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Superderivations for a Family of Lie Superalgebras of Special Type*

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Abstract: By means of generators, superderivations are completely determined for a family of Lie superalgebras of Special type, the tensor products of the exterior algebras and the finite-dimensional Special Lie algebras over a field of characteristic $p > 3$. In particular, the structure of the outer superderivation algebra is concretely formulated and the dimension of the first cohomology group is given.

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1 Introduction

The four families of finite-dimensional simple modular Lie superalgebras of Cartan type were constructed and studied by Zhang[1] in 1997. Now people have obtained many useful results relative to structures and representations of modular Lie superalgebras (see, for example, [2]–[5]). Determining the (super)derivation algebra for a modular Lie (super)algebra is of particular interest, since a centerless Lie (super)algebra, in general, can be embedded into its (super)derivation algebra, which possesses a natural p- or $(p, 2p)$-structure. As is well-known, Lie (super)algebras with such a structure are more manageable and more interesting than the usual ones. Moreover, the p-envelope contained in the (super)derivation algebra can be easily computed. Certain work on the superderivations of modular Lie superalgebras can be found in [5]–[7].

The tensor product of a finite-dimensional Special Lie algebra and an exterior algebra as an associative algebra is a Lie superalgebra, which is called of Special type. This Lie superalgebra is actually isomorphic to a subalgebra of the finite-dimensional Lie superalgebra of Cartan type S. The main result of this paper is the complete determination for

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the superderivation algebras of the special Lie superalgebras, which says that the outer superderivations come from the outer derivations of the Lie algebras of Cartan type S. In particular, the first cohomology groups are determined.

2 Basics

Throughout this paper $\mathbb{F}$ is a field of characteristic $p > 3$. Fix two integers $m, n \geq 2$ and an m-tuple $\underline{t} := (t_1, t_2, \cdots, t_m)$. Let $\mathcal{O}(m,\ \underline{t})$ be the divided power algebra with $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in \mathbb{A}(m,\ \underline{t})\}$, where

$$\mathbb{A} := \mathbb{A}(m,\ \underline{t}) := \{\alpha \in \mathbb{N}_0^m \mid \alpha_i \leq \pi_i,\ i = 1, 2, \cdots, m\}, \qquad \pi_i := p^{t_i} - 1.$$

Let $W(m,\ \underline{t})$ be the generalized Witt algebra, i.e.,

$$W(m,\ \underline{t}) := \sum_{i=1}^{m} \mathcal{O}(m,\ \underline{t})\partial_i,$$

where ∂_i is the derivation of $\mathcal{O}(m,\ \underline{t})$ determined by

$$\partial_i(x^{(\varepsilon_j)}) = \delta_{ij}, \qquad i = 1, 2, \cdots, m.$$

Denote by $\Lambda(n)$ the $\mathbb{F}$-exterior superalgebra in n variables $x_{m+1}, \cdots, x_{m+n}$. Then

$$\mathfrak{W} := \Lambda(n) \otimes W(m,\ \underline{t})$$

is a Lie superalgebra with bracket:

$$[f \otimes D,\ g \otimes H] = fg \otimes [D,\ H], \qquad f, g \in \Lambda(n),\ D, H \in W(m,\ \underline{t}).$$

The natural $\mathbb{Z}$-grading of $\Lambda(n)$ and the standard $\mathbb{Z}$-grading of $W(m,\ \underline{t})$ induce a $\mathbb{Z}$-grading structure of $\mathfrak{W}$ with

$$\mathfrak{W}_i = \sum_{k+l=i} \Lambda(n)_k \otimes W(m,\ \underline{t})_l.$$

Note that $\mathfrak{W}$ is isomorphic to a subalgebra of the generalized Witt superalgebra (see [4]). For simplicity we write fD for

$$f \otimes D, \qquad f \in \Lambda(n),\ D \in W(m,\ \underline{t}).$$

Put

$$\mathrm{I}_0 := \{1, 2, \cdots, m\}, \qquad \mathrm{I}_1 := \{m+1, \cdots, m+n\}$$

and

$$\mathrm{I} := \mathrm{I}_0 \cup \mathrm{I}_1.$$

Let

$$\mathbb{B} := \{\langle i_1, i_2, \cdots, i_k \rangle \mid m+1 \leq i_1 < i_2 < \cdots < i_k \leq m+n\}.$$

