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An Algorithm for Reducibility of 3-arrangements*

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Abstract: We consider a central hyperplane arrangement in a three-dimensional vector space. The definition of characteristic form to a hyperplane arrangement is given and we could make use of characteristic form to judge the reducibility of this arrangement. In addition, the relationship between the reducibility and freeness of a hyperplane arrangement is given.
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1 Introduction

In this paper, we study the reducibility and freeness of central arrangements in a three-dimensional vector space. In [1] and [2], the authors gave the necessary and sufficient conditions of the reducibility of an arrangement $\mathcal{A}$. They connected the reducibility with one degree-branch of $D(\mathcal{A})$ and Beta invariant $\beta(\mathcal{A})$ (see [3]) respectively. However, it is needed a great quantity of calculation to judge the reducibility of $\mathcal{A}$. We give a simple method to judge the reducibility of an arrangement $\mathcal{A}$, which needs only the characteristic form of $\mathcal{A}$.

The freeness is an important property of an arrangement. Recently, there are many papers studying the freeness of arrangements. For example, Ziegler[4] and Yuzvinsky[5] gave some necessary and sufficient conditions of freeness. In addition, some conclusions on freeness of arrangements in a vector space of dimension three or higher were given in [6] and [7]. We study the freeness of arrangements from another angle of view, and the relationship between reducibility and freeness of a hyperplane arrangement is given in this paper.

The notions and symbols in this paper are the same as in [8] and [9].

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2 Basic Notions

Let $\mathbb{K}$ be a field and V be a vector space of dimension l on $\mathbb{K}$. A hyperplane H in V is an affine subspace of dimension $(l-1)$. A hyperplane arrangement $\mathcal{A}$ is a finite set of hyperplanes in V. If $\mathcal{A}$ consists of n hyperplanes, we write that

$$|\mathcal{A}| = n.$$

We call polynomial

$$Q(\mathcal{A}) = \prod_{H\in\mathcal{A}} \alpha_H \qquad (\ker \alpha_H = H)$$

the defining polynomial of $\mathcal{A}$. If

$$\bigcap_{H\in\mathcal{A}} H \neq \emptyset,$$

then we call $\mathcal{A}$ central, otherwise, we call $\mathcal{A}$ non-central. The dimension of $\mathcal{A}$, $\dim\mathcal{A}$, is defined to be

$$\dim\mathcal{A} = l.$$

Let W be the space spanned by the normals to the hyperplanes in $\mathcal{A}$, and the rank of $\mathcal{A}$, $\mathrm{rank}(\mathcal{A})$, be the dimension of W. We say that $\mathcal{A}$ is essential if

$$\mathrm{rank}(\mathcal{A}) = \dim(\mathcal{A}).$$

Let

$$L = L(\mathcal{A})$$

be the set of nonempty intersections of elements of $\mathcal{A}$.

Let $(\mathcal{A}_1,\ V_1)$ and $(\mathcal{A}_2,\ V_2)$ be two arrangements and

$$V = V_1 \oplus V_2.$$

Define the product arrangement $(\mathcal{A}_1 \times \mathcal{A}_2,\ V)$ by

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\}.$$

Call the arrangement $(\mathcal{A},\ V)$ reducible if after a change of coordinates, we get

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2,$$

otherwise, call $\mathcal{A}$ irreducible.

Let

$$S = S(V^*)$$

be the symmetric algebra of the dual space V^* of V. If $x_1, \cdots, x_l$ is a basis for V^*, then

$$S \simeq \mathbb{K}[x_1,\ \cdots,\ x_l].$$

Let

$$\mathrm{Der}_{\mathbb{K}}(S) = \{\theta : S \to S \mid \theta(fg) = f\theta(g) + g\theta(f),\ f, g \in S\}.$$

