Communications in Mathematical Research
27(1)(2011), 37–46

A Pseudo-parabolic Type Equation with Nonlinear Sources*

Li Ying-hua[1], Cao Yang[2] and Wang Yi-fu[3]

(*1. School of Mathematical Sciences, South China Normal University, Guangzhou, 510631*)

(*2. School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024*)

(*3. Department of Mathematics, Beijing Institute of Technology, Beijing, 100081*)

Communicated by Yin Jing-xue

Abstract: This paper is concerned with the existence and uniqueness of nonnegative classical solutions to the initial-boundary value problems for the pseudo-parabolic equation with strongly nonlinear sources. Furthermore, we discuss the asymptotic behavior of solutions as the viscosity coefficient k tends to zero.

Key words: pseudo-parabolic equation, existence, uniqueness, asymptotic behavior

2000 MR subject classification: 35G30, 35B40

Document code: A

Article ID: 1674-5647(2011)01-0037-10

1 Introduction

In this paper, we investigate the existence, uniqueness and asymptotic behavior of solutions to the following initial-boundary value problem for the pseudo-parabolic equation in one spatial dimension:

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D^2 u + m(x,t)u^q, \qquad (x,t) \in Q, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \qquad t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in [0,1], \tag{1.3}$$

where $q > 1$, $Q \equiv (0,1) \times \mathbf{R}^+$, $D = \partial/\partial x$, $k > 0$ represents the viscosity coefficient, $m(x,t) \in C^{\alpha,\alpha/3}(\overline{Q})$ for some $\alpha \in (0,1)$ and satisfies $0 < \underline{m} \leq m(x,t) \leq \overline{m}$ for any $(x,t) \in Q$, $\underline{m}$ and $\overline{m}$ are positive constants.

The pseudo-parabolic equations are characterized by the occurrence of mixed third order derivatives, more precisely, second order in space and first order in time. Such equations are used to model heat conduction in two-temperature systems (see [1] and [2]), fluid flow in porous media (see [3] and [4]), two phase flow in porous media with dynamical capillary

*Received date: April 18, 2009.
Foundation item: The NSFC, CPSF, SRFDP and 973 Program (2010CB808002).

pressure (see [5]), and the populations with the tendency to form crowds (see [6] and [7]). Furthermore, according to experimental results, some researchers have recently proposed modifications to Cahn's model which incorporate out-of-equilibrium viscoelastic relaxation effects, and thus obtained this type of equations (see [8]). The pseudo-parabolic equations with strongly nonlinear sources we considered may arise in the study of nonstationary processes in semiconductors with sources of free-charge currents (see [9] and [10]). The processes can be described by the system that has the explicit form of the constitutive equations connecting the electric field strength E with the electric flux density G on the one hand, and the electric field strength E with the current density J in the semiconductor on the other hand, as follows:

$$\mathrm{div}G = 4\pi e n, \qquad E = -\nabla u, \qquad G = E + 4\pi P,$$
$$\frac{\partial n}{\partial t} = -\mathrm{div}J + Q, \quad J_i = \sigma_i E_i, \qquad i = 1, 2, \cdots, N,$$

where P is the polarization vector and in some models there has the following phenomenological relation $\mathrm{div}P = k_1 u$, $k_1 > 0$, n is the free electron concentration, u is the electric potential, and σ_i is the conductivity tensor. Finally, assume that, in a semiconductor, there are sources of free-charge currents whose distribution in the self-consistent electric field of the semiconductor is of the form $Q = m(x, t)u^p$. By differentiating both sides of the first equation with respect to t and taking account of the second equation, the above system can be reduced to the equation (1.1).

The pseudo-parabolic equations have been extensively investigated. In [11]–[13], the authors investigated the initial-boundary value problem and the Cauchy problem for the linear pseudo-parabolic equation and established the existence and uniqueness of solutions. The nonlinear pseudo-parabolic type equations with undefined or uninvertible operator at the highest derivative with respect to time were studied in [14]. The degenerate and quasilinear degenerate pseudo-parabolic type equations were investigated in [15] and [16]. For the local solvability of the pseudo-parabolic type equations with variety nonlocal boundary conditions, see [17]–[19].

