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On Homomorphism of Valuation Algebras*

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Abstract: In this paper, firstly, a necessary condition and a sufficient condition for an isomorphism between two semiring-induced valuation algebras to exist are presented respectively. Then a general valuation homomorphism based on different domains is defined, and the corresponding homomorphism theorem of valuation algebra is proved.

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1 Introduction

Valuation algebra is an abstract formalization from artificial intelligence including constraint systems (see [1]), Dempster-Shafer belief functions (see [2]), database theory, logic, and etc. There are three operations including labeling, combination and marginalization in a valuation algebra. With these operations on valuations, the system could combine information, and get information on a designated set of variables by marginalization. With further research and theoretical development, a new type of valuation algebra named domain-free valuation algebra is also put forward.

As valuation algebra is an algebraic structure, the concept of homomorphism between valuation algebras, which is derived from the classic study of universal algebra, has been defined naturally. Meanwhile, recent studies in [3], [4] have showed that valuation algebras induced by semirings play a very important role in applications. Based on the above fundamental factors, in this paper, we study the relationship between homomorphism of semiring-induced valuation algebras and that of semirings. Furthermore, in view of the lim-

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itation in the notion of valuation homomorphism, we give a general definition of valuation homomorphism which are based on different domain sets, and study its properties.

2 Notations and Preliminaries

The fundamental elements of a valuation algebra are valuations. In general, a valuation is a function that provides possible elements of a field for variables. Here a valuation represents some knowledge and information which may be a function, tuple or symbol.

In this study, variables will be designated by capital like X, Y, $\cdots$. The symbol Ω_X which contains at least two elements is the finite set of possible values of X, and it is called the frame of X. Lower-case letters such as x, y, $\cdots$, denote sets of variables. For a nonempty set s of variables, let Ω_s denote the Cartesian product of the frames Ω_X of the variables $X \in s$, i.e., $\Omega_s = \prod\limits_{X\in s} \Omega_X$, and Ω_s is called the frame of the set of variables s. If s is empty, for convenience, we denote $\Omega_\emptyset = \{\diamond\}$. We use lower-case, bold-faced letters such as $\mathbf{x}$, $\mathbf{y}$, $\cdots$ to designate the elements of Ω_s. If $\mathbf{x} \in \Omega_s$ and $t \subseteq s$, then $\mathbf{x}^{\downarrow t}$ denotes the projection of $\mathbf{x}$ to the subdomain t. In particular, we have $\mathbf{x}^{\downarrow\emptyset} = \diamond$.

Definition 2.1[1,3] *A semiring is a tuple $\mathcal{A} = \langle A, +, \times, 0, 1\rangle$ such that*

1. $+$, $\times$ *are commutative and associative*;
2. $\times$ *distributes over* $+$;
3. 0 *and* 1 *are the unit element of* $+$ *and* $\times$ *respectively. In addition,* 0 *is an absorbing element of* $\times$, *i.e.,*

$$0 \times a = a \times 0 = 0, \qquad a \in A.$$

For convenience, a semiring $\mathcal{A}$ will be denoted as A. If a semiring A satisfies

$$a + 1 = 1, \qquad a \in A,$$

then we call A a c-semiring. In a c-semiring A, we have

$$a + b = \sup\{a, b\}, \quad a \times b \le a, \qquad a, b \in A.$$

Now we consider a non-empty finite set r of variables with finite frames and a semiring A. A semiring valuation ϕ with domain $s \subseteq r$ is defined to be a function that associates a value from A with $\mathbf{x} \in \Omega_s$, i.e., $\phi : \Omega_s \to A$. The symbol $d(\phi)$ denotes the domain of the valuation ϕ, i.e., $d(\phi) = s$. Φ_s denotes the set of all valuations with domain s, and $\Phi = \bigcup\limits_{s\subseteq r} \Phi_s$. Let $D = \mathcal{P}(r)$ denote the lattice of subset of r. We define the operations in the pair (Φ, D) by using the operations $+$ and $\times$ in A:

(1) Combination: $\otimes : \Phi \times \Phi \to \Phi$: for $\phi, \psi \in \Phi$ and $\mathbf{x} \in \Omega_{d(\phi)\cup d(\psi)}$ we define

$$\phi \otimes \psi(\mathbf{x}) = \phi(\mathbf{x}^{\downarrow d(\phi)}) \times \psi(\mathbf{x}^{\downarrow d(\psi)});$$

(2) Marginalization: $\downarrow$: $\Phi \times D \to \Phi$ is defined by

$$\phi^{\downarrow t}(\mathbf{x}) = \sum_{\mathbf{z}\in\Omega_{d(\phi)}:\mathbf{z}^{\downarrow t}=\mathbf{x}} \phi(\mathbf{z}), \qquad \phi \in \Phi,\ t \subseteq d(\phi),\ \mathbf{x} \in \Omega_t.$$

In [3], it has shown that (Φ, D) satisfies the following axioms (1)–(6) in Definition 2.2, if A is a semiring. The system (Φ, D) is called a valuation algebra induced by the semiring A.

