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A Lower Bound of the Genus of a Self-amalgamated 3-manifolds*

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Abstract: Let M be a compact connected oriented 3-manifold with boundary, $Q_1, Q_2 \subset \partial M$ be two disjoint homeomorphic subsurfaces of ∂M, and $h : Q_1 \to Q_2$ be an orientation-reversing homeomorphism. Denote by M_h or $M_{Q_1=Q_2}$ the 3-manifold obtained from M by gluing Q_1 and Q_2 together via h. M_h is called a self-amalgamation of M along Q_1 and Q_2. Suppose Q_1 and Q_2 lie on the same component F' of $\partial M'$, and $F' - Q_1 \cup Q_2$ is connected. We give a lower bound to the Heegaard genus of M when M' has a Heegaard splitting with sufficiently high distance.

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1 Introduction

Let M be a compact connected oriented 3-manifold with boundary, $Q_1, Q_2 \subset \partial M$ be two disjoint homeomorphic subsurfaces of ∂M, and $h : Q_1 \to Q_2$ be an orientation-reversing homeomorphism. Denote by M_h or $M_{Q_1=Q_2}$ the 3-manifold obtained from M by gluing Q_1 and Q_2 together via h. M_h is called a self-amalgamation of M along Q_1 and Q_2. Usually, $Q = Q_1 = Q_2$ is a non-separating surface properly embedded in M_h, and M can be reobtained from M_h by cutting M_h open along Q.

An interesting problem is how the genus of M_h is related to that of M. Here are partial related results:

Theorem 1.1[1] *Let M be a compact orientable 3-manifold, and Q a non-separating incompressible closed surface in M. Let M' be the 3-manifold obtained by cutting M open along Q. Suppose M' admits a Heegaard splitting $V' \cup_{S'} W'$ with $d(S') \geq 2g(M')$. Then $g(M) \geq g(M') - g(F)$.*

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Theorem 1.2[2] *Let M be a closed orientable 3-manifold, and Q a non-separating incompressible closed surface in M. Let M' be the 3-manifold obtained by cutting M open along Q. Suppose M' admits a Heegaard splitting $V' \cup_{S'} W'$ relative to $\partial M'$ with $d(S') > 2(g(M', \partial M') + 2g(Q))$. Then M has a unique minimal Heegaard splitting, i.e., the self-amalgamation of $V' \cup_{S'} W'$.*

Both Theorems 1.1 and 1.2 deal with the case in which the non-separating surface is closed. In the present paper, we consider the situation in which the non-separating surface is with boundary. We obtain a lower bound of the genus of the self-amalgamated 3-manifold under some condition in terms of distances of the previous Heegaard splittings.

The paper is organized as follows. In Section 2, we review some preliminaries which is used later. The statement of the main result and its proof are included in Section 3. All 3-manifolds in this paper are assumed to be compact and orientable.

2 Preliminaries

In this section, we review some fundamental facts on surfaces in 3-manifolds. Definitions and terms which have not been defined are all standard, and the reader is referred to, for example, [3].

A Heegaard splitting of a 3-manifold M is a decomposition

$$M = V \cup_S W$$

in which V and W are compression bodies such that

$$V \cap W = \partial_+ V = \partial_+ W = S$$

and

$$M = V \cup W.$$

S is called a Heegaard surface of M. The genus $g(S)$ of S is called the genus of the splitting $V \cup_S W$. We use $g(M)$ to denote the Heegaard genus of M, which is equal to the minimal genus of all Heegaard splittings of M. A Heegaard splitting $V \cup_S W$ for M is minimal if $g(S) = g(M)$. $V \cup_S W$ is said to be weakly reducible (see [4]) if there are essential disks $D_1 \subset V$ and $D_2 \subset W$ with $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, $V \cup_S W$ is strongly irreducible.

