt{1+Mr}-1}{M}$. Therefore, if $0 < a < \dfrac{\sqrt{1+Mr}-1}{M}$, then the inequality (3.18) holds and, by (3.17), we have

$$\|T(t,\tau)|_{F_\tau^s}\| \le KLe_{\ominus a}(t,\tau). \tag{3.22}$$

If $x \in F_\tau^u$ for $\tau \in \mathbf{T}$, then, from (3.16), (H4) and (H5), we obtain

$$\begin{aligned}\|T(t,\tau)x\|^2 &\ge \frac{1}{L^2}|V(t,T(t,\tau)x)|^2 \\ &\ge \frac{1}{L^2}e_r(t,\tau)V^2(\tau,x) \ge \frac{1}{K^2L^2}e_r(t,\tau)\|x\|^2, \qquad t \ge \tau,\ t,\tau \in \mathbf{T}.\end{aligned}$$

Therefore,

$$\|T(\tau,t)|_{F_t^u}\|^2 \le K^2L^2 e_{\ominus r}(t,\tau).$$

By (3.18), we get

$$\|T(\tau,t)|_{F_t^u}\| \le KLe_{\ominus a}(t,\tau) = KLe_a(\tau,t). \tag{3.23}$$

Next, we estimate $\|P(t)\|$ and $\|Q(t)\|$. From the definitions of $P(t)$, $Q(t)$, the quadratic Lyapunov function and (H5), we have

$$V^2(t,P(t)x)=\langle S(t)P(t)x,P(t)x\rangle\geq\frac{1}{K^2}\|P(t)x\|^2, \tag{3.24}$$

$$V^2(t,Q(t)x)=-\langle S(t)Q(t)x,Q(t)x\rangle\geq\frac{1}{K^2}\|Q(t)x\|^2. \tag{3.25}$$

Since $S(t)$ is symmetric, by (3.24) and (3.25), we obtain

$$\begin{aligned}&\frac{1}{K^2}\Big\|P(t)x-\frac{K^2}{2}S(t)x\Big\|^2+\frac{1}{K^2}\Big\|Q(t)x+\frac{K^2}{2}S(t)x\Big\|^2\\ \leq{}&\langle S(t)P(t)x,P(t)x\rangle-\langle S(t)Q(t)x,Q(t)x\rangle\\ &+\frac{K^4}{2}\|S(t)x\|^2-\langle S(t)x,P(t)x\rangle+\langle S(t)x,Q(t)x\rangle\\ ={}&\frac{K^4}{2}\|S(t)x\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\|P(t)x\|&\leq\Big\|P(t)x-\frac{K^2}{2}S(t)x\Big\|+\Big\|\frac{K^2}{2}S(t)x\Big\|\\ &\leq\sqrt{2}K^2\|S(t)x\| &(3.26)\\ &\leq\sqrt{2}K^2L^2\|x\| &(3.27)\end{aligned}$$

and

$$\begin{aligned}\|Q(t)x\|&\leq\Big\|Q(t)x-\frac{K^2}{2}S(t)x\Big\|+\Big\|\frac{K^2}{2}S(t)x\Big\|\\ &\leq\sqrt{2}K^2\|S(t)x\| &(3.28)\\ &\leq\sqrt{2}K^2L^2\|x\|. &(3.29)\end{aligned}$$

By (3.22) and (3.27), we have

$$\|T(t,\tau)P(\tau)x\|\leq\|T(t,\tau)|_{F_\tau^s}\|\|P(\tau)x\|\leq\sqrt{2}K^3L^3e_{\ominus a}(t,\tau)\|x\|,\qquad t\geq\tau,\quad t,\tau\in\mathbf{T}.$$

Then

$$\|T(t,\tau)P(\tau)\|\leq De_{\ominus a}(t,\tau)\|x\|,\qquad t\geq\tau,\quad t,\tau\in\mathbf{T}, \tag{3.30}$$

where $D=\sqrt{2}K^3L^3$.

