Communications in Mathematical Research
27(1)(2011), ???–???

Hájek-Rényi-type Inequality for a Class of Random Variable Sequences and Its Applications*

Wang Xue-jun, Shen Yan, Hu Shu-he and Yang Wen-zhi
(*School of Mathematical Science, Anhui University, Hefei, 230039*)

Communicated by Wang De-hui

Abstract: In this paper, we obtain the Hájek-Rényi-type inequality for a class of random variable sequences and give some applications for associated random variable sequences, strongly positive dependent stochastic sequences and martingale difference sequences which generalize and improve the results of Prakasa Rao and Soo published in Statist. Probab. Lett., 57(2002) and 78(2008). Using this result, we get the integrability of supremum and the strong law of large numbers for a class of random variable sequences.

Key words: Hájek-Rényi-type inequality, associated random variable sequence, strongly positive dependent stochastic sequence, martingale difference sequence

2000 MR subject classification: 60F15, 60E15

Document code: A

Article ID: 1674-5647(2011)01-0???-11

1 Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega,\ \mathcal{F},\ P)$,

$$S_n = \sum_{i=1}^{n}(X_i - EX_i), \quad n \geq 1, \qquad S_0 = 0,$$

and $\{b_n,\ n \geq 1\}$ be a nondecreasing sequence of positive numbers. Hájek and Rényi[1] proved that: If $\{X_n,\ n \geq 1\}$ is a sequence of independent random variables with finite

*Received date: March 22, 2009.

Foundation item: The NSF (10871001, 60803059) of China, Talents Youth Fund (2010SQRL016ZD) of Anhui Province Universities, Youth Science Research Fund (2009QN011A) of Anhui University, Provincial Natural Science Research Project of Anhui Colleges (KJ2010A005), and Academic innovation team of Anhui University (KJTD001B).

second moment, then for any $\varepsilon > 0$ and $m < n$,

$$P\left\{\max_{m\le j\le n}\left|\frac{1}{b_j}\sum_{i=1}^{j}(X_i - EX_i)\right| \ge \varepsilon\right\} \le \frac{1}{\varepsilon^2}\left\{\sum_{j=1}^{m}\frac{Var(X_j)}{b_m^2} + \sum_{j=m+1}^{n}\frac{Var(X_j)}{b_j^2}\right\}. \tag{1.1}$$

Hájek-Rényi-type inequality has been studied by many authors; one can refer to [2]–[9]. In this paper, we study the Hájek-Rényi-type inequality under the general condition A1 below. In addition, we give some applications of Hájek-Rényi-type inequality which generalize and improve the results of Prakasa Rao[6] and Soo[9]. Let n and m be integers and C be a positive constant not depending on n and m in what follows.

A1 For any positive integers $m \le n$,

$$E\left\{\max_{m\le i\le n}\left|\sum_{j=m}^{i}(X_j - EX_j)\right|^2\right\} \le C\cdot E\left\{\sum_{j=m}^{n}(X_j - EX_j)\right\}^2, \tag{1.2}$$

$$Cov(X_i,\ X_j) \ge 0, \qquad i, j = 1, 2, \cdots \tag{1.3}$$

Lemma 1.1([5], Theorem 1.1) *Let $\beta_1,\ \beta_2,\ \cdots,\ \beta_n$ be a nondecreasing sequence of positive numbers, and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n$ be nonnegative numbers. Let r be a fixed positive number. Assume that for each m with $1 \le m \le n$,*

$$E\left(\max_{1\le l\le m}\left|\sum_{j=1}^{l}X_j\right|\right)^r \le \sum_{l=1}^{m}\alpha_l. \tag{1.4}$$

Then

$$E\left(\max_{1\le l\le n}\left|\frac{\sum_{j=1}^{l}X_j}{\beta_l}\right|\right)^r \le 4\sum_{l=1}^{n}\frac{\alpha_l}{\beta_l^r}. \tag{1.5}$$

2 Hájek-Rényi-type Inequality

Theorem 2.1 *Let $\{X_n,\ n \ge 1\}$ be a sequence of random variables satisfying A1 and $\{b_n,\ n \ge 1\}$ be a nondecreasing sequence of positive numbers. Then for any $\varepsilon > 0$ and $n \ge 1$,*

$$P\left\{\max_{1\le k\le n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \ge \varepsilon\right\} \le \frac{4C}{\varepsilon^2}\left\{\sum_{j=1}^{n}\frac{Var(X_j)}{b_j^2} + 2\sum_{1\le k<j\le n}\frac{Cov(X_k, X_j)}{b_j^2}\right\}, \tag{2.1}$$

where C is defined in (1.2).