Let

$$M = V \cup_S W$$

be a Heegaard splitting, α and β be two essential simple closed curves in S. The distance $d(\alpha, \beta)$ of α and β is the smallest integer $n \geq 0$ such that there is a sequence of essential simple closed curves

$$\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta$$

in S with $\alpha_{i-1} \cap \alpha_i = \emptyset$, for $1 \leq i \leq n$. The distance of the Heegaard splitting $V \cup_S W$ is defined to be

$$d(S) = \min\{d(\alpha, \beta)\},$$

where α bounds an essential disk in V and β bounds an essential disk in W. $d(S)$ was first defined by Hempel[5].

A properly embedded surface is essential if it is incompressible and not ∂-parallel.

Let P be a properly embedded separating surface in a 3-manifold M which cuts M into two 3-manifolds M_1 and M_2. Then P is bicompressible if P has compressing disks in both M_1 and M_2. P is strongly irreducible if it is bicompressible and each compressing disk in M_1 meets each compressing disk in M_2.

Scharlemann and Thompson[6] showed that any irreducible and ∂-irreducible Heegaard splitting

$$M = V \cup_S W$$

has an untelescoping as

$$V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m),$$

such that each $V_i \cup_{S_i} W_i$ is a strongly irreducible Heegaard splitting with

$$F_i = \partial_- W_i \cap \partial_- V_{i+1}, \qquad 1 \le i \le m-1,\ \partial_- V_1 = \partial_- V,\ \partial_- W_m = \partial_- W,$$

and for each i, each component of F_i is a closed incompressible surface of positive genus, and only one component of $M_i = V_i \cup_{S_i} W_i$ is not a product. It is easy to see that $g(S) \ge g(S_i), g(F_i)$, and when $m \ge 2$, $g(S) \ge g(S_i) + 1 \ge g(F_i) + 2$ for each i. From $V_1 \cup_{S_1} W_1, \cdots, V_m \cup_{S_m} W_m$, we can get a Heegaard splitting of M by a process called amalgamation (see [7]).

The following are some basic facts and results on Heegaard splittings.

Lemma 2.1[8] *Let V be a compression body and F a properly embedded incompressible surface in V. Then each component of $V \setminus F$ is a compression body.*

Lemma 2.2[9] (nested lemma) *Let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting. If α is an essential simple loop in S which bounds a disk D in M such that D is transverse to S, then α bounds an essential disk in V or W.*

Lemma 2.3[10] *Let $V \cup_S W$ be a Heegaard splitting for M and F a properly embedded incompressible surface (maybe not connected) in M. Then any component of F is parallel to ∂M or $d(S) \le 2 - \chi(F)$.*

Lemma 2.4[11] *Let $M = V \cup_S W$ and $M = V' \cup_{S'} W'$ be two different Heegaard splittings. Then $M = V' \cup_{S'} W'$ is a stabilization of $M = V \cup_S W$ or $d(S) \le 2 - g(S')$.*

Lemma 2.5[12] *Let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting and F a 2-side essential surface (not a disk or 2-sphere) in M. Then F can be isotoped so that*

(1) *Each component of $S \cap F$ is an essential loop in both F and S;*

(2) *At most one component of $\overline{S \setminus F}$ is compressible in $\overline{M \setminus F}$.*

3 The Main Result and Its Proof

The following is the main result of the present paper:

Theorem 3.1 *Let M' be a 3-manifold. Let F' be a component of $\partial M'$, and M' not a compression body with $\partial_+ M' = F'$. Let Q_1, Q_2 be two connected non-disk subsurfaces of F' with $Q_1 \cap Q_2 = \emptyset$, $F' - Q_1 \cup Q_2$ be connected, and $h : Q_1 \to Q_2$ be a homeomorphism. Let $M = M'/h$ be the 3-manifold obtained from M' by gluing Q_1 to Q_2 through the homeomorphism h. If M' has a Heegaard splitting $V' \cup_{S'} W'$ with*

$$d(S') \geq 2g(M') - 2g(F'),$$

then

$$g(M) \geq g(M') - g(F').$$

Proof. On the contrary, suppose that M has a Heegaard splitting $V \cup_S W$ such that

$$g(S) < g(M') - g(F').$$

Let

$$Q = h(Q_1) = Q_2.$$

Since M' is not a compression body with

$$\partial_+ M' = F', \qquad d(S') \geq 2g(M') - 2g(F') \geq 2,$$

F' is incompressible in M', and Q is an essential surface in M.