For pseudo-parabolic equations, classical maximum principle is invalid in general. For the nonnegativity of a solution, not only nonnegative initial data, but also an extra condition on the elliptic operator is needed (see [20]–[22]). Due to the special type of the problem (1.1)–(1.3) which is included in the studies of [22], we can prove the comparison principle of solutions, which enables us to obtain the existence of nonnegative solutions to the problem (1.1)–(1.3). For the asymptotic behavior of solutions, we know that in certain cases, the solution of a parabolic initial-boundary value problem can be obtained as a limit of solutions to the problem of the corresponding pseudo-parabolic equations, see [11]. In this paper, we show that the semilinear pseudo-parabolic equations still retain this character, namely, the solutions of the pseudo-parabolic equations converge to the solution of the parabolic equation as $k \to 0$.

This paper is organized as follows. In Section 2, we show the existence and uniqueness of nonnegative classical solutions to the initial-boundary value problem (1.1)–(1.3). Then, in Section 3, we discuss the asymptotic behavior of solutions as the viscosity coefficient k tends to zero.

2 The Initial-boundary Value Problem

In this section, we discuss the solvability of the initial-boundary value problem (1.1)–(1.3). Firstly, we prove the uniqueness of the solution.

Theorem 2.1 *If the initial-boundary value problem* (1.1)–(1.3) *admits classical solutions, then for any given initial datum* $u_0 \in C^{2+\alpha}[0,1]$ *with* $u_0(0) = u_0(1) = 0$*, the solution of the problem* (1.1)–(1.3) *is unique.*

Proof. Let u_1, u_2 be two solutions of the problem (1.1)–(1.3). Set $v = u_1 - u_2$. Then we have

$$\begin{aligned}
&\frac{\partial v}{\partial t} - k\frac{\partial D^2 v}{\partial t} = D^2 v + qm(x,t)\left(u_1 + \theta(u_2 - u_1)\right)^{q-1} v, && (x,t) \in Q_T,\\
&v(0,t) = v(1,t) = 0, && t \in [0,T],\\
&v(x,0) = 0, && x \in [0,1],
\end{aligned}$$

where $\theta \in (0,1)$, $T > 0$ is a given constant, and $Q_T = (0,1) \times (0,T)$. Multiplying the first equation by v, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(v^2 + k|Dv|^2\right) \mathrm{d}x + \int_0^1 |Dv|^2 \mathrm{d}x\\
=\ &\int_0^1 qm(x,t)(u_1 + \theta(u_2 - u_1))^{q-1} v^2 \mathrm{d}x.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(v^2 + k|Dv|^2\right) \mathrm{d}x &\le C\int_0^1 v^2 \mathrm{d}x\\
&\le C\int_0^1 (v^2 + k|Dv|^2)\mathrm{d}x, \qquad t \in (0,T).
\end{aligned}$$

Noticing that

$$v(x,0) = Dv(x,0) = 0,$$

by the Gronwall inequality, we obtain

$$\int_0^1 \left(v^2(x,t) + k|Dv(x,t)|^2\right) \mathrm{d}x \le 0, \qquad t \in (0,T).$$

Consequently,

$$\int_0^1 v^2(x,t)\mathrm{d}x \le 0, \qquad t \in (0,T),$$

which implies that $u_1 = u_2$ a.e. in Q_T. The proof of this theorem is completed.

Next, we prove that the solutions of the problem (1.1)–(1.3) admit comparison principle. DiBenedetto and Pierre[22] discussed the maximum principle for pseudo-parabolic equations including the equation

$$\frac{\partial}{\partial t}(u - k\Delta u) - \Delta u = f, \qquad (x,t) \in \Omega \times [0,T]$$

subject to the conditions

$$u|_{\partial\Omega} = g(t), \qquad u(0) = u_0,$$

where Ω is a bounded open set in $\mathbf{R}^n$, and $T > 0$ is a given constant. Denote

$$E = \{u \in H^2(\Omega);\ u_0|_{\partial\Omega} = g(0)\}.$$

Their argument contains the following conclusion (see [22], P.14).