Proof. Without loss of generality, we assume that $b_n \ge 1$. Let $\alpha = \sqrt{2}$. For $i \ge 0$, define

$$A_i = \{1 \le k \le n:\ \alpha^i \le b_k < \alpha^{i+1}\}.$$

For $A_i \ne \emptyset$, let

$$v(i) = \max\{k:\ k \in A_i\},$$

and t_n be the index of the last nonempty set A_i. Obviously,

$$A_iA_j = \emptyset, \qquad i \ne j$$

and

$$\sum_{i=0}^{t_n}A_i = \{1,\ 2,\ \cdots,\ n\}.$$

It is easy to see that

$$\alpha^i \le b_k \le b_{v(i)} < \alpha^{i+1}, \qquad k \in A_i.$$

By Markov's inequality and (1.2), we have

$$\begin{aligned} & P\Big\{\max_{1\le k\le n}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j-EX_j)\Big|\ge\varepsilon\Big\} \\ = & P\Big\{\max_{0\le i\le t_n, A_i\ne\emptyset}\max_{k\in A_i}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j-EX_j)\Big|\ge\varepsilon\Big\} \\ \le & \sum_{i=0,A_i\ne\emptyset}^{t_n} P\Big\{\frac{1}{\alpha^i}\max_{1\le k\le v(i)}\Big|\sum_{j=1}^{k}(X_j-EX_j)\Big|\ge\varepsilon\Big\} \\ \le & \frac{C}{\varepsilon^2}\sum_{j=1}^{n}\{Var(X_j)+2Cov(X_j,S_{j-1})\}\sum_{i=0,A_i\ne\emptyset,v(i)\ge j}^{t_n}\frac{1}{\alpha^{2i}}. \end{aligned} \tag{2.2}$$

Now we estimate $\displaystyle\sum_{i=0,A_i\ne\emptyset,v(i)\ge j}^{t_n}\frac{1}{\alpha^{2i}}$. Let

$$i_0=\min\{i: A_i\ne\emptyset,\ v(i)\ge j\}.$$

Then

$$b_j \le b_{v(i_0)} < \alpha^{i_0+1}$$

follows from the definition of $v(i)$. Therefore,

$$\sum_{i=0,A_i\ne\emptyset,v(i)\ge j}^{t_n}\frac{1}{\alpha^{2i}} < \sum_{i=i_0}^{\infty}\frac{1}{\alpha^{2i}} = \frac{1}{1-\frac{1}{\alpha^2}}\frac{1}{\alpha^{2i_0}} < \frac{\alpha^2}{1-\frac{1}{\alpha^2}}\frac{1}{b_j^2} = \frac{4}{b_j^2}. \tag{2.3}$$

Thus (2.1) follows from (2.2) and (2.3).

Theorem 2.2 *Let $\{X_n,\ n\ge 1\}$ be a sequence of random variables satisfying A1 and $\{b_n, n\ge 1\}$ be a nondecreasing sequence of positive numbers. Then for any $\varepsilon>0$ and any positive integers $m<n$,*

$$\begin{aligned} & P\Big\{\max_{m\le k\le n}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j-EX_j)\Big|\ge\varepsilon\Big\} \\ \le & \frac{4}{\varepsilon^2}\Big\{\sum_{j=1}^{m}\frac{Var(X_j)}{b_m^2}+2\sum_{1\le k<j\le m}\frac{Cov(X_k,X_j)}{b_m^2} \\ & +4C\sum_{j=m+1}^{n}\frac{Var(X_j)}{b_j^2}+8C\sum_{m+1\le k<j\le n}\frac{Cov(X_k,X_j)}{b_j^2}\Big\}, \end{aligned} \tag{2.4}$$

where C is defined in (1.2).

Proof. Observe that

$$\max_{m\le k\le n}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j-EX_j)\Big| \le \Big|\frac{1}{b_m}\sum_{j=1}^{m}(X_j-EX_j)\Big| + \max_{m+1\le k\le n}\Big|\frac{1}{b_k}\sum_{j=m+1}^{k}(X_j-EX_j)\Big|,$$

thus

$$
\begin{aligned}
&P\Big\{\max_{m\le k\le n}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j-EX_j)\Big|\ge\varepsilon\Big\}\\
\le\ &P\Big\{\Big|\frac{1}{b_m}\sum_{j=1}^{m}(X_j-EX_j)\Big|\ge\frac{\varepsilon}{2}\Big\}\\
&+P\Big\{\max_{m+1\le k\le n}\Big|\frac{1}{b_k}\sum_{j=m+1}^{k}(X_j-EX_j)\Big|\ge\frac{\varepsilon}{2}\Big\}\\
\doteq\ &I+II. \qquad (2.5)
\end{aligned}
$$