For $u = \langle i_1,\ i_2,\ \cdots,\ i_k \rangle \in \mathbb{B}$, write

$$|u| := k, \qquad x^u := x_{i_1} x_{i_2} \cdots x_{i_k}, \qquad |\emptyset| = 0, \qquad x^{\emptyset} = 1.$$

For $i, j \in \mathrm{I}_0$, define

$$\partial_{ij} : \Lambda(n) \otimes \mathcal{O}(m,\ \underline{t}) \longrightarrow \Lambda(n) \otimes W(m,\ \underline{t})$$

so that for $f \in \Lambda(n) \otimes \mathcal{O}(m,\ \underline{t})$,

$$\partial_{ij}(f) = \partial_j(f)\partial_i - \partial_i(f)\partial_j.$$

Let

$$S(m,\ \underline{t}) := \mathrm{span}_{\mathbb{F}}\{\partial_{ij}(f) \mid i,j \in \mathrm{I}_0,\ f \in \mathcal{O}(m,\ \underline{t})\}$$

and

$$\mathfrak{S} := \mathrm{span}_{\mathbb{F}}\{\partial_{ij}(f) \mid i,j \in \mathrm{I}_0,\ f \in \Lambda(n) \otimes \mathcal{O}(m,\ \underline{t})\}.$$

Then $\mathfrak{S}$ is a $\mathbb{Z}$-graded subalgebra of $\mathfrak{W}$. We note that, since

$$[x^u\partial_i,\ \partial_j] = 0, \qquad i,j \in \mathrm{I}_0,$$

one can see that $\mathfrak{S}$ is not transitive and not simple, however, it is centerless.

3 Superderivations

The following basic formulas will be used throughout without notice.

(1) $\partial_{ii}(f) = 0$, $i \in \mathrm{I}_0$, $f \in \Lambda(n) \otimes \mathcal{O}(m,\ \underline{t})$;

(2) $\partial_{ij}(f) = -\partial_{ji}(f)$, $i,j \in \mathrm{I}_0$, $f \in \Lambda(n) \otimes \mathcal{O}(m,\ \underline{t})$;

(3) $[\partial_{ij}(f),\ \partial_{kl}(g)] = \partial_{ik}(\partial_j(f)\partial_l(g)) + \partial_{il}(\partial_j(f)\partial_k(g)) + \partial_{jk}(\partial_i(f)\partial_l(g)) + \partial_{jl}(\partial_i(f)\partial_l(g))$, where $i,j,k,l \in \mathrm{I}_0$, $f,g \in \Lambda(n) \otimes \mathcal{O}(m,\ \underline{t})$.

A superderivation of a Lie superalgebra is completely determined by its action on the homogeneous generators. Thus one needs the generator set of $\mathfrak{S}$. The following proposition can be verified by a lengthy but straightforward computation (cf. [5]).

Proposition 3.1 *The Lie superalgebra $\mathfrak{S}$ is generated by $\mathcal{T} \cup \mathcal{M}$, where*

$$\mathcal{T} = \{\partial_{ij}(x^{(k\varepsilon_i)}) \mid k \le \pi_i,\ i,j \in \mathrm{I}_0\}, \qquad \mathcal{M} = \{\partial_{ij}(x_k x_i) \mid i,j \in \mathrm{I}_0,\ k \in \mathrm{I}_1\}.$$

The centralizer of $\mathfrak{S}_{-1}$ in $\mathfrak{S}$ is

$$N := C_{\mathfrak{S}}(\mathfrak{S}_{-1}) = \mathrm{span}_{\mathbb{F}}\{x^u\partial_i \mid u \in \mathbb{B},\ i \in \mathrm{I}_0\}.$$

It is clear that N is a $\mathbb{Z}$-graded subalgebra of $\mathfrak{S}$.

Let L be a $\mathbb{Z}$-graded subalgebra of $\mathfrak{S}$. View $\mathfrak{S}$ as an L-module by means of the adjoint representation. Denote by $\mathrm{Der}(L,\ \mathfrak{S})$ the superderivation space of L into L-module $\mathfrak{S}$. One can directly verify the following lemma.