Lemma 2.1 *The following propositions are equivalent*:

(i) *For any* $u_0 \in E,\ \ f \in L^1(0,T;L^2(\Omega))$,

$$u_0 \geq 0,\ f(t) \geq 0 \quad a.e.\ t \in [0,T] \Rightarrow u_k(t,u_0,f) \geq 0, \qquad t \in [0,T].$$

(ii) *For any* $t \in [0,T],\ \ g(t) \geq \mathrm{e}^{-t/k}g(0)$.

In virtue of that the boundary value conditions (1.2) imply $g(t) = 0$ for any $t \in [0,T]$, the proposition (ii) of Lemma 2.1 holds. Therefore, we obtain the comparison principle as follows.

Proposition 2.1 *Let u_1 and u_2 be two solutions of the initial-boundary value problem* (1.1)–(1.3) *in Q_T with initial data u_{10} and u_{20} respectively, $u_{10}, u_{20} \in C^{2+\alpha}[0,1]$. If $u_{10} \geq u_{20} \geq 0$ in $(0,1)$, then $u_1 \geq u_2 \geq 0$ in Q_T.*

Proof. We first prove that u_1 and u_2 are nonnegative. For this purpose, we consider the following problem:

$$\frac{\partial \hat{u}_1}{\partial t} - k\frac{\partial D^2\hat{u}_1}{\partial t} = D^2\hat{u}_1 + m(x,t)(\hat{u}_{1+})^q, \qquad (x,t) \in Q_T, \tag{2.1}$$

$$\hat{u}_1(0,t) = \hat{u}_1(1,t) = 0, \qquad t \in [0,T],$$

$$\hat{u}_1(x,0) = u_{10}(x), \qquad x \in [0,1],$$

where $s_+ = \max\{s,\ 0\}$. Since

$$m(x,t)(\hat{u}_{1+})^q \geq 0, \qquad u_{10}(x) \geq 0,$$

and the zero boundary value conditions admit the proposition (ii) of Lemma 2.1, by virtue of Lemma 2.1 we see that $\hat{u}_1 \geq 0$ in Q_T. Hence, $\hat{u}_{1+}$ in the equation (2.1) is $\hat{u}_1$ itself. Because the solution of the problem (1.1)–(1.3) is unique, we have

$$u_1 = \hat{u}_1 \geq 0 \qquad \text{in } Q_T.$$

Similarly, we can also prove

$$u_2 \geq 0 \qquad \text{in } Q_T.$$

Set

$$v = u_1 - u_2.$$

Then v satisfies

$$\frac{\partial v}{\partial t} - k\frac{\partial D^2 v}{\partial t} = D^2 v + qm(x,t)(\theta u_1 + (1-\theta)u_2)^{q-1}v, \qquad (x,t) \in Q_T, \tag{2.2}$$

$$v(0,t) = v(1,t) = 0, \qquad t \in [0,T], \tag{2.3}$$

$$v(x,0) = u_{10}(x) - u_{20}(x), \qquad x \in [0,1], \tag{2.4}$$

where $\theta \in (0,1)$. Consider the following equation:

$$\frac{\partial \hat{v}}{\partial t} - k\frac{\partial D^2\hat{v}}{\partial t} = D^2\hat{v} + qm(x,t)(\theta u_1 + (1-\theta)u_2)^{q-1}\hat{v}_+, \qquad (x,t) \in Q_T. \tag{2.5}$$

Noticing that $u_1 \geq 0$, $u_2 \geq 0$, $u_{10}(x) \geq u_{20}(x)$ and the boundary value of v equals to zero, by Lemma 2.1 we have the solution of the problem (2.5)–(2.3)–(2.4) $\hat{v} \geq 0$ in Q_T. Consequently, we see that $\hat{v}_+ = \hat{v}$ in the equation (2.5). Similarly to the proof of Theorem 2.1, we can prove that the solution of the problem (2.2)–(2.4) is unique. Therefore, the solution of the problem (2.2)–(2.4) $v = \hat{v} \geq 0$ in Q_T, i.e., $u_1 \geq u_2$ in Q_T. The proof of this proposition is completed.

In what follows, we prove the existence of solutions to the problem (1.1)–(1.3). For this purpose, we first construct a uniformly bounded supersolution of the problem (1.1)–(1.3), which enables us to obtain the upper bound of solutions. The definition of a supersolution is as follows.