For I, by Markov's inequality, we have

$$
\begin{aligned}
I&\le\frac{4}{\varepsilon^2}E\Big\{\frac{1}{b_m}\sum_{j=1}^{m}(X_j-EX_j)\Big\}^2\\
&=\frac{4}{\varepsilon^2}\Big\{\sum_{j=1}^{m}\frac{Var(X_j)}{b_m^2}+2\sum_{1\le k<j\le m}\frac{Cov(X_k,X_j)}{b_m^2}\Big\}. \qquad (2.6)
\end{aligned}
$$

For II, we apply Theorem 2.1 to $\{X_{m+i},\ 1\le i\le n-m\}$ and $\{b_{m+i},\ 1\le i\le n-m\}$. Noting that

$$
\max_{m+1\le k\le n}\Big|\frac{1}{b_k}\sum_{j=m+1}^{k}(X_j-EX_j)\Big|=\max_{1\le k\le n-m}\Big|\frac{1}{b_{m+k}}\sum_{j=1}^{k}(X_{m+j}-EX_{m+j})\Big|,
$$

by (1.2) and Theorem 2.1, we get

$$
\begin{aligned}
II&\le\frac{4C}{(\varepsilon/2)^2}\Big\{\sum_{j=1}^{n-m}\frac{Var(X_{m+j})}{b_{m+j}^2}+2\sum_{1\le k<j\le n-m}\frac{Cov(X_{m+k},X_{m+j})}{b_{m+j}^2}\Big\}\\
&=\frac{16C}{\varepsilon^2}\Big\{\sum_{j=m+1}^{n}\frac{Var(X_j)}{b_j^2}+2\sum_{m+1\le k<j\le n}\frac{Cov(X_k,X_j)}{b_j^2}\Big\}. \qquad (2.7)
\end{aligned}
$$

Therefore, the desired result (2.4) follows from (2.5)–(2.7) immediately.

Theorem 2.3 *Let $\{X_n,\ n\ge1\}$ be a sequence of random variables satisfying A1 and $\{b_n,\ n\ge1\}$ be a nondecreasing sequence of positive numbers. Assume that*

$$
\sum_{j=1}^{\infty}\frac{Var(X_j)}{b_j^2}+2\sum_{1\le k<j}\frac{Cov(X_k,X_j)}{b_j^2}<\infty. \qquad (2.8)
$$

Then

$$
\begin{aligned}
E\Big(\sup_{n\ge1}\Big|\frac{S_n}{b_n}\Big|^r\Big)&\le1+\frac{4Cr}{2-r}\Big\{\sum_{j=1}^{\infty}\frac{Var(X_j)}{b_j^2}+2\sum_{1\le k<j}\frac{Cov(X_k,X_j)}{b_j^2}\Big\}\\
&<\infty,\qquad r\in(0,2), \qquad (2.9)
\end{aligned}
$$

$$
E\Big(\sup_{n\ge1}\Big|\frac{S_n}{b_n}\Big|^2\Big)\le4C\Big\{\sum_{j=1}^{\infty}\frac{Var(X_j)}{b_j^2}+2\sum_{1\le k<j}\frac{Cov(X_k,X_j)}{b_j^2}\Big\}<\infty, \qquad (2.10)
$$

where C is defined in (1.2). *Furthermore, if $\lim_{n\to\infty}b_n=+\infty$, then*

$$
\lim_{n\to\infty}\frac{1}{b_n}\sum_{j=1}^{n}(X_j-EX_j)=0\qquad a.s.
$$

Proof. By the Continuity of Probability and Theorem 2.1, we get

$$\begin{aligned} E\left(\sup_{n\ge 1}\left|\frac{S_n}{b_n}\right|^r\right) &\le 1+\int_1^\infty \lim_{N\to\infty} P\left(\max_{1\le n\le N}\left|\frac{S_n}{b_n}\right|>t^{1/r}\right)\mathrm{d}t \\ &\le 1+\frac{4Cr}{2-r}\left\{\sum_{j=1}^\infty \frac{Var(X_j)}{b_j^2}+2\sum_{1\le k<j}\frac{Cov(X_k,X_j)}{b_j^2}\right\} \\ &< \infty. \end{aligned}$$