By (3.23) and (3.29), we have

$$\|T(t,\tau)Q(\tau)x\|\leq\|T(t,\tau)|_{F_\tau^u}\|\|Q(\tau)x\|\leq\sqrt{2}K^3L^3e_{a}(t,\tau)\|x\|,\qquad t\leq\tau,\quad t,\tau\in\mathbf{T}.$$

Then

$$\|T(t,\tau)Q(\tau)\|\leq De_{a}(t,\tau)\|x\|,\qquad t\leq\tau,\quad t,\tau\in\mathbf{T}, \tag{3.31}$$

where $D=\sqrt{2}K^3L^3$. Therefore, by (3.30) and (3.31), the equation (3.1) admits an exponential dichotomy.

4 Instability of the Solution to the Nonlinear Perturbed Equation

In this section, we discuss the instability of the nonlinear equation

$$x^{\Delta}=A(t)x+f(t,x), \tag{4.1}$$

where $f : \mathbf{T} \times \mathbf{R}^n \to \mathbf{R}^n$ is an rd-continuous function in $\mathbf{T}$ satisfying $f(t, 0) = 0$. We have the following theorem.

Theorem 4.1 *Assume that the equation* (3.1) *admits a strong exponential dichotomy with projections* $P(t)$ *for* $t \in \mathbf{T}$ *and constants* $b \geq a > 0$ *and* $D > 1$. *There exists* $R > 0$ *such that*

$$\|f(t, x) - f(t, y)\| \leq R\|x - y\|, \qquad t \in \mathbf{T}, \quad x, y \in \mathbf{R}^n.$$

Assume also that $Q(\sigma(t))f(t, x) = Q(\sigma(t))f(t, Q(t)x)$, *where* $Q(t) = I - P(t)$. *If* R *is sufficiently small, then there exist constants* $N > 0$ *and* $d > 0$ *such that*

$$\|z(t)\|^2 \geq N e_d(t, \tau)\|z(\tau)\|^2, \qquad t \geq \tau, \quad t \in \mathbf{T}, \tag{4.2}$$

whenever $z(\tau) \in F^u_\tau$. *Here* F^u_τ *is the same as the one in the proof of Theorem* 3.2.

Proof. Set $H(t, x) = \langle S(t)x, x\rangle$, $t \in \mathbf{T}$, $x \in \mathbf{R}^n$ with $S(t)$ given by (3.8). If $x \in F^u_\tau$, we have $x(t) = T(t, \tau)x \in F^u_t$ for each $t \geq \tau$, $t, \tau \in \mathbf{T}$, and

$$\begin{aligned} H(t, x(t)) &= -\int_{-\infty}^{t} \|T(\sigma(v), \tau)Q(\tau)x\|^2 e_{a\ominus c}(t, \sigma(v))\Delta v \\ &\leq -\int_{-\infty}^{\tau} \|T(\sigma(v), \tau)Q(\tau)x\|^2 e_{a\ominus c}(\tau, \sigma(v))\Delta v e_{a\ominus c}(t, \tau) \\ &= e_{a\ominus c}(t, \tau)H(\tau, x). \end{aligned}$$

Then

$$\begin{aligned} \mu(t)H^\Delta(t, x(t)) &= H(\sigma(t), x(\sigma(t))) - H(t, x(t)) \\ &\leq [e_{a\ominus c}(\sigma(t), t) - 1]H(t, x(t)) \\ &= \Big[\exp\Big\{\int_t^{\sigma(t)} \frac{1}{\mu(t)} \ln(1 + \mu(v)a \ominus c)\Delta v\Big\} - 1\Big]H(t, x(t)) \\ &= \mu(t)(a \ominus c)H(t, x(t)). \end{aligned}$$

Thus,

$$H^\Delta(t, x(t)) \leq (a \ominus c)H(t, x(t)). \tag{4.3}$$

Since $H(t, x(t)) = \langle S(t)x(t), x(t)\rangle$, we have

$$\begin{aligned} H^\Delta(t, x(t)) &= \langle S(t)x(t), x(t)\rangle^\Delta \\ &= \langle S^\Delta(t)x(t), x(t)\rangle + \langle S(\sigma(t))x^\Delta(t), x(t)\rangle + \langle S(\sigma(t))x(\sigma(t)), x^\Delta(t)\rangle \\ &= \langle [S^\Delta(t) + S(\sigma(t))A(t) + A^*(t)S(\sigma(t)) + \mu(t)A^*(t)S(\sigma(t))A(t)]x(t), x(t)\rangle. \end{aligned}$$