Definition 2.1 *A function $\overline{u}$ is called a supersolution of the initial-boundary value problem* (1.1)–(1.3) *provided that*

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial t} - k\frac{\partial D^2\overline{u}}{\partial t} \geq D^2\overline{u} + m(x,t)\overline{u}^q, && (x,t) \in Q,\\
&\overline{u}(0,t) \geq 0, \quad \overline{u}(1,t) \geq 0, && t \geq 0,\\
&\overline{u}(x,0) \geq u_0(x), && x \in [0,1].
\end{aligned}$$

Let $\overline{u}(x,t) = \alpha(2-x^2)$, where α is a positive constant to be determined. Then, $\overline{u}$ satisfies

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial t} - k\frac{\partial D^2\overline{u}}{\partial t} - D^2\overline{u} - m(x,t)\overline{u}^q = 2\alpha - m(x,t)\alpha^q(2-x^2)^q, && (x,t) \in Q,\\
&\overline{u}(0,t) = 2\alpha, \quad \overline{u}(1,t) = \alpha, && t \geq 0,\\
&\overline{u}(x,0) = \alpha(2-x^2), && x \in [0,1].
\end{aligned}$$

We choose α to satisfy $0 < \alpha \leq \frac{1}{2}\left(\frac{1}{\overline{m}}\right)^{\frac{1}{q-1}}$, where $\overline{m}$ is the upper bound of $m(x,t)$. When $\max\limits_{(0,1)} u_0 \leq \frac{1}{2}\left(\frac{1}{\overline{m}}\right)^{\frac{1}{q-1}}$, it is easy to verify that $\overline{u}$ is a supersolution of the problem (1.1)–(1.3).

If u is the solution of the problem (1.1)–(1.3), and set $v = \overline{u} - u$, then we have

$$\begin{aligned}
&\frac{\partial v}{\partial t} - k\frac{\partial D^2 v}{\partial t} - D^2 v \geq m(x,t)(\theta\overline{u} + (1-\theta)u)^{q-1}v, && (x,t) \in Q_T,\\
&v(0,t) = 2\alpha, \quad v(1,t) = \alpha, && t \in [0,T],\\
&v(x,0) = \alpha(2-x^2) - u_0(x) \geq 0, && x \in [0,1].
\end{aligned}$$

Noticing that $v(0,t)$ and $v(1,t)$ satisfy the proposition (ii) of Lemma 2.1, similar to the proof of Proposition 2.1, we can obtain $v(x,t) \geq 0$ in Q_T, i.e., $\overline{u}(x,t) \geq u(x,t) \geq 0$ in Q_T. Consequently, the following lemma holds.

Lemma 2.2 *For any given initial datum $u_0 \in C^{2+\alpha}[0,1]$ with $u_0(0) = u_0(1) = 0$, $u_0 \geq 0$, if u_0 is appropriately small, u is the solution of the initial-boundary value problem* (1.1)–(1.3)*, then*

$$0 \leq u(x,t) \leq M_2, \qquad (x,t) \in Q_T, \tag{2.6}$$

where M_2 is a positive constant independent of u and k.

By means of the above lemma, we show the global existence of solutions to the problem (1.1)–(1.3).

Theorem 2.2 *For any given initial datum $u_0 \in C^{2+\alpha}[0,1]$ with $u_0(0) = u_0(1) = 0$, $u_0 \geq 0$, if u_0 is appropriately small, then the problem (1.1)–(1.3) admits at least one nonnegative classical solution.*

Proof. We prove this theorem by employing the Leray-Schauder fixed point theorem. Consider a family of relevant equations with parameter, namely

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D^2 u + \sigma m(x,t)u^q, \qquad (x,t) \in Q_T \tag{2.7}$$

subject to the conditions

$$u(0,t) = u(1,t) = 0, \qquad t \in [0,T], \tag{2.8}$$

$$u(x,0) = \sigma u_0(x), \qquad x \in [0,1], \tag{2.9}$$

where $\sigma \in [0,1]$ is a parameter. Here and below, we denote by C a constant, whose value may be different from line to line and is independent of u and σ. Multiplying (2.7) by u and integrating the result over Q_t, we have

$$\begin{aligned}&\frac{1}{2}\int_0^1 u^2(x,t)\mathrm{d}x + \frac{k}{2}\int_0^1 |Du(x,t)|^2\mathrm{d}x + \iint_{Q_t} |Du|^2\mathrm{d}x\mathrm{d}s\\ &= \frac{\sigma^2}{2}\int_0^1 u_0^2(x)\mathrm{d}x + \frac{k}{2}\sigma^2\int_0^1 |Du_0(x)|^2\mathrm{d}x + \sigma\iint_{Q_t} m(x,s)u^{q+1}\mathrm{d}x\mathrm{d}s.\end{aligned}$$