By (1.2), we have

$$E\left(\max_{1\le i\le n} S_i^2\right)\le CES_n^2 \doteq \sum_{j=1}^n \alpha_j, \tag{2.11}$$

where

$$\alpha_j = C(ES_j^2-ES_{j-1}^2)=C(Var(X_j)+2Cov(X_j,S_{j-1}))\ge 0, \qquad j=1,2,\cdots,n.$$

By (2.11) and Lemma 1.1,

$$E\left(\max_{1\le i\le n}\left|\frac{S_i}{b_i}\right|^2\right)\le 4\sum_{j=1}^n\frac{\alpha_j}{b_j^2}=4C\sum_{j=1}^n\frac{Var(X_j)+2Cov(X_j,S_{j-1})}{b_j^2}. \tag{2.12}$$

Thus, by Monotone Convergence Theorem,

$$\begin{aligned} E\left(\sup_{n\ge 1}\left|\frac{S_n}{b_n}\right|^2\right) &= E\left\{\lim_{n\to\infty}\left(\max_{1\le i\le n}\left|\frac{S_i}{b_i}\right|^2\right)\right\} \\ &= \lim_{n\to\infty} E\left(\max_{1\le i\le n}\left|\frac{S_i}{b_i}\right|^2\right) \\ &\le 4C\left\{\sum_{j=1}^\infty\frac{Var(X_j)}{b_j^2}+2\sum_{1\le k<j}\frac{Cov(X_k,X_j)}{b_j^2}\right\} \\ &< \infty. \end{aligned}$$

Observe that

$$\begin{aligned} P\left(\bigcup_{n=m}^\infty\left(\left|\frac{S_n}{b_n}\right|>\varepsilon\right)\right) &= P\left(\bigcup_{N=m}^\infty\left(\max_{m\le n\le N}\left|\frac{S_n}{b_n}\right|>\varepsilon\right)\right) \\ &= \lim_{N\to\infty}P\left(\max_{m\le n\le N}\left|\frac{S_n}{b_n}\right|>\varepsilon\right). \end{aligned}$$

By Theorem 2.2, we have that

$$\begin{aligned} P\left(\max_{m\le n\le N}\left|\frac{S_n}{b_n}\right|>\varepsilon\right) \le{}& \frac{4}{\varepsilon^2}\left\{\sum_{j=1}^m\frac{Var(X_j)}{b_m^2}+2\sum_{1\le k<j\le m}\frac{Cov(X_k,X_j)}{b_m^2}\right\} \\ &+\frac{16C}{\varepsilon^2}\left\{\sum_{j=m+1}^N\frac{Var(X_j)}{b_j^2}+2\sum_{m+1\le k<j\le N}\frac{Cov(X_k,X_j)}{b_j^2}\right\}. \end{aligned}$$

Hence, by Kronecker's Lemma, we get

$$\lim_{m\to\infty}P\left(\bigcup_{n=m}^\infty\left(\left|\frac{S_n}{b_n}\right|>\varepsilon\right)\right)=0,$$

which is equivalent to

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{j=1}^n(X_j-EX_j)=0 \qquad \text{a.s.},$$

so the desired results are proved.

3 Applications

3.1 Application for associated random variable sequences

Definition 3.1 *A finite collection of random variables $X_1,\ X_2,\ \cdots,\ X_m$ is said to be associated if*

$$Cov\{f(X_1,\cdots,X_m),\ g(X_1,\cdots,X_m)\} \geq 0 \tag{3.1}$$

for any two coordinatewise nondecreasing functions $f,\ g$ on $\mathbf{R}^m$ such that the covariance is well defined. An infinite sequence $\{X_n,\ n\geq 1\}$ is associated if every finite subcollection is associated.

Lemma 3.1 *Let $\{X_n,\ n\geq 1\}$ be a sequence of associated random variables, and $f_n(x)$ be a nondecreasing function of x for each $n\geq 1$. Then $\{f_n(X_n),\ n\geq 1\}$ and $\{-X_n,\ n\geq 1\}$ are also sequences of associated random variable.*