Definition 2.2[5] *A system $(\varPhi, D)$ with three operations is defined as follows, where D is the power set of a variable set r :*

1. *Labeling*: $\varPhi \to D$; $\phi \mapsto d(\phi)$;
2. *Combination*: $\varPhi \times \varPhi \to \varPhi$; $(\phi, \psi) \mapsto \phi \otimes \psi$;
3. *Projection:* $\varPhi \times D \to \varPhi$; $(\phi, x) \mapsto \phi^{\downarrow x}$.

If the following axioms hold, we call $(\varPhi, D)$ a labeled valuation algebra:

(1) *Semigroup*: *$\varPhi$ is associative and commutative under combination. For all $s \in D$, $\varPhi_s$ has a unit element e_s with the operation $\otimes$, and e_s is called a neutral element of $\varPhi_s$*;

(2) *Labeling*: *For $\phi, \psi \in \varPhi$, $d(\phi \otimes \psi) = d(\phi) \cup d(\psi)$*;

(3) *Marginalization*: *For $\phi \in \varPhi$ and $x \subseteq d(\phi)$, $d(\phi^{\downarrow x}) = x$*;

(4) *Transitivity*: *For $\phi \in \varPhi$ and $x \subseteq y \subseteq d(\phi)$, $(\phi^{\downarrow y})^{\downarrow x} = \phi^{\downarrow x}$*;

(5) *Combination*: *For $\phi, \psi \in \varPhi$ with $d(\phi) = x$, $d(\psi) = y$, $(\phi \otimes \psi)^{\downarrow x} = \phi \otimes \psi^{\downarrow x \cap y}$*;

(6) *Neutrality*: *For $x, y \in D$, $e_x \otimes e_y = e_{x \cup y}$.*

Moreover, a labeled valuation algebra, satisfying the stability and idempotency, is called an information algebra:

(7) *Stability*: *For $x, y \in D$, $x \subseteq y$, $e_y^{\downarrow x} = e_x$*;

(8) *Idempotency*: *For $\phi \in \varPhi$ and $x \in D$, $x \subseteq d(\phi)$, $\phi \otimes \phi^{\downarrow x} = \phi$.*

If $(\varPhi, D)$ is an information algebra, we write $\phi \le \psi$ when $\phi \otimes \psi = \psi$. The order $\le$ is a partial order on an information algebra. Since, for the set D, the axioms in the definition of valuation algebra involve only its two operations $\cap$, $\cup$, we can take the domains of valuations as elements of a general lattice. For other unexplained (or undefined) notions or more details on valuation algebras, please refer to [5] and [6].

3 Homomorphism Between Semiring-induced Valuation Algebras

Definition 3.1[4] *A mapping f from a semiring A to a semiring B is said to be a semiring homomorphism, if for all $a, b \in A$, it satisfies that $f(a+b) = f(a) + f(b)$, $f(a \times b) = f(a) \times f(b)$, $f(0) = 0$ and $f(1) = 1$. If f is a bijective, then it is called a semiring isomorphism.*

Definition 3.2[5] *Let $(\varPhi, D)$, $(\varPsi, D)$ be two labeled valuation algebras. A mapping h : $(\varPhi, D) \to (\varPsi, D)$ is called a homomorphism, if, for ϕ_1, ϕ_2, $\phi \in \varPhi$,*

1. $h(\phi_1 \otimes \phi_2) = h(\phi_1) \otimes h(\phi_2)$;
2. $h(\phi^{\downarrow x}) = h(\phi)^{\downarrow x}$, *if $x \subseteq d(\phi)$*;
3. $h(e_x) = e_x$ *for all $x \in D$.*

If h is injective, then h is called a monomorphism. If h is bijective, then h is called an isomorphism.

It is clear that $(h(\varPhi),\ D)$, the homomorphic image of $(\varPhi,\ D)$, is also a valuation algebra. If h is a homomorphism between two valuation algebras with the stability, then h maintains the domain of valuation, i.e.,

$$d(h(\phi)) = d(\phi), \qquad \phi \in \Phi.$$

Lemma 3.1 *Let $(\Phi,\ D)$, $(\Psi,\ D)$ be the valuation algebras induced by semirings A and B respectively, and let $f : A \to B$ be a semiring homomorphism. For $\phi \in \Phi$, let $h(\phi) = f \circ \phi$. Then $(h(\Phi),\ D)$ is a valuation algebra.*

Moreover, if $(\Phi,\ D)$ is an information algebra, then $(h(\Phi),\ D)$ is also an information algebra.

Proof. Firstly, we show that h is a homomorphism.