Lemma 3.1 *Let L be a $\mathbb{Z}$-graded subalgebra of $\mathfrak{S}$ satisfying $L_{-1} = \mathfrak{S}_{-1}$. Suppose $\phi \in \mathrm{Der}(L,\ \mathfrak{S})$ and $\phi(L_{-1}) = 0$. If $E \in L$ and $[E,\ \mathfrak{S}_{-1}] \in \ker\phi$, then $\phi(E) \in N$.*

Lemma 3.2 *Let $\phi \in \mathrm{Der}_t\mathfrak{S}$, $t \ge 0$. If $\phi(\mathfrak{S}_0 \oplus \mathfrak{S}_{-1}) = 0$, then $\phi = 0$.*

Proof. Assert that

$$\phi(\partial_{ij}(x^{(k\varepsilon_i)})) = 0, \qquad i,j \in \mathrm{I}_0.$$

It suffices to show that

$$\phi(D_{ij}(x^{(k\varepsilon_i)})) = 0 \qquad \text{for } k \ge 3.$$

If $k = 3$, then by Lemma 3.1 one can write

$$\phi(\partial_{ij}x^{(3\varepsilon_i)}) = \sum_{l \in \mathrm{I}_0;\, w \in \mathbb{B}} a_{wl}x^w\partial_l, \qquad a_{wl} \in \mathbb{F}.$$

Since

$$[\partial_{kt}(x^{(\varepsilon_k+\varepsilon_t)}),\ \partial_{ij}(x^{(3\varepsilon_i)})]=0,\qquad \{k,t\}\in \mathrm{I}_0\backslash\{i,j\},$$

we have

$$a_{wt}=a_{wk}=0,$$

and then

$$\phi(\partial_{ij}(x^{(3\varepsilon_i)}))=a_{wi}x^w\partial_i+a_{wj}x^w\partial_j,\qquad w\in\mathbb{B},\ i,j\in\mathrm{I}_0.$$

Note that

$$[\partial_{ti}(x^{(2\varepsilon_i)}),\ \partial_{ij}(x^{(3\varepsilon_i)})]=0,\qquad t\in\mathrm{I}_0\backslash\{i,j\}.$$

One has

$$a_{wi}=0$$

and so

$$\phi(\partial_{ij}(x^{(3\varepsilon_i)}))=a_{wj}x^w\partial_j.$$

Similarly, one can write

$$\phi(\partial_{ti}(x^{(3\varepsilon_i)}))=b_{vt}x^v\partial_t,\qquad t\in\mathrm{I}_0\backslash\{i,j\},\ v\in\mathbb{B}.$$

Using the equation

$$[\partial_{tj}(x^{(2\varepsilon_j)}),\ \partial_{ij}(x^{(3\varepsilon_i)})]=\partial_{ti}(x^{(3\varepsilon_i)}),$$

one gets

$$a_{wj}=0.$$

So

$$\phi(\partial_{ij}(x^{(3\varepsilon_i)}))=0,\qquad i,j\in\mathrm{I}_0.$$

By induction hypothesis and Lemma 3.1, $\phi(\partial_{ij}(x^{(k\varepsilon_i)}))\in N$. Hence one can assume that

$$\phi(\partial_{ij}(x^{(k\varepsilon_i)}))=\sum_{q\in\mathrm{I}_0;u\in\mathbb{B}}a_{uq}x^u\partial_q,\qquad a_{uq}\in\mathbb{F}.$$

Noticing that

$$[\partial_{tl}(x^{(\varepsilon_t+\varepsilon_l)}),\ \partial_{ij}(x^{(k\varepsilon_i)})]=0,\qquad t,l\in\mathrm{I}_0\backslash\{i,j\},$$

one deduces that

$$\phi(\partial_{ij}x^{(k\varepsilon_i)})=a_{ui}x^u\partial_i+a_{uj}x^u\partial_j,\qquad a_{ui}\in\mathbb{F},\ u\in\mathbb{B},\ i,j\in\mathrm{I}_0.$$

From the equation

$$[\partial_{ti}(x^{(2\varepsilon_i)}),\ \partial_{ij}(x^{(k\varepsilon_i)})]=0,\qquad t\in\mathrm{I}_0\backslash\{i,j\}$$

one gets

$$a_{ui}=0,$$

and then

$$\phi(\partial_{ij}(x^{(k\varepsilon_i)}))=a_{uj}x^u\partial_j.$$

Write

$$\phi(\partial_{ti}(x^{(k\varepsilon_i)}))=c_{vt}x^v\partial_t,\qquad t\in\mathrm{I}_0\backslash\{i,j\},\ v\in\mathbb{B}.$$