By (2.6), we obtain

$$\int_0^1 |Du(x,t)|^2\mathrm{d}x \leq C, \qquad t \in (0,T), \tag{2.10}$$

$$\iint_{Q_T} |Du|^2\mathrm{d}x\mathrm{d}t \leq C. \tag{2.11}$$

Multiplying (2.7) with D^2u and integrating over Q_t, we get

$$\begin{aligned}&\frac{1}{2}\int_0^1 |Du(x,t)|^2\mathrm{d}x + \frac{k}{2}\int_0^1 |D^2u(x,t)|^2\mathrm{d}x + \iint_{Q_t} |D^2u|^2\mathrm{d}x\mathrm{d}s\\ &= \frac{\sigma^2}{2}\int_0^1 |Du_0(x)|^2\mathrm{d}x + \frac{k}{2}\sigma^2\int_0^1 |D^2u_0(x)|^2\mathrm{d}x - \sigma\iint_{Q_t} m(x,s)u^q D^2u\mathrm{d}x\mathrm{d}s\\ &\leq \frac{1}{2}\iint_{Q_t} |D^2u|^2\mathrm{d}x\mathrm{d}s + C,\end{aligned}$$

which implies that

$$\int_0^1 |D^2u(x,t)|^2\mathrm{d}x \leq C, \qquad t \in (0,T), \tag{2.12}$$

$$\iint_{Q_T} |D^2u|^2\mathrm{d}x\mathrm{d}t \leq C. \tag{2.13}$$

Combining (2.10) with (2.12), we conclude that

$$\|Du\|_{L^\infty(Q_T)} \leq C. \tag{2.14}$$

Multiplying (2.7) with $\dfrac{\partial u}{\partial t}$ and integrating over Q_t yield

$$\begin{aligned}
&\iint_{Q_t}\left|\frac{\partial u}{\partial t}\right|^2\mathrm{d}x\mathrm{d}s+k\iint_{Q_t}\left|\frac{\partial Du}{\partial t}\right|^2\mathrm{d}x\mathrm{d}s+\frac{1}{2}\int_0^1|Du(x,t)|^2\mathrm{d}x\\
&=\frac{\sigma^2}{2}\int_0^1|Du_0(x)|^2\mathrm{d}x+\sigma\iint_{Q_t}m(x,t)u^q\frac{\partial u}{\partial t}\mathrm{d}x\mathrm{d}s\\
&\le\frac{1}{2}\iint_{Q_t}\left|\frac{\partial u}{\partial t}\right|^2\mathrm{d}x\mathrm{d}s+C.
\end{aligned}$$

It follows that

$$\iint_{Q_T}\left|\frac{\partial u}{\partial t}\right|^2\mathrm{d}x\mathrm{d}t\le C. \tag{2.15}$$

We rewrite the equation (2.7) into the form

$$\frac{\partial D^2u}{\partial t}=\frac{1}{k}\frac{\partial u}{\partial t}-\frac{1}{k}D^2u-\frac{\sigma}{k}m(x,t)u^q.$$

Then, we have

$$\iint_{Q_T}\left|\frac{\partial D^2u}{\partial t}\right|^2\mathrm{d}x\mathrm{d}t\le C. \tag{2.16}$$

In the following, we claim that

$$|u(x_1,t_1)-u(x_2,t_2)|\le C(|x_1-x_2|+|t_1-t_2|^{1/3}) \tag{2.17}$$

for any $(x_i,t_i)\in\overline{Q}_T$ $(i=1,2)$. It is obvious that the above inequality is equivalent to

$$|u(x_1,t)-u(x_2,t)|\le C|x_1-x_2|,\qquad t\in[0,T],\quad x_1,x_2\in[0,1], \tag{2.18}$$

$$|u(x,t_1)-u(x,t_2)|\le C|t_1-t_2|^{1/3},\qquad x\in[0,1],\quad t_1,t_2\in[0,T]. \tag{2.19}$$