1. Let $\phi,\ \psi \in \Phi$ with $d(\phi) = s$, $d(\psi) = t$. For all $\mathbf{x} \in \Omega_{s\cup t}$, since f is a semiring homomorphism, we have

$$\begin{aligned} h(\phi \otimes \psi)(\mathbf{x}) &= f((\phi \otimes \psi)(\mathbf{x})) \\ &= f(\phi(\mathbf{x}^{\downarrow s})) \times f(\psi(\mathbf{x}^{\downarrow t})) \\ &= h(\phi)(\mathbf{x}^{\downarrow s}) \times h(\psi)(\mathbf{x}^{\downarrow t}) \\ &= (h(\phi) \otimes h(\psi))(\mathbf{x}). \end{aligned}$$

So $h(\phi \otimes \psi) = h(\phi) \otimes h(\psi)$.

2. Let $\phi \in \Phi$ with $d(\phi) = s$. If $t \subseteq s$, for all $\mathbf{x} \in \Omega_t$, we have

$$h(\phi)^{\downarrow t}(\mathbf{x}) = \sum_{\mathbf{y}\in\Omega_{s-t}} (f \circ \phi)(\mathbf{x},\ \mathbf{y}) = f\Big(\sum_{\mathbf{y}\in\Omega_{s-t}} \phi(\mathbf{x},\ \mathbf{y})\Big) = f(\phi^{\downarrow t}(\mathbf{x})) = h(\phi^{\downarrow t})(\mathbf{x}).$$

Then $h(\phi)^{\downarrow t} = h(\phi^{\downarrow t})$.

3. In a semiring-induced valuation algebra, the neutral element e_s of the semigroup Φ_s is defined by $e_s(\mathbf{x}) = 1$ for all $\mathbf{x} \in \Omega_s$. Because f is a semiring homomorphism, then

$$h(e_s)(\mathbf{x}) = f(e_s(\mathbf{x})) = f(1) = 1.$$

Therefore, $h(e_s) = e_s$.

In summary, h is a homomorphism, and then $(h(\Phi),\ D)$ is a valuation algebra.

Assume that $(\Phi,\ D)$ is an information algebra. For all $\phi \in \Phi$ with $d(\phi) = s$, if $t \subseteq s$ and $\mathbf{x} \in \Omega_s$, then

$$\begin{aligned} (h(\phi) \otimes h(\phi)^{\downarrow t})(\mathbf{x}) &= h(\phi)(\mathbf{x}) \times h(\phi)^{\downarrow t}(\mathbf{x}^{\downarrow t}) \\ &= f(\phi(\mathbf{x})) \times \sum_{\mathbf{y}\in\Omega_{s-t}} f(\phi(\mathbf{x}^{\downarrow t}, \mathbf{y})) \\ &= f\Big(\phi(\mathbf{x}) \times \sum_{\mathbf{y}\in\Omega_{s-t}} \phi(\mathbf{x}^{\downarrow t}, \mathbf{y})\Big) \\ &= f(\phi \otimes \phi^{\downarrow t}(\mathbf{x})) \\ &= h(\phi)(\mathbf{x}). \end{aligned}$$

So

$$h(\phi) \otimes h(\phi)^{\downarrow t} = h(\phi).$$

Thus the property of idempotency holds.

Homomorphism preserves the neutral element e_s of every semigroup Φ_s, so the stability of $(h(\Phi),\ D)$ holds. Hence $(h(\Phi),\ D)$ is an information algebra.

Theorem 3.1 *Let $(\varPhi,\ D)$, $(\varPsi,\ D)$ be the valuation algebras induced by semirings A and B respectively. If $f : A \to B$ is a semiring isomorphism, then $(\varPhi,\ D)$ is isomorphic to $(\varPsi,\ D)$.*

Proof. According to Lemma 3.1, we need only to show that $h : \varPhi \to \varPsi$ is bijective.

If $h(\phi_1) = h(\phi_2)$, then

$$d(\phi_1) = d(f \circ \phi_1) = d(h(\phi_1)) = d(h(\phi_2)) = d(f \circ \phi_2) = d(\phi_2).$$

Suppose $d(\phi_1) = d(\phi_2) = s$. We have

$$f(\phi_1(\mathbf{x})) = (f \circ \phi_1)(\mathbf{x}) = (f \circ \phi_2)(\mathbf{x}) = f(\phi_2(\mathbf{x})), \qquad \mathbf{x} \in \varOmega_s.$$

So

$$\phi_1(\mathbf{x}) = \phi_2(\mathbf{x}), \qquad \mathbf{x} \in \varOmega_s,$$

that is,

$$\phi_1 = \phi_2.$$

For all $\psi : \varOmega_s \to B$, we take a valuation $\phi : \varOmega_s \to A$ as $\phi(\mathbf{x}) = f^{-1}(\psi(\mathbf{x}))$. Then

$$h(\phi) = \psi.$$

Therefore, h is an isomorphism.