Let S_p denote the $\mathbb{K}$-vector subspace of S consisting of 0 and the homogeneous polynomials of degree p for $p \geq 0$. A nonzero element $\theta \in \mathrm{Der}_{\mathbb{K}}(S)$ is homogeneous of polynomial degree p if

$$\theta = \sum_{k=1}^{l} f_k \frac{\partial}{\partial x_k} \in \mathrm{Der}_{\mathbb{K}}(S),$$

and $f_k \in S_p$ for $1 \leq k \leq l$. In this case we write

$$\mathrm{pdeg}\theta = p.$$

Let

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}.$$

We call $\mathcal{A}$ a free arrangement if $D(\mathcal{A})$ is a free module over S. Let $\mathcal{A}$ be a free arrangement and let $\{\theta_1, \cdots, \theta_l\}$ be a homogeneous basis for $D(\mathcal{A})$. We call $\mathrm{pdeg}\theta_1, \cdots, \mathrm{pdeg}\theta_l$ the exponents of $\mathcal{A}$ and write

$$\exp \mathcal{A} = \{\mathrm{pdeg}\theta_1, \ \cdots, \ \mathrm{pdeg}\theta_l\}.$$

Then we know that

$$\sum_{i=1}^{l} \mathrm{pdeg}\theta_i = |\mathcal{A}|.$$

3 Main Results

Since every non-essential arrangement can be considered as the product arrangement of its essentialization and an empty arrangement, we assume that $\mathcal{A}$ is an essential arrangement in this paper.

Definition 3.1 *Let $\mathcal{A}$ be an arrangement with $|\mathcal{A}| = n$. Let*

$$S = \{\mathcal{B} \mid \mathcal{B} \textit{ is the maximal subset of } \mathcal{A} \textit{ with } \mathrm{rank}(\mathcal{B}) = 2\},$$

$$k_i = \#\{\mathcal{B} \in S \mid \#\mathcal{B} = i + 1\},$$

where $\#\mathcal{B}$ stands for the number of elements in $\mathcal{B}$. It follows from the definitions of S and k_i that $k_i = 0$ when $i \leq 0$ and $i \geq n$. We call set $(2^{k_1}, \cdots, n^{k_{n-1}})$ the characteristic form of $\mathcal{A}$.

Example 3.1 Let $\mathcal{A}$ be the arrangement defined by

$$Q(\mathcal{A}) = xyz(x + y).$$

The characteristic form of $\mathcal{A}$ is $(2^3, \ 3^1)$.

If the characteristic form of $\mathcal{A}$ is $(2^{k_1}, \cdots, n^{k_{n-1}})$, it is clear that the following equality holds:

$$\binom{n}{2} = k_1\binom{2}{2} + \cdots + k_{n-1}\binom{n}{2}.$$

The above equality can be used to examine whether a given characteristic form is right or not.

Lemma 3.1[2] *The arrangement $\mathcal{A}$ is reducible if and only if $\beta(\mathcal{A}) = 0$, where*

$$\beta(\mathcal{A}) = (-1)^{\mathrm{rank}(\mathcal{A})-1} \left.\frac{\mathrm{d}\chi(\mathcal{A}, t)}{\mathrm{d}t}\right|_{t=1}$$

is the Beta invariant of $\mathcal{A}$ and $\chi(\mathcal{A}, t)$ is the characteristic polynomial of $\mathcal{A}$.

Theorem 3.1 *Let $\mathcal{A}$ be an arrangement with $|\mathcal{A}| = n$. If the characteristic form of $\mathcal{A}$ is $(2^{k_1}, \cdots, n^{k_{n-1}})$, then $\mathcal{A}$ is reducible if and only if*

$$\sum_{i=1}^{n-1} ik_i = 2n - 3.$$

Proof. It follows from Lemma 3.1 that $\mathcal{A}$ is reducible if and only if

$$\frac{\mathrm{d}\chi(\mathcal{A}, t)}{\mathrm{d}t}\bigg|_{t=1} = 0.$$

Since $|\mathcal{A}| = n$, by the definition of the characteristic polynomial, we assume that

$$\chi(\mathcal{A}, t) = t^3 - nt^2 + a_1 t - a_0,$$

where a_0, $a_1 \in \mathbb{N}$. Then

$$\frac{\mathrm{d}\chi(\mathcal{A}, t)}{\mathrm{d}t}\bigg|_{t=1} = 3 - 2n + a_1 = 0,$$

that is,

$$a_1 = 2n - 3.$$

In addition, if $\mathcal{B} \in S$ and $\#\mathcal{B} = i$, then, for $x = \bigcap\limits_{H \in \mathcal{B}} H$, we know that

$$\mu(x) = -\sum_{z<x} \mu(z) = -(1 - i) = i - 1,$$

where μ is the Möbius function of $L(\mathcal{A})$. So

$$a_1 = \sum_{\substack{x \in L(\mathcal{A}) \\ \dim x = 1}} \mu(x) = \sum_{\substack{\cap\mathcal{B}=x \\ \mathcal{B} \in S}} \mu(x) = \sum_{\substack{\#\mathcal{B}=i \\ \mathcal{B} \in S}} (i - 1)(k_{i-1}) = \sum_{i=1}^{n-1} ik_i.$$