We can easily obtain (2.18) from (2.14). To prove (2.19), we need only to consider the case of $0\le x\le 1/2$. Let $\Delta t=t_2-t_1>0$ satisfying $(\Delta t)^{1/3}\le 1/4$. Integrating (2.7) over $(y,y+(\Delta t)^{1/3})\times(t_1,t_2)$ gives

$$\begin{aligned}
&\int_y^{y+(\Delta t)^{1/3}}(u(z,t_2)-u(z,t_1))\mathrm{d}z\\
&=k\int_y^{y+(\Delta t)^{1/3}}\int_{t_1}^{t_2}D^2u_t\mathrm{d}t\mathrm{d}z+\int_{t_1}^{t_2}(Du(y+(\Delta t)^{1/3},t)-Du(y,t))\mathrm{d}t\\
&\quad+\sigma\int_y^{y+(\Delta t)^{1/3}}\int_{t_1}^{t_2}m(z,t)u^q\mathrm{d}t\mathrm{d}z.
\end{aligned}$$

Integrating the above equality with respect to y over $(x,x+(\Delta t)^{1/3})$, by (2.6), (2.14), (2.16) and the mean value theorem, we have

$$|u(x^*,t_2)-u(x^*,t_1)|\le C|t_2-t_1|^{1/3},$$

where

$$x^*=y^*+\theta^*(\Delta t)^{1/3},\quad y^*\in(x,x+(\Delta t)^{1/3}),\qquad \theta^*\in(0,1).$$

Combining the above inequality with (2.18), we arrive at

$$\begin{aligned}
|u(x,t_1)-u(x,t_2)|&\le|u(x,t_1)-u(x^*,t_1)|+|u(x^*,t_1)-u(x^*,t_2)|+|u(x,t_2)-u(x^*,t_2)|\\
&\le C|t_1-t_2|^{1/3}.
\end{aligned}$$

Now, we define a linear space X by

$$X = \{u \in C^{1,1/3}(\overline{Q}_T) : \ u(0,t) = u(1,t) = 0, \ \forall t \in [0,T]\},$$

and an operator

$$F : X \longrightarrow X, \qquad u \longmapsto w,$$

where w is a solution of the following linear problem:

$$\begin{aligned}
&\frac{\partial w}{\partial t} - k\frac{\partial D^2 w}{\partial t} = D^2 w + m(x,t)u^q, && (x,t) \in Q_T,\\
&w(0,t) = w(1,t) = 0, && t \in [0,T],\\
&w(x,0) = u_0(x), && x \in [0,1].
\end{aligned}$$

By the classical theory (see [23]), the above linear problem admits a unique solution $w \in C^{2+\beta,1+\beta/3}(\overline{Q}_T)$, $\beta \in (0,1)$. So, the operator F is well-defined. We can also obtain that the operator F is compact by means of the compact embedding theorem. Moreover, if $u = \sigma F u$ holds for some $\sigma \in [0,1]$, then u satisfies the problem (2.7)–(2.9). Clearly $\sigma = 0$ implies $u \equiv 0$. If $\sigma \neq 0$, from the above argument we see that $\|u\|_{C^{1,1/3}(\overline{Q}_T)}$ is bounded and the bound is independent of u and σ. By the Leray-Schauder fixed point theorem, the operator F has a fixed point u, which is the desired classical solution of the problem (1.1)–(1.3) in Q_T. As above, we consider the problem in $Q_{(T,2T)}$, $Q_{(2T,3T)}$, $\cdots$, $Q_{((n-1)T,nT)}$ in turn. Then, we infer that the problem (1.1)–(1.3) admits a classical solution in Q. The proof of this theorem is completed.

3 Asymptotic Behavior

In this section, we discuss the asymptotic behavior of solutions as the viscosity coefficient k tends to zero. Here, we denote by C a constant independent of u and k, and by $C(k)$ a constant independent of u.