Corollary 3.1 *If $(\varPhi,\ D)$ is the information algebra induced by a semiring A, and A is isomorphic to a semiring B, then $(\varPsi,\ D)$ induced by B is also an information algebra.*

Proof. By Theorem 3.1 and Lemma 3.1, we obtain that $(\varPsi,\ D)$ is an information algebra immediately.

Proposition 3.1 *Let $g : B \to A$ be an injective semiring homomorphism, $(\varPhi,\ D)$ and $(\varPsi,\ D)$ be two valuation algebras induced by semirings A and B respectively. If $h : (\varPhi,\ D) \to (\varPsi,\ D)$ is a mapping such that $g \circ h(\phi) = \phi$ for all $\phi \in \varPhi$, then $h : (\varPhi,\ D) \to (\varPsi,\ D)$ is a monomorphism.*

Proof. Since g is injective, h is well-defined. In fact, for a valuation $\phi \in \varPhi$, if $h(\phi) = \psi_1$ and $h(\phi) = \psi_2$, then

$$g \circ \psi_1 = \phi = g \circ \psi_2.$$

Because g is injective, we have

$$\psi_1 = \psi_2.$$

In the following we show that h is a homomorphism.

First, for all $\phi \in \varPhi$, by the definition of g, we have

$$d(h(\phi)) = d(g \circ h(\phi)) = d(\phi).$$

1. h preserves the neutral elements of $\varPhi_s$ for all $s \in D$: for any $\mathbf{x} \in \varOmega_s$, we have

$$(g \circ h(e_s))(\mathbf{x}) = e_s(\mathbf{x}) = 1,$$

i.e.,

$$g(h(e_s)(\mathbf{x})) = 1.$$

Since g is an injective semiring homomorphism, we have

$$h(e_s)(\mathbf{x}) = 1.$$

Thus $h(e_s) = e_s$.

2. For all ϕ_1, $\phi_2 \in \Phi$ with $d(\phi_1) = s$, $d(\phi_2) = t$, we have

$$d(h(\phi_1 \otimes \phi_2)) = d(\phi_1 \otimes \phi_2) = d(h(\phi_1) \otimes h(\phi_2)) = s \cup t.$$

For all $\mathbf{x} \in s \cup t$, we obtain that

$$\begin{aligned}(g \circ (h(\phi_1) \otimes h(\phi_2)))(\mathbf{x}) &= g(h(\phi_1)(\mathbf{x}^{\downarrow s}) \times h(\phi_2)(\mathbf{x}^{\downarrow t}))\\ &= g(h(\phi_1)(\mathbf{x}^{\downarrow s})) \times g(h(\phi_2)(\mathbf{x}^{\downarrow t}))\\ &= \phi_1(\mathbf{x}^{\downarrow s}) \times \phi_2(\mathbf{x}^{\downarrow t})\\ &= (\phi_1 \otimes \phi_2)(\mathbf{x})\\ &= g(h(\phi_1 \otimes \phi_2)(\mathbf{x}))\\ &= (g \circ h(\phi_1 \otimes \phi_2))(\mathbf{x}).\end{aligned}$$

Then

$$(h(\phi_1) \otimes h(\phi_2))(\mathbf{x}) = h(\phi_1 \otimes \phi_2)(\mathbf{x}).$$

Thus

$$h(\phi_1) \otimes h(\phi_2) = h(\phi_1 \otimes \phi_2).$$

3. For $\phi \in \Phi$ with $d(\phi) = t$, and $s \subseteq t$, $\mathbf{x} \in \Omega_s$, we have

$$\begin{aligned}g(h(\phi)^{\downarrow s}(\mathbf{x})) &= g\Big(\sum_{\mathbf{y}\in\Omega_{t-s}} h(\phi)(\mathbf{x}, \mathbf{y})\Big)\\ &= \sum_{\mathbf{y}\in\Omega_{t-s}} \phi(\mathbf{x}, \mathbf{y})\\ &= \phi^{\downarrow s}(\mathbf{x})\\ &= g(h(\phi^{\downarrow s})(\mathbf{x})).\end{aligned}$$

Then

$$h(\phi)^{\downarrow s}(\mathbf{x}) = h(\phi^{\downarrow s})(\mathbf{x}).$$

So

$$h(\phi)^{\downarrow s} = h(\phi^{\downarrow s}).$$

It is easy to see that h is injective. By the proof above, we get that h is a monomorphism.

In the above discussion, we have shown that valuation algebras induced by semirings are isomorphic if semirings are isomorphic. Now, in turn, we will give a necessary condition for an isomorphism between valuation algebras to exist.