Lemma 3.2([10], Theorem 2) *Let $X_1,\ X_2,\ \cdots,\ X_n$ be associated random variables with mean zero and finite second moment. Then*

$$E\Big(\max_{1\leq j\leq n} S_j\Big)^2 \leq ES_n^2. \tag{3.2}$$

Theorem 3.1 *Let $\{X_n,\ n\geq 1\}$ be a sequence of associated random variables with finite second moment and $\{b_n,\ n\geq 1\}$ be a nondecreasing sequence of positive numbers. Then for any $\varepsilon>0$ and any positive integers $m<n$,*

$$\begin{aligned} &P\Big\{\max_{1\leq k\leq n}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j-EX_j)\Big|\geq\varepsilon\Big\}\\ &\leq \frac{8}{\varepsilon^2}\Big\{\sum_{j=1}^{n}\frac{Var(X_j)}{b_j^2}+2\sum_{1\leq k<j\leq n}\frac{Cov(X_k,X_j)}{b_j^2}\Big\}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &P\Big\{\max_{m\leq k\leq n}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j-EX_j)\Big|\geq\varepsilon\Big\}\\ &\leq \frac{4}{\varepsilon^2}\Big\{\sum_{j=1}^{m}\frac{Var(X_j)}{b_m^2}+2\sum_{1\leq k<j\leq m}\frac{Cov(X_k,X_j)}{b_m^2}\\ &\qquad +8\sum_{j=m+1}^{n}\frac{Var(X_j)}{b_j^2}+16\sum_{m+1\leq k<j\leq n}\frac{Cov(X_k,X_j)}{b_j^2}\Big\}. \end{aligned} \tag{3.4}$$

Proof. Since

$$f_n(x) \doteq x - EX_n$$

is a nondecreasing function of x, by Lemma 3.1, we can see that $\{X_n-EX_n,\ n\geq 1\}$ is also a sequence of associated random variables. Denote

$$Y_n \doteq X_n - EX_n, \quad T_n \doteq \sum_{i=1}^{n} Y_i \qquad \text{for each } n\geq 1.$$

Observe that

$$E\Big(\max_{1\le j\le n} T_j^2\Big) \le E\Big(\max_{1\le j\le n} T_j\Big)^2 + E\Big(\max_{1\le j\le n}(-T_j)\Big)^2, \tag{3.5}$$

and by Lemma 3.2,

$$E\Big(\max_{1\le j\le n} T_j\Big)^2 \le ET_n^2. \tag{3.6}$$

By Lemma 3.1, $\{-Y_n,\ n\ge 1\}$ is a sequence of associated random variables. By Lemma 3.2,

$$E\Big(\max_{1\le j\le n}(-T_j)\Big)^2 \le E(-T_n)^2 = ET_n^2. \tag{3.7}$$

By (3.5)–(3.7), we have

$$E\bigg(\max_{1\le i\le n}\Big|\sum_{j=1}^{i}(X_j-EX_j)\Big|^2\bigg) \le 2E\bigg(\sum_{j=1}^{n}(X_j-EX_j)\bigg)^2. \tag{3.8}$$

Similarly to the proof of (3.8), we can get (1.2) for $C=2$. Therefore, (3.3) and (3.4) follow from Theorem 2.1 and Theorem 2.2, respectively.

Remark 3.1 Under the conditions of Theorem 3.1, Prakasa Rao[6] obtained the following results:

$$P\bigg\{\max_{1\le k\le n}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j-EX_j)\Big|\ge\varepsilon\bigg\}$$
$$\le \frac{4}{\varepsilon^2}\bigg\{\sum_{j=1}^{n}\frac{Var(X_j)}{b_j^2}+\sum_{1\le k\ne j\le n}\frac{Cov(X_k,X_j)}{b_kb_j}\bigg\}, \tag{3.9}$$

$$P\bigg\{\max_{m\le k\le n}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j-EX_j)\Big|\ge\varepsilon\bigg\}$$
$$\le \frac{4}{\varepsilon^2}\bigg\{\sum_{j=1}^{m}\frac{Var(X_j)}{b_m^2}+\sum_{1\le k\ne j\le m}\frac{Cov(X_k,X_j)}{b_m^2}$$
$$+\sum_{j=m+1}^{n}\frac{Var(X_j)}{b_j^2}+\sum_{m+1\le k\ne j\le n}\frac{Cov(X_k,X_j)}{b_kb_j}\bigg\}. \tag{3.10}$$

But there are some typos in (3.9) and (3.10). (3.9) and (3.10) should be replaced by

$$P\bigg\{\max_{1\le k\le n}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j-EX_j)\Big|\ge\varepsilon\bigg\}$$
$$\le \frac{8}{\varepsilon^2}\bigg\{\sum_{j=1}^{n}\frac{Var(X_j)}{b_j^2}+\sum_{1\le k\ne j\le n}\frac{Cov(X_k,X_j)}{b_kb_j}\bigg\}, \tag{3.11}$$