Then by (4.3) we get

$$S^\Delta(t) + S(\sigma(t))A(t) + A^*(t)S(\sigma(t)) + \mu(t)A^*(t)S(\sigma(t))A(t) \leq (a \ominus c)S(t).$$

Suppose that $y(t)$ is a solution of the equation (4.1) satisfying $y(\tau) \in F^u_\tau$. Let $z(t) := Q(t)y(t)$. By (3.2), we know that

$$Q(t) = T(t, s)Q(s)T(s, t). \tag{4.4}$$

Differentiating (4.4) on both sides, we obtain

$$Q^\Delta(t) = A(t)Q(t) - Q(\sigma(t))A(t).$$

Then

$$z^\Delta(t) = Q^\Delta(t)y(t) + Q(\sigma(t))y^\Delta(t)$$

$$= A(t)Q(t)y(t) + Q(\sigma(t))f(t, y(t)).$$

Hence, we have

$$\begin{aligned}
H^\Delta(t, z(t)) &= \langle S(t)Q(t)y(t), Q(t)y(t)\rangle^\Delta \\
&= \langle S^\Delta(t)Q(t)y(t), Q(t)y(t)\rangle + \langle S(\sigma(t))Q^\Delta(t)y(t), Q(t)y(t)\rangle \\
&\quad + \langle S(\sigma(t))Q(\sigma(t))y^\Delta(t), Q(t)y(t)\rangle \\
&\quad + \langle S(\sigma(t))Q(\sigma(t))y(\sigma(t)), Q^\Delta(t)y(t)\rangle \\
&\quad + \langle S(\sigma(t))Q(\sigma(t))y(\sigma(t)), Q(\sigma(t))y^\Delta(t)\rangle. \\
&= \langle S^\Delta(t)z(t), z(t)\rangle + \langle S(\sigma(t))A(t)z(t), z(t)\rangle \\
&\quad + \langle S(\sigma(t))Q(\sigma(t))f(t, z(t)), z(t)\rangle \\
&\quad + \langle S(\sigma(t))z(\sigma(t)), A(t)z(t)\rangle \\
&\quad + \langle S(\sigma(t))z(\sigma(t)), Q(\sigma(t))f(t, z(t))\rangle \\
&= \langle [S^\Delta(t) + S(\sigma(t))A(t) + A^*(t)S(\sigma(t)) \\
&\quad + \mu(t)A^*(t)S(\sigma(t))A(t)]z(t), z(t)\rangle \\
&\quad + \langle [\mu(t)A(t)z(t) + z(t) + z(\sigma(t))], S(\sigma(t))Q(\sigma(t))f(t, z(t))\rangle \\
&\le (a \ominus c)H(t, z(t)) + \langle 2\mu(t)A(t)z(t), S(\sigma(t))Q(\sigma(t))f(t, z(t))\rangle \\
&\quad + \langle 2z(t), S(\sigma(t))Q(\sigma(t))f(t, z(t))\rangle \\
&\quad + \langle \mu(t)S(\sigma(t))Q(\sigma(t))f(t, z(t)), Q(\sigma(t))f(t, z(t))\rangle. \qquad (4.5)
\end{aligned}$$

Since $S(t)$ is defined by (3.8) and $P(t)Q(t) = 0$, we have

$$\begin{aligned}
&\langle \mu(t)S(\sigma(t))Q(\sigma(t))f(t, z(t)), Q(\sigma(t))f(t, z(t))\rangle \\
= &-\mu(t)\int_{-\infty}^{\sigma}(t)\|T(\sigma(v), \sigma(\tau))Q(\sigma(\tau))Q(\sigma(t))f(t, z(t))\|^2 e_{a\ominus c}(\sigma(t), \sigma(v))\Delta v \le 0. \qquad (4.6)
\end{aligned}$$