From the equation

$$[\partial_{tj}(x^{(2\varepsilon_j)}),\ \partial_{ij}(x^{(k\varepsilon_i)})] = \partial_{ti}(x^{(k\varepsilon_i)})$$

one gets

$$a_{uj} = 0,$$

and then

$$\phi(\partial_{ij}(x^{(k\varepsilon_i)})) = 0, \qquad i, j \in \mathrm{I}_0.$$

By Proposition 3.1, $\phi(\mathfrak{S}) = 0$ and $\phi = 0$.

Proposition 3.2 *Suppose that* $\phi \in \mathrm{Der}_{-1}\mathfrak{S}$ *and* $\phi(\mathfrak{S}_0) = 0$*. Then* $\phi = 0$*.*

Proof. By Proposition 3.1, it suffices to show that

$$\phi(\partial_{ij}(x^{(k\varepsilon_i)})) = 0, \qquad i, j \in \mathrm{I}_0.$$

But this can be verified as in the proof of Lemma 3.2.

Lemma 3.3 *Let* $\phi \in \mathrm{Der}_{-t}\mathfrak{S}$*,* $t > 1$*. If* $\phi(\partial_{ij}(x^{((t+1)\varepsilon_i)})) = 0$ *for any* $i, j \in \mathrm{I}_0$*, then* $\phi = 0$*.*

Proof. If $k \leq t$, then

$$\phi(\partial_{ij}(x^{k\varepsilon_i})) = 0, \qquad i, j \in \mathrm{I}_0.$$

Note that

$$\phi(\partial_{ij}(x^{(t+1)\varepsilon_i)}) = 0, \qquad i, j \in \mathrm{I}_0.$$

If $k > t + 1$, using induction on k one gets as in the proof of Lemma 3.2 that

$$\phi(\partial_{ij}(x^{(k\varepsilon_i)}) = 0, \qquad i, j \in \mathrm{I}_0.$$

Lemma 3.4 *Let* L *be a* $\mathbb{Z}$*-graded subalgebra of* $\mathfrak{W}$*. If* $\phi \in \mathrm{Der}_t(L,\ \mathfrak{W})$*,* $t \geq 0$*, then there exists* $E \in \mathrm{N}_{\mathfrak{W}}(L) := \{x \in \mathfrak{W} \mid [x,\ L] \subseteq L\}$ *such that* $(\phi - \mathrm{ad}E) \mid_{L_{-1}} = 0$*.*

Proof. The proof is completely analogous to the one of Proposition 2.6 in [5].

A direct verification shows that

$$\mathrm{N}_{\mathfrak{W}}(\mathfrak{S}) = \mathfrak{S} \oplus \sum_{i \in \mathrm{I}_0} x^{(\pi - \pi_i\varepsilon_i)} x^{\delta} \partial_i \oplus \sum_{k \in \mathrm{I}_0} \mathbb{F} x_k \partial_k.$$

Now we can prove the main result in this paper.

Theorem 3.1 (i)

$$\mathrm{ad}:\ \mathfrak{S} \oplus \sum_{i \in \mathrm{I}_0} x^{(\pi - \pi_i\varepsilon_i)} x^{\delta} \partial_i \oplus \sum_{k \in \mathrm{I}_0} \mathbb{F} x_k \partial_k \longrightarrow \mathrm{Der}(\mathfrak{S}),$$
$$X \longmapsto \mathrm{ad}X$$

is an embedding, where $\delta = \langle m+1, \cdots, m+n \rangle$*;*

(ii)

$$\mathrm{Der}(\mathfrak{S}) = \mathfrak{S} \oplus \sum_{i \in \mathrm{I}_0} x^{(\pi - \pi_i\varepsilon_i)} x^{\delta} \partial_i \oplus \sum_{k \in \mathrm{I}_0} \mathbb{F} x_k \partial_k \oplus P(m, \underline{t}),$$

where

$$P(m,\ \underline{t}) := \mathrm{span}_{\mathbb{F}}\{(\mathrm{ad}\partial_i)^{p^{r_i}} \mid i \in \mathrm{I}_0,\ 1 \leq r_i \leq t_i - 1\}.$$

Proof. (i) can be directly verified.