Theorem 3.1 *If u_k is a nonnegative classical solution of the initial-boundary value problem* (1.1)–(1.3)*, then $u_k(x,t)$ is uniformly convergent in Q_T as $k \to 0$, and the limit function $u(x,t)$ is a nonnegative classical solution of the following initial-boundary value problem*:

$$\frac{\partial u}{\partial t} = D^2 u + m(x,t)u^q, \qquad (x,t) \in Q_T, \tag{3.1}$$

$$u(0,t) = u(1,t) = 0, \qquad t \in [0,T], \tag{3.2}$$

$$u(x,0) = u_0(x), \qquad x \in [0,1]. \tag{3.3}$$

Proof. From Lemma 2.2, we see that

$$\|u_k\|_{L^\infty(Q_T)} \le M_2.$$

Multiplying (1.1) by $D^2 u_k$ and integrating over Q_t, we have

$$\int_0^1 |Du_k(x,t)|^2 \mathrm{d}x \le C + C(k), \qquad t \in (0,T), \tag{3.4}$$

$$\iint_{Q_T} |D^2 u_k|^2 \mathrm{d}x\mathrm{d}t \le C + C(k). \tag{3.5}$$

Multiplying (1.1) by $\dfrac{\partial u_k}{\partial t}$ and integrating over Q_T, we get

$$\iint_{Q_T}\left|\frac{\partial u_k}{\partial t}\right|^2 \mathrm{d}x\mathrm{d}t \le C. \tag{3.6}$$

Rewrite the equation (1.1) into the following form:

$$k\frac{\partial D^2 u_k}{\partial t} = \frac{\partial u_k}{\partial t} - D^2 u_k - m(x,t)u_k^q.$$

Then, we arrive at

$$k\iint_{Q_T}\left|\frac{\partial D^2 u_k}{\partial t}\right|^2 \mathrm{d}x\mathrm{d}t \le C + C(k). \tag{3.7}$$

By a proof similar to that of Theorem 2.2, from (3.4)–(3.7), we can obtain

$$|u_k(x_1,t_1) - u_k(x_2,t_2)| \le (C + C(k))(|x_1 - x_2|^{1/2} + |t_1 - t_2|^{1/6}) \tag{3.8}$$

for any $(x_i,t_i) \in \overline{Q}_T$ $(i = 1,2)$. Therefore, there exists a function $u \in H^{2,1}(Q_T) \cap C^{1/2,1/6}(\overline{Q}_T)$ such that

$$u_k \to u \quad \text{uniformly in } \ Q_T, \qquad \frac{\partial u_k}{\partial t} \rightharpoonup \frac{\partial u}{\partial t},\ D^2 u_k \rightharpoonup D^2 u \quad \text{in } L^2(Q_T), \tag{3.9}$$

as $k \to 0$. Recalling the equation (1.1), we see that for any $\varphi \in C^2(\overline{Q}_T)$ satisfying $\varphi(0,t) = \varphi(1,t) = 0$, we have

$$\begin{aligned}&\iint_{Q_T}\frac{\partial u_k}{\partial t}\varphi \mathrm{d}x\mathrm{d}t - k\iint_{Q_T}\frac{\partial u_k}{\partial t}D^2\varphi \mathrm{d}x\mathrm{d}t\\ =&\iint_{Q_T} D^2 u_k \varphi \mathrm{d}x\mathrm{d}t + \iint_{Q_T} m(x,t)u_k^q \varphi \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Taking $k \to 0$, by (3.9), we have

$$\iint_{Q_T}\frac{\partial u}{\partial t}\varphi \mathrm{d}x\mathrm{d}t = \iint_{Q_T} D^2 u\varphi \mathrm{d}x\mathrm{d}t + \iint_{Q_T} m(x,t)u^q\varphi \mathrm{d}x\mathrm{d}t,$$

which implies that u satisfies the equation (3.1) in the sense of distribution. Noticing that $u \in C^{1/2,1/6}(\overline{Q}_\omega)$, by the classical theory (see [23]) we can deduce that $u \in C^{2+\beta,1+\beta/3}(\overline{Q}_\omega)$, $\beta \in (0,1/2)$. It is obvious that u is nonnegative and satisfies the conditions (3.2) and (3.3). Hence, $u(x,t)$ is a nonnegative classical solution of the problem (3.1)–(3.3). The proof of this theorem is completed.