Hence, we have

$$a_1 = \sum_{i=1}^{n-1} ik_i = 2n - 3.$$

It follows from Definition 3.1 that we may give the characteristic form of an arbitrary arrangement $\mathcal{A}$ only by the defining polynomial $Q(\mathcal{A})$, and after that we could judge the reducibility of $\mathcal{A}$ quickly by the equality in Theorem 3.1.

Example 3.2 Let $\mathcal{A}$ be the arrangement defined by

$$Q(\mathcal{A}) = (x + y + z)(x + 2y + 3z)(2x - y - z)(x + 2y + 2z).$$

The characteristic form of $\mathcal{A}$ is $(2^3,\ 3^1)$. Then $\mathcal{A}$ is reducible by a direct examination.

Corollary 3.1 *Let $\mathcal{A}$ be an arrangement with $|\mathcal{A}| = n$. If the characteristic form of $\mathcal{A}$ is $(2^{k_1}, \ldots, n^{k_{n-1}})$, then $\mathcal{A}$ is factored if and only if $(n+1)^2 - 4\Big(\sum\limits_{i=1}^{n-1} ik_i + 1\Big)$ is a square of an integer. We say that the arrangement $\mathcal{A}$ is factored if the characteristic polynomial $\chi(\mathcal{A}, t)$ has a complete factorization over $\mathbb{Z}$.*

Proof. Since $\mathcal{A}$ is a central arrangement, we have

$$\chi(\mathcal{A}, 1) = \sum_{x \in L(\mathcal{A})} \mu(x) = \sum_{\hat{0} \le x \le \hat{1}} \mu(x) = 0,$$

that is, $(t-1)$ is a factor of $\chi(\mathcal{A}, t)$. We may let

$$\chi(\mathcal{A}, t) = t^3 - nt^2 + a_1 t - (1 - n + a_1) = (t-1)[t^2 + (1-n)t + (1 - n + a_1)].$$

Then we know that $\mathcal{A}$ is factored if and only if $t^2 + (1-n)t + (1-n+a_1)$ has a complete factorization over $\mathbb{Z}$, that is,

$$(1-n)^2 - 4(1-n+a_1) = (n+1)^2 - 4(a_1+1) = (n+1)^2 - 4\Big(\sum_{i=1}^{n-1} ik_i + 1\Big)$$

is a square of an integer.

Next, we use the following results on freeness. For the detailed proofs, please see [8].

Lemma 3.2[8] *Let $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ be two arrangements. The product arrangement $(\mathcal{A}_1 \times \mathcal{A}_2,\ V_1 \oplus V_2)$ is free if and only if both $(\mathcal{A}_1,\ V_1)$ and $(\mathcal{A}_2,\ V_2)$ are free. In this case,*

$$\exp(\mathcal{A}_1 \times \mathcal{A}_2) = \{\exp \mathcal{A}_1,\ \exp \mathcal{A}_2\}.$$

Lemma 3.3[8] *If $\mathcal{A}$ is a free arrangement with*

$$\exp \mathcal{A} = \{b_1,\ \cdots,\ b_l\},$$

then

$$\chi(\mathcal{A},\ t) = \prod_{i=1}^{l}(t - b_i).$$

The following theorem gives the relationship between reducibility and freeness of $\mathcal{A}$.