To show (ii), let us consider three cases separately.

Case I: Suppose that $t \geq -1$ and assert that

$$\mathrm{Der}_t\mathfrak{S} = \mathfrak{S}_t \oplus \Big(\sum_{i\in \mathrm{I}_0} x^{(\pi-\pi_i\varepsilon_i)}x^{\delta}\partial_i \oplus \sum_{k\in \mathrm{I}_0} \mathbb{F}x_k\partial_k\Big)_t.$$

Let $t \geq 0$, $\phi \in \mathrm{Der}_t\mathfrak{S}$. By Lemma 3.4, there exists $E \in \mathrm{N}_{\mathfrak{W}}(\mathfrak{S})$ such that

$$(\phi - \mathrm{ad}E)|_{\mathfrak{S}_{-1}} = 0.$$

Let $\psi := \phi - \mathrm{ad}E$. One gets

$$\psi(\partial_{ij}(x^{(\varepsilon_i+\varepsilon_j)})) \in N_t, \qquad i, j \in \mathrm{I}_0.$$

So then we may assume that

$$\psi(\partial_{ij}(x^{(\varepsilon_i+\varepsilon_j)})) = \sum_{k\in \mathrm{I}_0; u\in\mathbb{B}} a_{uk}x^u\partial_k, \qquad a_{uk} \in \mathbb{F}.$$

Since

$$[\partial_{tq}(x^{(\varepsilon_t+\varepsilon_q)}),\ \partial_{ij}(x^{(\varepsilon_i+\varepsilon_j)})] = 0, \qquad \{t, q\} \in \mathrm{I}_0\backslash\{i, j\},$$

one easily gets

$$a_{ut} = a_{uq} = 0, \qquad \{t, q\} \in \mathrm{I}_0\backslash\{i, j\},$$

and thereby

$$\psi(\partial_{ij}(x^{(\varepsilon_i+\varepsilon_j)})) = a_{ui}x^u\partial_i + a_{uj}x^u\partial_j.$$

Similarly, since $\psi(\partial_{ij}(x^{(2\varepsilon_i)})) \in N_t$, $i, j \in \mathrm{I}_0$, one can write

$$\psi(\partial_{ij}(x^{(2\varepsilon_i)})) = \sum_{q\in \mathrm{I}_0; v\in\mathbb{B}} b_{vq}x^v\partial_q, \qquad b_{vq} \in \mathbb{F}.$$

From the equation

$$[\partial_{ij}(x^{(2\varepsilon_i)}),\ \partial_{ij}(x^{(2\varepsilon_j)})] = -D_{ij}(x^{(\varepsilon_i+\varepsilon_j)}), \qquad i, j \in \mathrm{I}_0$$

one can deduce that

$$a_{ui} = a_{uj} = 0.$$

Hence

$$\psi(\partial_{ij}(x^{(\varepsilon_i+\varepsilon_j)}) = 0.$$

Since

$$[\partial_{ij}(x^{(\varepsilon_i+\varepsilon_j)}),\ \partial_{ij}(x^{(2\varepsilon_i)})] = 2\partial_{ij}(x^{2\varepsilon_i}),$$

one has

$$b_{vq} = 0, \qquad q \in \mathrm{I}_0,$$

and then

$$\psi(\partial_{ij}(x^{(2\varepsilon_i)})) = 0.$$

Noting that

$$\psi(\partial_{ij}(x^{(\varepsilon_i)}x_k)) \in N_t, \qquad i, j \in \mathrm{I}_0,\ k \in \mathrm{I}_1,$$

one can assume that

$$\psi(\partial_{ij}(x^{(\varepsilon_i)}x_k)) = \sum_{q\in \mathrm{I}_0; w\in\mathbb{B}} c_{wq}x^w\partial_q, \qquad c_{wq} \in \mathbb{F}.$$

Note that

$$[\partial_{ij}(x^{(\varepsilon_i)}x_k),\ \partial_{ij}(x^{(\varepsilon_i+\varepsilon_j)})] = -\partial_{ij}(x^{(\varepsilon_i)}x_k), \qquad i,j\in \mathrm{I}_0,\ k\in \mathrm{I}_1.$$