Proposition 3.2 *Let $(\Phi,\ D)$, $(\Psi,\ D)$ be the valuation algebras with stability induced by semirings A and B respectively. If $(\Phi,\ D)$ is isomorphic to $(\Psi,\ D)$, then the semigroup $\langle A,\ \times\rangle$ is isomorphic to the semigroup $\langle B,\ \times\rangle$.*

Proof. Suppose that $h : (\Phi, D) \to (\Psi, D)$ is an isomorphism. We denote the set of valuations with empty domain by $\Phi_\emptyset$ and $\Psi_\emptyset$ in the two valuation algebras respectively. It is obvious that $h : \Phi_\emptyset \to \Psi_\emptyset$ is bijective.

For all $a \in A$, there exists only a valuation $\phi^{(a)} \in \Phi_\emptyset$ such that $\phi^{(a)}(\diamond) = a$. Let a mapping $f : A \to B$ be defined by $f(a) = h(\phi^{(a)})(\diamond)$. Now we show that f is an isomorphism.

If $f(a) = f(b)$, then

$$h(\phi^{(a)})(\diamond) = h(\phi^{(b)})(\diamond).$$

Since h is bijective, we have $\phi^{(a)} = \phi^{(b)}$, i.e., $a = b$. So f is injective. For all $b \in B$, there exists only a valuation $\psi^{(b)} \in \Psi_\emptyset$ such that $\psi^{(b)}(\diamond) = b$. Since h is bijective, there exists a $\phi^{(a)} \in \Phi_\emptyset$ such that $h(\phi^{(a)}) = \psi^{(b)}$. Therefore,

$$f(a) = h(\phi^{(a)})(\diamond) = \psi^{(b)}(\diamond) = b.$$

Hence f is bijective.

For $a_1, a_2 \in A$, we have

$$(\phi^{(a_1)} \otimes \phi^{(a_2)})(\diamond) = a_1 \times a_2.$$

Then

$$\phi^{(a_1)} \otimes \phi^{(a_2)} = \phi^{(a_1 \times a_2)}.$$

Therefore,

$$\begin{aligned} f(a_1 \times a_2) &= h(\phi^{(a_1 \times a_2)})(\diamond) \\ &= h(\phi^{(a_1)} \otimes \phi^{(a_2)})(\diamond) \\ &= (h(\phi^{(a_1)}) \otimes h(\phi^{(a_2)}))(\diamond) \\ &= h(\phi^{(a_1)})(\diamond) \times h(\phi^{(a_2)})(\diamond) \\ &= f(a_1) \times f(a_2). \end{aligned}$$

By what we have proven, we obtain that f is an isomorphism between $\langle A, \times \rangle$ and $\langle B, \times \rangle$.

In a valuation algebra (Φ, D), we can define a sum operation "$\oplus$" on each semigroup Φ_s as follows: for all $\phi, \psi \in \Phi_s$, $\phi \oplus \psi \in \Phi_s$ and

$$(\phi \oplus \psi)(\mathbf{x}) = \phi(\mathbf{x}) + \psi(\mathbf{x}), \qquad \mathbf{x} \in \Omega_s.$$

If $h : \Phi_s \to \Psi_s$ is a mapping such that

$$h(\phi \oplus \psi) = h(\phi) \oplus h(\psi), \qquad \phi, \psi \in \Phi_s,$$

then we say that h preserves the sum operation.

Proposition 3.3 *Let $(\Phi,\ D)$ be the valuation algebra induced by a c-semiring A. If $h : \Phi_s \to \Psi_s$ preserves the sum operation, then it preserves the order $\leq$ on the set Φ_s.*

Proof. First, for $\phi, \psi \in \Phi_s$, we have $\phi \leq \psi$ if and only if $\phi \oplus \psi = \phi$. In fact, if $\phi \leq \psi$, then for all $\mathbf{x} \in \Omega_s$, we have

$$\psi(\mathbf{x}) = (\phi \otimes \psi)(\mathbf{x}) = \phi(\mathbf{x}) \times \psi(\mathbf{x}) \leq \phi(\mathbf{x}).$$

Thus

$$(\phi \oplus \psi)(\mathbf{x}) = \phi(\mathbf{x}) + \psi(\mathbf{x}) = \phi(\mathbf{x}).$$

So we have $\phi \oplus \psi = \phi$. And vice versa.

If $\phi \leq \psi$, then

$$h(\phi) = h(\phi \oplus \psi) = h(\phi) \oplus h(\psi).$$

We have $h(\phi) \leq h(\psi)$. Thus h is order-preserving.