Theorem 3.2 *Let $\mathcal{A}$ be an arrangement with $|\mathcal{A}| = n$. Then $\mathcal{A}$ is reducible if and only if $\mathcal{A}$ is free and*

$$\exp \mathcal{A} = \{1,\ 1,\ n-2\}.$$

Proof. If $\mathcal{A}$ is reducible, then there exist two non-empty arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$, and after a change of coordinates, we have

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2.$$

Since

$$\dim \mathcal{A} = 3,$$

we may assume that

$$\dim \mathcal{A}_1 = 1, \qquad \dim \mathcal{A}_2 = 2.$$

Since $\mathcal{A}$ is central, we have

$$|\mathcal{A}_1| = 1.$$

Thus $\theta = x\dfrac{\partial}{\partial x}$ is the basis of $D(\mathcal{A}_1)$, and then $D(\mathcal{A}_1)$ is a free module, so $\mathcal{A}_1$ is free and

$$\exp \mathcal{A}_1 = \{1\}.$$

Recall that all 2-arrangements are free. Thus $\mathcal{A}_2$ is free and

$$\exp \mathcal{A}_2 = \{1,\ |\mathcal{A}_2| - 1\} = \{1,\ n-2\}.$$

It follows from Lemma 3.2 that $\mathcal{A}$ is free and

$$\exp \mathcal{A} = \{\exp \mathcal{A}_1,\ \exp \mathcal{A}_2\} = \{1,\ 1,\ n-2\}.$$

Conversely, if $\mathcal{A}$ is free and

$$\exp\mathcal{A} = \{1,\ 1,\ n-2\},$$

then by Lemma 3.3,

$$\chi(\mathcal{A},\ t) = (t-1)^2(t-n+2) = t^3 - nt^2 + (2n-3)t - (n-2),$$

and we have

$$a_1 = 2n-3.$$

Thus $\mathcal{A}$ is reducible by Theorem 3.1.

4 Applications

There is another use of the characteristic form of an arrangement $\mathcal{A}$. It can be used to judge the freeness of some arrangements quickly. By Lemma 3.3, if the characteristic polynomial of $\mathcal{A}$ cannot factor completely over $\mathbb{Z}$, then $\mathcal{A}$ is not free. Then by Corollary 3.1, if $(n+1)^2 - 4\Big(\sum\limits_{i=1}^{n-1} ik_i + 1\Big)$ is not a square of an integer, then $\mathcal{A}$ is not free. We make use of the above result to judge the freeness of the following arrangements.

Example 4.1 Let $\mathcal{A}$ be the arrangement defined by

$$Q(\mathcal{A}) = xyz(x+y-z).$$

The characteristic form of $\mathcal{A}$ is (2^6). Then

$$(|\mathcal{A}|+1)^2 - 4\Big(\sum_{i=1}^{n-1} ik_i + 1\Big) = -3$$

is not a square of an integer, and thus $\mathcal{A}$ is not free.

Lemma 4.1[8] *Suppose $\mathcal{A}$ is not empty. Let $(\mathcal{A},\ \mathcal{A}',\ \mathcal{A}'')$ be a triple, where*

$$\mathcal{A}' = \mathcal{A} - \{H_0\}, \qquad \mathcal{A}'' = \{H_0 \cap H \neq \emptyset \mid H \in \mathcal{A}'\},$$

and $H_0 \in \mathcal{A}$ is a distinguished hyperplane. Any two of the following statements imply the third:

(1) $\mathcal{A}$ *is free with* $\exp\mathcal{A} = \{b_1,\ \cdots,\ b_{l-1}, b_l\}$;

(2) $\mathcal{A}'$ *is free with* $\exp\mathcal{A}' = \{b_1,\ \cdots,\ b_{l-1},\ b_l - 1\}$;

(3) $\mathcal{A}''$ *is free with* $\exp\mathcal{A}'' = \{b_1,\ \cdots,\ b_{l-1}\}$.

Example 4.2 Let $\mathcal{A}$ be the arrangement defined by

$$Q(\mathcal{A}) = xyz(x+y)(x+y-z).$$

Choose

$$H_0 = \ker z.$$

Then

$$Q(\mathcal{A}') = xy(x+y)(x+y-z),$$

and the characteristic form of $\mathcal{A}'$ is $(2^3,\ 3^1)$. By Theorem 3.1, $\mathcal{A}'$ is reducible, and then by Theorem 3.2, $\mathcal{A}'$ is free with

$$\exp\mathcal{A}' = \{1,\ 1,\ 2\}.$$

Since

$$\dim(\mathcal{A}'') = 2, \qquad |\mathcal{A}''| = 3,$$

$\mathcal{A}''$ is free with

$$\exp \mathcal{A}'' = \{1,\ 2\}.$$

By Lemma 4.1, $\mathcal{A}$ is free with

$$\exp \mathcal{A} = \{1,\ 2,\ 2\}.$$

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