References

[1] Chen, P. J. and Gurtin, M. E., On a theory of heat conduction involving two temperatures, *Z. Angew. Math. Phys.*, **19**(1968), 614–627.

[2] Ting, T. W., A cooling process according to two-temperature theory of heat conduction, *J. Math. Anal. Appl.*, **45**(1974), 23–31.

[3] Barenblatt, G. I., Entov, V. M. and Ryzhik, V. M., Theory of Fluid Flow Through Natural Rocks, Kluwer, Dordrecht, 1990.

[4] Barenblatt, G. I., Zheltov, I. and Kochina, I. N., Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks, *J. Appl. Math. Mech.*, **24**(1960), 1286–1303.

[5] Cuesta, C., van Duijn, C. J. and Hulshof, J., Infiltration in porous media with dynamic capillary pressure: Traveling waves, *European J. Appl. Math.*, **4**(2000), 381–397.

[6] Padrón, V., Sobolev regularization of a nonlinear ill-posed parabolic problem as a model for aggregating populations, *Comm. Partial Differential Equations*, **23**(1998), 457–486.

[7] Padrón, V., Effect of aggregation on population recovery modeled by a forward-backward pesudoparabolic equation, *Trans. Amer. Math. Soc.*, **356**(2004), 2739–2756.

[8] Novick-Cohen, A. and Pego, R. L., Stable patterns in a viscous diffusion equation, *Trans. Amer. Math. Soc.*, **324**(1991), 331–351.

[9] Bonch-Bruevich, V. L. and Kalashnikov, S. G., The Physics of Semiconductors [in Russian], Nauka, Moscow, 1990.

[10] Korpusov, M. O. and Sveshnikov, A. G., On the "blow-up" of the solution of a strongly nonlinear equation of pseudoparabolic type with a double nonlinearity (Russian), *Mat. Zametki*, **79**(6)(2006), 879–899; translation in *Math. Notes*, **79**(5-6)(2006), 820–840.

[11] Ting, T. W., Parabolic and pseudo-parabolic partial differential equations, *J. Math. Soc. Japan*, **21**(1969), 440–453.

[12] Showalter, R. E. and Ting, T. W., Pseudoparabolic partial differential equations, *SIAM J. Math. Anal.*, **1**(1970), 1–26.

[13] Gopala Rao V. and Ting, T. W., Solutions of pseudo-heat equations in the whole space, *Arch. Rational Mech. Anal.*, **49**(1972/73), 57–78.

[14] Egorov, I. E., Pyatkov, S. G. and Popov, S. V., Nonclassical Operator-Differential Equations, Nauka, Novosibirsk, 2000.

[15] Ptashnyk, M., Degenerate quasilinear pseudoparabolic equations with memory terms and vairational inequalities, *Nonlinear Anal.*, **66**(2007), 2653–2675.

[16] Scarpini, F., Degenerate and pseudoparabolic variational inequalities, *Numer. Funct. Anal. Optim.*, **9**(1987), 859–879.

[17] Mesloub, S., A nonlinear nonlocal mixed problem for a second order pseudoparabolic equation, *J. Math. Anal. Appl.*, **316**(2006), 189–209.

[18] Dai, D. Q. and Huang, Y., On a nonlocal problem for nonlinear pseudoparabolic equations, *Nonlinear Anal.*, **64**(2006), 499–512.

[19] Bouziani, A., Initial-boundary value problems for a class of pseudoparabolic equations with integral boundary conditions, *J. Math. Anal. Appl.*, **291**(2004), 371–386.

[20] Rundell, W., and Stecher, M., The nonpositivity of solutions to pseudoparabolic equations, *Proc. Amer. Math. Soc.*, **75**(1979), 251–254.

[21] Showalter, R. E., Local regularity, boundary values and maximum principles for pseudoparabolic equations, *Appl. Anal.*, **16**(1983), 235–241.

[22] DiBenedetto, E. and Pierre, M., On the maximum principle for pseudoparabolic equations, *Indiana Univ. Math. J.*, **6**(30)(1981), 821–854.

[23] Ladyzenskaja, O., Solonnikov, V. and Uraltseva, N., Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs, vol. **23**, Amer. Math. Soc., Providence, 1968.