Proposition 3.4 *Let $(\Phi,\ D)$, $(\Psi,\ D)$ be two valuation algebras induced by two c-semirings A and B respectively. If there exists a bijective mapping $h : \Phi_\emptyset \to \Psi_\emptyset$ preserving the operations combination $(\otimes)$ and sum $(\oplus)$, then A is isomorphic to B.*

Proof. Let $f : A \to B$ be defined by $f(a) = h(\phi^{(a)})(\diamond)$ as in the proof of Proposition 3.2, where $\phi^{(a)} : \Omega_\emptyset \to A$ is such that $\phi^{(a)}(\diamond) = a$. Since h preserves the operation $\oplus$, we have

$$\begin{aligned}
f(a+b) &= h(\phi^{(a+b)})(\diamond)\\
&= h(\phi^{(a)} \oplus \phi^{(b)})(\diamond)\\
&= (h(\phi^{(a)}) \oplus h(\phi^{(b)}))(\diamond)\\
&= h(\phi^{(a)})(\diamond) + h(\phi^{(b)})(\diamond)\\
&= f(a) + f(b).
\end{aligned}$$

By Proposition 3.2, we obtain that f is an isomorphism between A and B.

In the end of this section, we pose an open problem.

Problem Let $(\Phi,\ D)$, $(\Psi,\ D)$ be two valuation algebras induced by semirings A and B respectively. If $(\Phi,\ D)$ is isomorphic to $(\Psi,\ D)$, whether A is isomorphic to B or not?

4 General Homomorphism between Valuation Algebras

In the above section, valuation homomorphism is defined between two valuation algebras which have a same variable set. More generally, in this section, we discuss a new homomorphism which is defined between two valuation algebras on different domains.

First, let us see another type of valuation algebras, which is called domain-free valuation algebra.

Definition 4.1[5] *A system $(\Psi,\ D)$ with two operations is defined as follows, where Ψ is a set of valuations and D is the power set of a variable set r :*

1. *Combination*: $\Psi \times \Psi \to \Psi$; $(\phi, \psi) \mapsto \phi \otimes \psi$;
2. *Focusing*: $\Psi \times D \to \Psi$; $(\psi, x) \mapsto \psi^{\Rightarrow x}$.

$(\Psi,\ D)$ is called a domain-free valuation algebra, if the following axioms hold:

(1) *Semigroup*: *Ψ is associative and commutative under combination. There is a neutral element e such that $e \otimes \psi = \psi \otimes e = \psi$ for all $\psi \in \Psi$;*

(2) *Transitivity*: *For $\psi \in \Phi$ and $x, y \in D$, $(\psi^{\Rightarrow y})^{\Rightarrow x} = \psi^{\downarrow x\cap y}$;*

(3) *Combination*: *For $\phi, \psi \in \Psi$ and $x \in D$,*

$$(\phi^{\Rightarrow x} \otimes \psi)^{\Rightarrow x} = \phi^{\Rightarrow x} \otimes \psi^{\Rightarrow x};$$

(4) *Support*: *For $\psi \in \Psi$, there is an $x \in D$ such that $\psi^{\Rightarrow x} = \psi$.*

If a domain-free valuation algebra $(\Phi,\ D)$ also satisfies the axiom of idempotency, we say that it is a domain-free information algebra:

(5) *Idempotency*: *For $\psi \in \Psi$ and $x \in D$, $\psi \otimes \psi^{\Rightarrow x} = \psi$.*

Definition 4.2 *Let $(\Phi,\ D)$ and $(\Psi,\ E)$ be two domain-free valuation algebras. $(\Phi,\ D)$ and $(\Psi,\ E)$ are homomorphic. If there is a mapping $h:(\Phi,\ D)\to(\Psi,\ E)$ such that, for a mapping $f:D\to E$ and all $\phi_1,\phi_2,\phi\in\Phi$:*

1. $h(\phi_1\otimes\phi_2)=h(\phi_1)\otimes h(\phi_2)$;
2. $h(\phi^{\Rightarrow x})=[h(\phi)]^{\Rightarrow f(x)}$, *if* $x\in D$;
3. $h(e)=e$,

then we call h a homomorphism between $(\Phi,\ D)$ and $(\Psi,\ E)$ with respect to f.

Similarly, suppose that $(\Phi,\ D)$, $(\Psi,\ E)$ are two labeled valuation algebras. $(\Phi,\ D)$ and $(\Psi,\ E)$ are homomorphic, if there is a mapping $h:(\Phi,\ D)\to(\Psi,\ E)$ such that, for a mapping $f:D\to E$ and all $\phi_1,\phi_2,\phi\in\Phi$:

1. $d(h(\phi))=f(d(\phi))$;
2. $h(\phi_1\otimes\phi_2)=h(\phi_1)\otimes h(\phi_2)$;
3. $h(\phi^{\downarrow x})=[h(\phi)]^{\downarrow f(x)}$, *if* $x\subseteq d(\phi)$;
4. $h(e_x)=e_{f(x)}$ *for all* $x\in D$.

If h and f are injective, then $(\Phi,\ D)$ is said to be embedded in $(\Psi,\ E)$. If the two mappings f and h are bijective, we say that $(\Phi,\ D)$ and $(\Psi,\ E)$ are isomorphic.