By the formula of variation of constant, it follows that

$$T(\sigma(t), t) = I + \int_t^{\sigma(t)} A(s)T(s, t)\Delta s = Id + \mu(t)A(t)T(t, t) = Id + \mu(t)A(t).$$

From (3.11), we have

$$\|T(\sigma(t), t)\| \le De_{\ominus a}(\sigma(t), t) + De_b(\sigma(t), t) = \frac{D}{1 + \mu(t)a} + D + Db\mu(t).$$

Therefore,

$$\|Id + \mu(t)A(t)\| \le \frac{D}{1 + \mu(t)a} + D + Db\mu(t) \le 2D + DbM.$$

Then we have

$$\|\mu(t)A(t)\| \le \|Id + \mu(t)A(t)\| + \|Id\| \le 2D + DbM + 1. \qquad (4.7)$$

From (3.10), we get

$$\|S(t)\| \le \Big(\frac{2}{a + c} + M\Big)D^2. \qquad (4.8)$$

By (4.5)–(4.8),

$$\begin{aligned}
H^\Delta(t, z(t)) \le (a \ominus c)H(t, z(t)) &+ 2\|\mu(t)A(t)\|\|z(t)\|\|S(\sigma(t))\|\|Q(\sigma(t))\|\|f(t, z(t))\| \\
&+ 2\|z(t)\|\|S(\sigma(t))\|\|Q(\sigma(t))\|\|f(t, z(t))\|
\end{aligned}$$

$$\begin{aligned}
&\le (a\ominus c)H(t,z(t)) + 2(2D+DbM+2)DR\Big(\frac{2}{a+c}+M\Big)D^2\|z(t)\|^2\\
&\le (a\ominus c)H(t,z(t)) + 2K^2D^3R(2D+DbM+2)\Big(\frac{2}{a+c}+M\Big)|H(t,z(t))|\\
&= (a\ominus c)H(t,z(t)) + E|H(t,z(t))| \le (E-a+c)|H(t,z(t))|, \qquad (4.9)
\end{aligned}$$

where

$$E = 2K^2D^3R(2D+DbM+2)\Big(\frac{2}{a+c}+M\Big).$$

Set $u(t) := H(t,z(t))$. Let R be sufficiently small so that $a-c-E>0$. Thus, by (4.9) we have $u^{\Delta}(t) \le (a-c-E)u(t)$. Then

$$\begin{aligned}
[u(t)e_{\ominus(a-c-E)}(t,\tau)]_t^{\Delta} &= u^{\Delta}(t)e_{\ominus(a-c-E)}(\sigma(t),\tau) + u(t)(\ominus(a-c-E))e_{\ominus(a-c-E)}(t,\tau)\\
&\le [u^{\Delta}(t) - (a-c-E)u(t)]e_{\ominus(a-c-E)}(t,\tau) \le 0.
\end{aligned}$$

Therefore,

$$u(t)e_{\ominus(a-c-E)}(t,\tau) - u(\tau) = \int_\tau^t [u(v)e_{\ominus(a-c-E)}(v,\tau)]_v^{\Delta}\Delta v \le 0.$$

That is, $H(t,z(t)) \le e_{a-c-E}(t,\tau)H(\tau,z(\tau)) \le 0$. It follows that

$$|H(t,z(t))| \ge e_{a-c-E}(t,\tau)|H(\tau,z(\tau))|. \qquad (4.10)$$

By (3.10), (3.15) and (4.10), we have

$$\|z(t)\|^2 \ge \frac{|H(t,z(t))|}{\Big(\frac{2}{a+c}+M\Big)D^2} \ge \frac{e_{a-c-E}(t,\tau)|H(\tau,z(\tau))|}{\Big(\frac{2}{a+c}+M\Big)D^2} \ge \frac{e_{a-c-E}(t,\tau)\|z(\tau)\|^2}{K^2D^2\Big(\frac{2}{a+c}+M\Big)}.$$

Let $N := \dfrac{1}{K^2D^2\Big(\frac{2}{a+c}+M\Big)}$ and $d = a-c-E$. Then (4.2) holds. The proof is completed.

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