$$P\bigg\{\max_{m\le k\le n}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j-EX_j)\Big|\ge\varepsilon\bigg\}$$
$$\le \frac{4}{\varepsilon^2}\bigg\{\sum_{j=1}^{m}\frac{Var(X_j)}{b_m^2}+\sum_{1\le k\ne j\le m}\frac{Cov(X_k,X_j)}{b_m^2}$$
$$+8\sum_{j=m+1}^{n}\frac{Var(X_j)}{b_j^2}+8\sum_{m+1\le k\ne j\le n}\frac{Cov(X_k,X_j)}{b_kb_j}\bigg\}, \tag{3.12}$$

respectively. Since $\{b_n,\ n \ge 1\}$ is a nondecreasing sequence of positive numbers, the right-hand side of (3.3) and (3.4) are dominated by the right-hand side of (3.11) and (3.12), respectively. Hence, Theorem 3.1 improves the result of [6].

Remark 3.2 Under the conditions of Theorem 3.1, Soo[9] obtained the following result:

$$P\left\{\max_{m\le k\le n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \ge \varepsilon\right\} \le \frac{8}{\varepsilon^2}\sum_{j=1}^{m}\frac{Cov(X_j, S_j)}{b_m^2} + \frac{16}{\varepsilon^2}\sum_{j=m+1}^{n}\frac{Cov(X_j, S_j)}{b_j^2}.$$

There is also a typo in [9] (the factor 16 should be 64). Since $Cov(X_j, X_k) \ge 0$ by the definition of associated random variables, the right-hand side of (3.4) is dominated by

$$\frac{8}{\varepsilon^2}\sum_{j=1}^{m}\frac{Cov(X_j, S_j)}{b_m^2} + \frac{64}{\varepsilon^2}\sum_{j=m+1}^{n}\frac{Cov(X_j, S_j)}{b_j^2}.$$

Hence, Theorem 3.1 improves the result of [9].

Remark 3.3 According to the proof of Theorem 3.1, we can see that (1.2) and (1.3) are satisfied for associated random variable sequences. Thus, Theorem 2.3 holds for associated random variable sequences. Furthermore, since

$$\sum_{j=1}^{n}\frac{Var(X_j)}{b_j^2} + 2\sum_{1\le k<j\le n}\frac{Cov(X_k, X_j)}{b_j^2} \le \sum_{j=1}^{n}\frac{Var(X_j)}{b_j^2} + \sum_{1\le k\ne j\le n}\frac{Cov(X_k, X_j)}{b_j b_k},$$

$$\sum_{j=1}^{n}\frac{Var(X_j)}{b_j^2} + 2\sum_{1\le k<j\le n}\frac{Cov(X_k, X_j)}{b_j^2} \le 2\sum_{j=1}^{n}\frac{Cov(X_j, S_j)}{b_j^2},$$

and the integrability of supremum for $r = 2$ is obtained, the results of Theorem 2.3 for associated random variable sequences generalize and improve Theorems 3.3–3.4 of [6] and Theorems 3.1–3.2 of [9].

3.2 Application for strongly positive dependent stochastic sequences

Definition 3.2 *A finite collection of random variables $X_1,\ X_2,\ \cdots,\ X_n$ is said to be strongly positive dependent if*

$$P(\bar{X}_1 \in \Lambda_1;\ \bar{X}_2 \in \Lambda_2) \ge P(\bar{X}_1 \in \Lambda_1)P(\bar{X}_2 \in \Lambda_2) \tag{3.13}$$

for all Borel measurable and increasing (or decreasing) set pairs $(\Lambda_1, \Lambda_2) \subset R_1 \times R_2$ (A set Λ is said to be increasing (or decreasing) if $x \le$ (or $\ge$)y implies $y \in \Lambda$ for any $x \in \Lambda$), where

$$\bar{X}_1 = (X_i,\ i \in I),\quad \bar{X}_2 = (X_i, i \in I^c),\quad I \subset (1, 2, \cdots, n),\quad I^c = (1, 2, \cdots, n)\backslash I,$$

$$R_1 = \mathbf{R}^{|I|},\quad R_2 = \mathbf{R}^{|I^c|}\quad (|I| \text{ stands for the base of } I).$$

An infinite sequence $\{X_n, n \ge 1\}$ is strongly positive dependent if every finite subcollection is strongly positive dependent.