Remark 4.1 Let $(\Phi,\ D)$ and $(\Psi,\ E)$ be two labeled valuation algebras. If $x\subseteq d(\phi)$, by the first principle of the definition, we have $f(x)\subseteq f(d(\phi))=d(h(\phi))$. Thus the statement of the third principle has no problem.

Let us see an abstract example for homomorphism between valuation algebras.

Example 4.1 Let $\Phi=\{\bot,\ \top,\ c,\ d_1,\ d_2,\ \cdots,\ d_n,\ \cdots\}$ and $\Psi=\{\bot,\ \top,\ a,\ b\}$, where $\bot$, $\top$ are the minimum element and the maximum element of Φ and Ψ respectively, $d_i\le d_j$ if and only if $i\le j$. Let $D=\{z\}$ and $E=\{x,y\}$, where $x\subseteq y$.

Define $\phi\otimes\psi=\phi\vee\psi$ for all $\phi,\psi\in\Phi$. The operation focusing is defined by $\phi^{\Rightarrow z}=\phi$ for all $\phi\in\Phi$. It is clear that $\bot$ is the neutral element and $(\Phi,\ D)$ is an information algebra.

The combination in Ψ is same as that in Φ. The operation focusing is defined as follows:

$$\psi^{\Rightarrow x}=\begin{cases}\top, & \psi=\top;\\ \bot, & \text{otherwise};\end{cases}$$
$$\psi^{\Rightarrow y}=\psi,\qquad \psi\in\Psi.$$

In order to show that $(\Psi,\ E)$ is an information algebra, it suffices to show the axioms of transitivity, combination and idempotency, while the other axioms are obvious.

(1) Transitivity: For $\psi\in\Psi$ and $x,y\in E$, we have

$$\psi^{\Rightarrow x\cap y}=\psi^{\Rightarrow x}=(\psi^{\Rightarrow x})^{\Rightarrow y};$$

(2) Combination: Let $\phi,\psi\in\Psi$ and $x\in E$. If $\phi=\top$, then

$$(\phi^{\Rightarrow x}\otimes\psi)^{\Rightarrow x}=(\top\otimes\psi)^{\Rightarrow x}=\top=\phi^{\Rightarrow x}\otimes\psi^{\Rightarrow x}.$$

Otherwise,

$$(\phi^{\Rightarrow x}\otimes\psi)^{\Rightarrow x}=(\bot\otimes\psi)^{\Rightarrow x}=\psi^{\Rightarrow x}=\phi^{\Rightarrow x}\otimes\psi^{\Rightarrow x}.$$

(3) Idempotency: For $\psi\in\Psi$, we have

$$\psi\otimes\psi^{\Rightarrow y}=\psi\otimes\psi=\psi.$$

Since $\psi^{\Rightarrow x} \le \psi$, we have

$$\psi \otimes \psi^{\Rightarrow x} = \psi.$$

Thus the axiom of idempotency holds.

Define two mappings $f : D \to E$ and $h : \Phi \to \Psi$ as follows:

$$f(z) = y,$$

and

$$h(\perp) = \perp, \qquad h(c) = a, \qquad h(d_i) = b \quad (i = 1, 2, \cdots), \qquad h(\top) = \top.$$

By the monotonicity of h, we have

$$h(\phi \otimes \psi) = h(\phi) \otimes h(\psi).$$

Meanwhile,

$$h(\phi^{\Rightarrow z}) = h(\phi) = h(\phi)^{\Rightarrow y} = h(\phi)^{\Rightarrow f(z)}, \qquad \phi \in \Phi.$$

Then h is a homomorphism with respect to f.

Proposition 4.1 *If h is a homomorphism between two domain-free information algebras $(\Phi,\ D)$ and $(\Psi,\ E)$, then h is order-preserving.*

Moreover, if h is a monomorphism, then h is order-reflecting, that is, for any $\phi, \psi \in \Phi$, $h(\phi) \le h(\psi)$ implies $\phi \le \psi$.

Proof. If $\phi \le \psi$, then $\phi \otimes \psi = \psi$. Since h is a homomorphism, then

$$h(\psi) = h(\phi \otimes \psi) = h(\phi) \otimes h(\psi).$$

So $h(\phi) \le h(\psi)$, that is, h is order-preserving.

Since h is a monomorphism, if $h(\phi) \le h(\psi)$, then

$$h(\phi \otimes \psi) = h(\phi) \otimes h(\psi) = h(\psi).$$

Thus $\phi \otimes \psi = \psi$. We have $\phi \le \psi$.