Applying ψ, one gets $c_{wi}=0$ and hence

$$\psi(\partial_{ij}(x^{(\varepsilon_i)}x_k)) = c_{wj}x^w\partial_j.$$

Since

$$[\partial_{ij}(x^{(\varepsilon_i)}x_k),\ \partial_{ij}(x^{(2\varepsilon_j)})] = 0, \qquad i,j\in \mathrm{I}_0,\ k\in \mathrm{I}_1,$$

one can deduce that

$$c_{wj}=0.$$

And therefore,

$$\psi(D_{ij}(x^{(\varepsilon_i)}x_k)) = 0.$$

Summarizing, we have

$$\psi(\mathfrak{S}_0)=0$$

showing that

$$\psi(\mathfrak{S}_0\oplus\mathfrak{S}_{-1})=0.$$

This combining with Lemma 3.2 gives $\psi(\mathfrak{S})=0$, that is, $\phi=\mathrm{ad}E$.

Now suppose $\phi\in\mathrm{Der}_{-1}\mathfrak{S}$. Noticing that $\phi(\partial_{ij}(x^{(\varepsilon_i+\varepsilon_j)}))\in N_{-1}$, we may suppose

$$\phi(\partial_{ij}(x^{(\varepsilon_i+\varepsilon_j)}) = \sum_{r\in \mathrm{I}_0} c_r\partial_r, \qquad c_r\in\mathbb{F},\ i,j\in \mathrm{I}_0.$$

Applying ψ to the equation

$$[\partial_{tq}(x^{(\varepsilon_t+\varepsilon_q)}),\ \partial_{ij}(x^{(\varepsilon_i+\varepsilon_j)})] = 0, \qquad \{t,q\}\in \mathrm{I}_0\setminus\{i,j\},$$

one obtain by a direct computation that

$$\phi(\partial_{ij}(x^{(\varepsilon_i+\varepsilon_j)})) = c_i\partial_i + c_j\partial_j.$$

Similarly, suppose

$$\phi(\partial_{ij}(x^{(2\varepsilon_i)})) = \sum_{q\in \mathrm{I}_0} b_q\partial_q, \qquad b_q\in\mathbb{F},\ i,j\in \mathrm{I}_0.$$

From the equation

$$[\partial_{ij}(x^{(\varepsilon_i+\varepsilon_j)}),\ \partial_{ij}(x^{(2\varepsilon_i)})] = 2\partial_{ij}(x^{(2\varepsilon_i)})$$

it follows that $b_i=0$. Thus

$$\phi(\partial_{ij}(x^{(2\varepsilon_i)})) = b_j\partial_j.$$

Write

$$\phi(\partial_{ij}(x_ix_k)) = \sum_{r\in \mathrm{I}_0} a_r\partial_r, \qquad a_r\in\mathbb{F},\ i,j\in \mathrm{I}_0,\ k\in \mathrm{I}_1.$$

Apply ϕ to the equation below

$$[\partial_{ij}(x^{(2\varepsilon_i)}),\ \partial_{ij}(x^{(\varepsilon_i)}x_k)] = 0.$$

This yields $a_i=0$, $i\in \mathrm{I}_0\setminus\{j\}$ and

$$\phi(\partial_{ij}(x_ix_k)) = a_j\partial_j.$$

One can write

$$\phi(\partial_{ti}(x_ix_k)) = d_t\partial_t, \qquad t\in \mathrm{I}_0\setminus\{i,j\},\ k\in \mathrm{I}_1.$$

Since

$$[\partial_{ti}(x^{(2\varepsilon_i)}),\ \partial_{ij}(x^{(\varepsilon_j)}x_k)] = \partial_{ti}(x_i x_k),$$

it follows that

$$\phi(\partial_{ij}(x_i x_k)) = 0.$$

Put

$$\psi := \phi - \sum_{r\in \mathrm{I}_0} a_r(\mathrm{ad}\partial_r).$$

Then $\psi(\mathfrak{S}_0) = 0$. By Proposition 3.3, $\psi = 0$ and then

$$\phi = \sum_{r\in \mathrm{I}_0} a_r(\mathrm{ad}\partial_r) \in \mathrm{ad}\mathfrak{S}_{-1}.$$