Remark 3.4 Zheng[11] has proved that (3.13) is equivalent to

$$Ef_1(\bar{X}_1)f_2(\bar{X}_2) \ge Ef_1(\bar{X}_1)Ef_2(\bar{X}_2)$$

for all Borel measurable and nonincreasing (or nondecreasing) function pairs $(f_1,\ f_2)$ such that the expectations above are well defined. Thus

$$Cov(X_j, X_k) = EX_jX_k - EX_jEX_k \ge 0 \qquad \text{for all } j, k \ge 1.$$

Lemma 3.3([11], Theorem 1) *Let $\{X_n,\ n \geq 1\}$ be a mean zero strongly positive dependent stochastic sequence with finite second moment and $q > 1$. Then for each $n \geq 1$,*

$$E\Big(\max_{1\leq i\leq n} |S_i|\Big)^q \leq \left(\frac{q}{q-1}\right)^q E|S_n|^q. \tag{3.14}$$

Theorem 3.2 *Let $\{X_n,\ n \geq 1\}$ be a sequence of strongly positive dependent random variables with finite second moment and $\{b_n,\ n \geq 1\}$ be a nondecreasing sequence of positive numbers. Then for any $\varepsilon > 0$ and any positive integers $m < n$,*

$$P\Big\{\max_{1\leq k\leq n}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\Big| \geq \varepsilon\Big\}$$
$$\leq \frac{16}{\varepsilon^2}\Big\{\sum_{j=1}^{n}\frac{Var(X_j)}{b_j^2} + 2\sum_{1\leq k<j\leq n}\frac{Cov(X_k, X_j)}{b_j^2}\Big\}, \tag{3.15}$$

$$P\Big\{\max_{m\leq k\leq n}\Big|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\Big| \geq \varepsilon\Big\}$$
$$\leq \frac{4}{\varepsilon^2}\Big\{\sum_{j=1}^{m}\frac{Var(X_j)}{b_m^2} + 2\sum_{1\leq k<j\leq m}\frac{Cov(X_k, X_j)}{b_m^2}$$
$$+ 16\sum_{j=m+1}^{n}\frac{Var(X_j)}{b_j^2} + 32\sum_{m+1\leq k<j\leq n}\frac{Cov(X_k, X_j)}{b_j^2}\Big\}. \tag{3.16}$$

Proof. By the definition of a strongly positive dependent random variable, we can see that $\{X_n - EX_n,\ n \geq 1\}$ is also a sequence of strongly positive dependent random variables. By Lemma 3.3 for $q = 2$, we have

$$E\bigg(\max_{1\leq i\leq n}\Big|\sum_{j=1}^{i}(X_j - EX_j)\Big|^2\bigg) \leq 4E\Big(\sum_{j=1}^{n}(X_j - EX_j)\Big)^2. \tag{3.17}$$

Similarly to the proof of (3.17), we can get (1.2) for $C = 4$. Therefore, (3.15) and (3.16) follow from Theorem 2.1 and Theorem 2.2, respectively.

3.3 Application for martingale difference sequences

Definition 3.3 *Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables and $\{\mathcal{F}_n,\ n \geq 1\}$ be an increasing sequence of σ fields with $\mathcal{F}_n \subset \mathcal{F}$ for each $n \geq 1$. If X_n is $\mathcal{F}_n$ measurable for each $n \geq 1$, then σ fields $\{\mathcal{F}_n,\ n \geq 1\}$ are said to be adapted to the sequence $\{X_n,\ n \geq 1\}$ and $\{X_n,\ \mathcal{F}_n,\ n \geq 1\}$ is said to be an adapted stochastic sequence.*

If $\{X_n,\ \mathcal{F}_n,\ n \geq 1\}$ is an adapted stochastic sequence with

$$E(X_n|\mathcal{F}_{n-1}) = 0 \qquad a.s.\ for\ each\ n \geq 2,$$

then the sequence $\{X_n,\ \mathcal{F}_n,\ n \geq 1\}$ is called a martingale difference sequence.

If $\{X_n,\ \mathcal{F}_n,\ n \geq 1\}$ is an adapted stochastic sequence with

$$E(X_n|\mathcal{F}_{n-1}) = X_{n-1} \qquad a.s.\ for\ each\ n \geq 2,$$

then the sequence $\{X_n,\ \mathcal{F}_n,\ n \geq 1\}$ is called a martingale.