If $V \cup_S W$ is strongly irreducible, then by Proposition 2.5 in [12], we can isotope S and Q so that:

1) $S \cap Q$ are essential circles on both S and Q;

2) $\overline{S \setminus Q}$ has at most one compressible component.

In addition to the above conditions, we may assume that $|S \cap Q|$ is minimal and we can take $N(Q)$ to be sufficiently thin so that $S \cap M' \cong \overline{S \setminus N(Q)}$ has at most one compressible component, say C if there is, and $S \cap N(Q)$ is a collection of annuli. Again denote the two cutting sections of $M \setminus N(Q)$ by Q_1 and Q_2, respectively, and denote $F \setminus (Q_1 \cup Q_2)$ by $\widetilde{F}$.

Claim 1. C does exist.

If otherwise, each component of $S \cap M'$ is incompressible, and some of them, say C', is not boundary parallel. Since Q is incompressible, each component of $S \cap M'$ has non-positive Euler characteristic, and $S \cap N(Q)$ is a collection of annuli, $\chi(C') \geq \chi(S)$, so

$$d(S') \leq 2 - \chi(C') \leq 2 - \chi(S) = 2g(S) < 2g(M') - 2g(F'),$$

contradicting the assumption that

$$d(S') \geq 2g(M') - 2g(F').$$

Hence each component of $S \cap M'$ is boundary parallel to a subsurface of F'. After an isotopy, S is disjoint from a copy of F', and it is easy to see that F' is essential in M, which means that a compression V or W contains a closed essential surface, a contradiction. This completes the proof of Claim1.

Claim 2. $S \cap M' = C$, i.e., $S \cap M'$ has only one component.

If otherwise, there would exist $C^* \subset S \cap M' \setminus C$, which is boundary parallel to a subsurface $\widetilde{C^*} \subset F'$. If $\widetilde{C^*} \subset Q_1$ or Q_2, we can isotope S so that $|S \cap Q|$ is reduced, contradicting the assumption. Hence $\widetilde{C^*} \supset \widetilde{F}$. It is easy to see that

$$C^* \cap Q_1 \neq \emptyset \neq C^* \cap Q_2,$$

and $C^* \cap Q_i$ is separating in Q_i $(i = 1, 2)$. Denote by Q_i^1 the component of $Q_i \setminus (C^* \cap Q_i)$ which is adjacent to $\widetilde{F}$, and Q_i^2 the other. $Q_i^{1,j}$ is similarly defined, $j = 1, 2$. If $h(C^* \cap Q_1) \neq C^* \cap Q_2$, since S cannot intersect itself, without loss of generality, we may assume that $h(C^* \cap Q_1) \subset Q_2^1$. Denote the handlebody bounded by C^* and $\widetilde{C^*}$ by H_{C^*}. It is easy to see that

$$C \cap H_{C^*} = \emptyset,$$

and $h(C^* \cap Q_1) \subset \partial C^{**}$, where $C^{**} \subset S \cap M' \setminus (C \cup C^*)$. If $C^{**} \cap Q_1 = C^{**} \cap Q_1^1 = \emptyset$, we can isotope S to reduce $|S \cap Q|$, so that

$$C^{**} \cap Q_1^1 \neq \emptyset.$$

Continuing this process, we conclude that $S \cap M'$ has infinitely many components. But this is impossible. Hence

$$h(C^*) \cap Q_1 = C^* \cap Q_2,$$

which contradicts the connectedness of S. This completes the proof of Claim2.