We conclude that

$$\mathrm{Der}_{-1}\mathfrak{S} = \mathrm{ad}\mathfrak{S}_{-1}.$$

Since

$$\mathrm{ad}:\ \mathfrak{S} \oplus x^{(\pi-\pi_i\varepsilon_i)}x^{\delta}\partial_i \oplus \sum_{k\in \mathrm{I}_0} \mathbb{F}x_k\partial_k \longrightarrow \mathrm{Der}\mathfrak{S}$$

is a monomorphism, one has for $t \geq -1$,

$$\mathrm{Der}_t\mathfrak{S} = \mathfrak{S}_t \oplus \Big(\sum_{i\in \mathrm{I}_0} x^{(\pi-\pi_i\varepsilon_i)}x^{\delta}\partial_i \oplus \sum_{k\in \mathrm{I}_0} \mathbb{F}x_k\partial_k\Big)_t.$$

Case II: Suppose $k > 1$ and k is not a p-power. Then, as in the proof of Proposition 3.5 in [5], one can show that

$$\mathrm{Der}_{-k}\mathfrak{S} = 0.$$

Case III: Suppose $k = p^r$ where r is a positive integer. Claim that

$$\mathrm{Der}_{-k}\mathfrak{S} = \mathrm{span}_{\mathbb{F}}\{(\mathrm{ad}\partial_i)^k \mid i \in \mathrm{I}_0\}.$$

Let $\phi \in \mathrm{Der}_{-k}\mathfrak{S}$. Since

$$\mathrm{zd}(\phi(\partial_{ij}(x^{((k+1)\varepsilon_i)})) = -1,$$

one may suppose that

$$\phi(\partial_{ij}(x^{(k+1\varepsilon_i)})) = \sum_{r\in \mathrm{I}_0} a_r\partial_r, \qquad a_r \in \mathbb{F},\ i,j \in \mathrm{I}_0.$$

Note that

$$[\partial_{tl}(x^{(\varepsilon_t+\varepsilon_l)}),\ \partial_{ij}(x^{((k+1)\varepsilon_i)})] = 0, \qquad t,l \in \mathrm{I}_0 \setminus \{i,j\}.$$

Applying ϕ to the equation we obtain that

$$a_t = a_l = 0,$$

and consequently,

$$\phi(\partial_{ij}(x^{(k+1)\varepsilon_i)})) = a_i\partial_i + a_j\partial_j.$$

Similarly, since

$$[\partial_{ij}(x^{(2\varepsilon_i)}),\ \partial_{ij}(x^{((k+1)\varepsilon_i)})] = 0,$$

we can get $a_i = 0$ and then

$$\phi(\partial_{ij}(x^{(k+1)\varepsilon_i)}) = a_j\partial_j.$$

Putting

$$\psi := \phi - \sum_{r\in \mathrm{I}_0} a_r(\mathrm{ad}\partial_r)^k,$$

we have

$$\psi(\partial_{ij}(x^{((k+1)\varepsilon_i)})) = 0, \qquad i, j \in \mathrm{I}_0.$$

Now Lemma 3.3 ensures that $\psi = 0$. The proof is complete.

Theorem 3.2 *The outer superderivation algebra of $\mathfrak{S}$ is isomorphic to the Lie algebra whose underlying space is*

$$L := \mathit{\Gamma} \oplus V \oplus W,$$

where $\mathit{\Gamma}$ is an m-dimensional vector space with $\mathbb{F}$-basis $\{\gamma_1, \cdots, \gamma_m\}$, V is an m-dimensional vector space with $\mathbb{F}$-basis $\{v_1, \cdots, v_m\}$, W is a vector space of dimension $\sum_{i=1}^{m} t_i - m$ and the Lie bracket is given by

$$[\gamma_i,\ v_j] = -\delta_{ij}, \qquad i, j \in \overline{1,\ m};$$

$$[\mathit{\Gamma} \oplus V,\ W] = [V,\ V] = [W,\ W] = 0.$$

Proof. This follows from Theorem 3.1.

Remark 3.1 By Theorem 3.2, the first cohomology space $H^1(\mathfrak{S};\ \mathfrak{S})$ is of dimension $m + \sum_{i\in\mathbf{I}_0} t_i$.

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