Remark 3.5 By the definition of a martingale difference sequence, we can see that

$$EX_n = 0, \qquad n \geq 1.$$

Let $j < k$, $X_j \in \mathcal{F}_j \subset \mathcal{F}_{k-1}$. Then

$$EX_jX_k = E\{E(X_jX_k|\mathcal{F}_{k-1})\} = E\{X_jE(X_k|\mathcal{F}_{k-1})\} = 0.$$

Thus

$$Cov(X_j, X_k) = EX_jX_k - EX_jEX_k = 0, \qquad j \neq k.$$

Lemma 3.4([12], Corollary 3.3.2) *Let $\{T_n,\ n \geq 1\}$ be a martingale or nonnegative submartingale. For fixed $q > 1$, suppose that*

$$E|T_n|^q < \infty, \qquad n \geq 1.$$

Then

$$E\Big(\max_{1\leq i\leq n} |T_i|\Big)^q \leq \Big(\frac{q}{q-1}\Big)^q E|T_n|^q, \qquad n \geq 1.$$

Corollary 3.1 *Let $\{X_n,\ n \geq 1\}$ be a martingale difference sequence with finite second moment and $\{b_n,\ n \geq 1\}$ be a nondecreasing sequence of positive numbers. Then for any $\varepsilon > 0$ and any positive integers $m < n$,*

$$P\Big\{\max_{1\leq k\leq n}\Big|\frac{1}{b_k}\sum_{j=1}^{k} X_j\Big| \geq \varepsilon\Big\} \leq \frac{16}{\varepsilon^2}\sum_{j=1}^{n}\frac{Var(X_j)}{b_j^2}, \tag{3.18}$$

$$P\Big\{\max_{m\leq k\leq n}\Big|\frac{1}{b_k}\sum_{j=1}^{k} X_j\Big| \geq \varepsilon\Big\} \leq \frac{4}{\varepsilon^2}\Big\{\sum_{j=1}^{m}\frac{Var(X_j)}{b_m^2} + 16\sum_{j=m+1}^{n}\frac{Var(X_j)}{b_j^2}\Big\}. \tag{3.19}$$

Proof. It is a simple fact that

$$EX_n = 0.$$

Denote

$$W_n \doteq \sum_{i=1}^{n} X_i, \qquad n \geq 1.$$

Thus, $\{W_n,\ n \geq 1\}$ is a martingale satisfying the conditions of Lemma 3.4. By Lemma 3.4 for $q = 2$, we have

$$E\Big(\max_{1\leq i\leq n}\Big|\sum_{j=1}^{i} X_j\Big|^2\Big) \leq 4E\Big(\sum_{j=1}^{n} X_j\Big)^2. \tag{3.20}$$

Similarly to the proof of (3.20), we can get (1.2) for $C = 4$. Therefore, (3.18) and (3.19) follow from Theorems 2.1 and 2.2, respectively.

Remark 3.6 According to the proof of Theorems 3.2 and 3.3, we can see that (1.2) and (1.3) are satisfied for strongly positive dependent stochastic sequences and martingale difference sequences, and thus, Theorem 2.3 also holds for them.

References

[1] Hájek, J. and Rényi, A., A generalization of an inequality of Kolmogorov, *Acta Math. Acad. Sci. Hungar.*, **6**(1955), 281–284.

[2] Gan, S., The Hájeck-Rényi inequality for Banach space valued martingales and the p smoothness of Banach spaces, *Statist. Probab. Lett.*, **32**(1997), 245–248.

[3] Liu, J., Gan, S. and Chen, P., The Hájeck-Rényi inequality for the NA random variables and its application, *Statist. Probab. Lett.*, **43**(1999), 99–105.

[4] Christofides, T. C., Maximal inequalities for demimartingale and a strong law of large numbers, *Statist. Probab. Lett.*, **50**(2000), 357–363.

[5] Fazekas, L. and Klesov, O., A general approach to the strong law of large numbers, *Theory Probab. Appl.*, **45**(2001), 436–449.

[6] Prakasa Rao, B. L. S., Hájek-Rényi type inequality for associated sequences, *Statist. Probab. Lett.*, **57**(2002), 139–143.

[7] Hu, S. H., Hu, X. P. and Zhang, L. S., Hájek-Rényi-type inequality under second moment conditions and its applications (in Chinese), *Acta Math. Appl. Sinica*, **28**(2005), 227–235.

[8] Tómács, T. and Líbor, Z., A Hájek-Rényi type inequality and its applications, *Ann. Math. Inform.*, **33**(2006), 141–149.

[9] Soo, H. S., A note on the Hájek-Rényi inequality for associated random variables, *Statist. Probab. Lett.*, **78**(2008), 885–889.

[10] Newman, C. M. and Wright, A. L., An invariance principle for certain dependent sequences, *Ann. Probab.*, **9**(1981), 671–675.

[11] Zheng, Y. H., Inequalities of moment and convergence theorem of order statistics of partial sums for a class of strongly positive dependent stochastic sequence (in Chinese), *Acta Math. Appl. Sinica*, **24**(2001), 168–176.

[12] Stout, W. F., Almost Sure Convergence, Academic Press, Inc, New York, 1974, pp.137–158.