Recalling that $S \cap N(Q)$ is a collection of annuli, we have $\chi(S) = \chi(C)$.

Without loss of generality, we may assume that C is compressible in $V \cap M'$. Maximally compress C in $V \cap M'$, obtaining C_V. By nested lemma, we know that C_V is incompressible in M'. If some component of C_V, say C', is not boundary parallel, then

$$d(S') \leq 2 - \chi(C') \leq 2 - \chi(C) = 2 - \chi(S) = 2g(S) < 2g(M') - 2g(F'),$$

a contradiction. So each component of C_V is boundary parallel in M'. By the argument in [1], we know that no two components of C_V are nested. Since $Q \cap V$ ($Q \cap W$, resp.) is essential in V (W, resp.), $V \cap M'$ ($W \cap M'$, resp.) is a compression body. Denote the components of $F' \setminus (C \cap F')$ by $A_1, \cdots, A_k, B_1, \cdots, B_s$, where A_i lies in V, B_j lies in W, $i = 1, \cdots, k$, $j = 1, \cdots, s$. Let S^* be the surface obtained by uniting C and $\bigcup_{j=1}^{k} B_j$ and push it slightly into the interior of M'. S^* cuts M' into two parts, denoted by V^* and W^*, where V^* is homeomorphic to (surfaces) $\times I \cup$1-handles, so V^* is a compression body. W^* is homeomorphic to $(W \cap M') \setminus \bigcup_{j=1}^{k} (B_j \times I)$, which is also a compression body. Hence $V^* \cup_{S^*} W^*$ is Heegaard splitting for M'. So we have $g(S^*) \geq g(M')$, i.e.,

$$\chi(S^*) = 2 - 2g(S^*) \leq 2 - 2g(M').$$

On the other hand, we have

$$\chi(S^*) = \chi(C) + \sum_{j=1}^{k} \chi(B_j) \geq \chi(S) + \chi(F').$$

Combining the above two inequalities we have

$$g(S) \geq g(M') - g(F') + 1,$$

a contradiction.

Hence, $V \cup_S W$ is weakly reducible.

Let $(V_1 \cup_{S_1} W_1) \cup_{H_1} (V_2 \cup_{S_2} W_2) \cup_{H_2} \cdots \cup_{H_{n-1}} (V_n \cup_{S_n} W_n)$ be an untelescoping for $V \cup_S W$ of minimal length. If $Q \cap \bigcup_{i=1}^{n-1} H_i = \emptyset$, then for an arbitrary connected component H_0 of $\bigcup_{i=1}^{n-1} H_i$, which is an essential closed surface in M, that either (1) H_0 is also essential in M', or (2) H_0 is parallel to F', would occur. If (1) occurs, then

$$d(S') \leq 2 - \chi(H_0) \leq 2 - \chi(S) = 2g(S) < 2g(M') - 2g(F'),$$

a contradiction. If (2) occurs, i.e., each component of $\bigcup_{i=1}^{n-1} H_i$ is parallel to F', then there exists an i_0 such that

$$V_{i_0} \cup_{S_{i_0}} W_{i_0} \cong M',$$

and hence

$$g(S) \geq g(S_{i_0}) + 1 \geq g(M') + 1,$$

contradicting the assumption that $g(S) < g(M') - g(F')$.

By similar arguments to the previous paragraphs, we have $n = 2$, H_1 is connected, $H_1 \cap Q \neq \emptyset$, and $H_1 \cap M'$ is parallel to a subsurface of F', which contains $\widetilde{F}$. Now, H_1 cuts off from M a 3-manifold N, which is a compression body with $\partial_+ N = H_1$, $\partial_- N = F$, which contradicts the incompressibility of H_1.

This completes the proof of Thoerem 3.1.

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