For a domain-free valuation algebra $(\Psi,\ D)$, let

$$\Psi^* = \{(\psi, x) : \psi \in \Psi,\ \psi^{\Rightarrow x} = \psi\},$$

and three operations are defined on Ψ^* as follows:

1. Labeling: For $(\psi, x) \in \Psi^*$ define $d(\psi, x) = x$;
2. Combination: For $(\phi, x), (\psi, y) \in \Psi^*$, define $(\phi, x) \otimes (\psi, y) = (\phi \otimes \psi, x \cup y)$;
3. Marginalization: For $(\psi, x) \in \Psi^*$ and $y \subseteq x$, define $(\psi, x)^{\downarrow y} = (\psi^{\Rightarrow y}, y)$.

Then $(\Psi^*,\ D)$ is a labeled valuation algebra, and we say that it is the associated labeled valuation algebra with $(\Psi,\ D)$ (see [5]).

Theorem 4.1 *Let $(\Phi,\ D)$ and $(\Psi,\ E)$ be two domain-free valuation algebras, and $(\Phi^*,\ D)$ and $(\Psi^*,\ E)$ be the associated labeled valuation algebras respectively. If $h : (\Phi,\ D) \to (\Psi,\ E)$ is a homomorphism with respect to $f : D \to E$, and we define $h^*(\phi, x) = (h(\phi), f(x))$, then $h^* : (\Phi^*, D) \to (\Psi^*, E)$ is a homomorphism with respect to f.*

Proof. 1. If $(\phi, x) \in (\Phi^*, D)$, then, by the definition,

$$h(\phi) = h(\phi^{\Rightarrow x}) = h(\phi)^{\Rightarrow f(x)},$$

and

$$d(h^*(\phi, x)) = d(h(\phi), f(x)) = f(x) = f(d(\phi, x)).$$

2. If $(\phi_1, x_1), (\phi_2, x_2) \in (\Phi^*, D)$, then

$$\begin{aligned} h^*((\phi_1, x_1) \otimes (\phi_2, x_2)) &= h^*(\phi_1 \otimes \phi_2, x_1 \cup x_2) \\ &= (h(\phi_1 \otimes \phi_2), f(x_1 \cup x_2)) \\ &= (h(\phi_1) \otimes h(\phi_2), f(x_1) \cup f(x_2)) \\ &= (h(\phi_1), f(x_1)) \otimes (h(\phi_2), f(x_2)) \\ &= h^*(\phi_1, x_1) \otimes h^*(\phi_2, x_2). \end{aligned}$$

3. If $(\phi, x) \in (\Phi^*,\ D)$ and $y \subseteq x$, then

$$\begin{aligned} h^*((\phi, x)^{\downarrow y}) &= h^*(\phi^{\Rightarrow y}, y) \\ &= (h(\phi^{\Rightarrow y}), f(y)) \\ &= (h(\phi)^{\Rightarrow f(y)}), f(y)) \\ &= (h(\phi), f(x))^{\downarrow f(y)} \\ &= [h^*(\phi, x)]^{\downarrow f(y)}. \end{aligned}$$

4. Let e be the neutral element of Φ. Then (e, x) is the neutral element of Φ^*_x and $h^*(e, x) = (h(e), f(x))$. It is clear that, $(h(e), f(x))$ is the neutral element of $\Psi^*_{f(x)}$. This proves that h^* is a homomorphism with respect to f.

In a domain-free valuation algebra $(\Phi,\ D)$, an equivalence relation θ in $(\Phi,\ D)$ is called a congruence, if for all $\phi, \psi \in \Phi$, $x \in D$:

1. $\phi \equiv \psi (\text{mod } \theta)$ implies $\phi^{\Rightarrow x} \equiv \psi^{\Rightarrow x} (\text{mod } \theta)$;
2. $\phi_1 \equiv \psi_1 (\text{mod } \theta)$, $\phi_2 \equiv \psi_2 (\text{mod } \theta)$ implies $\phi_1 \otimes \phi_2 \equiv \psi_1 \otimes \psi_2 (\text{mod } \theta)$.

If θ is a congruence, we define a new combination and focusing in Φ/θ as follows:

1. Combination: $[\phi]_\theta \otimes [\psi]_\theta = [\phi \otimes \psi]_\theta$;
2. Focusing: $[\phi]_\theta^{\Rightarrow x} = [\phi^{\Rightarrow x}]_\theta$.

Then $(\Phi/\theta,\ D)$ is still a domain-free valuation algebra.

Lemma 4.1 *Let (Φ, D), $(\Psi,\ E)$ be two domain-free valuation algebras. A mapping h : $(\Phi,\ D) \to (\Psi,\ E)$ is a homomorphism with respect to $f : D \to E$. We define the relation θ in Φ as follows*:

$$\phi \equiv \psi (\text{mod } \theta) \text{ if and only if } h(\phi) = h(\psi).$$

Then θ is a congruence.

Proof. It is clear that θ is an equivalence relation.

1. If $\phi \equiv \psi (\text{mod } \theta)$, then $h(\phi) = h(\psi)$. For all $x \in D$, we have

$$h(\phi^{\Rightarrow x}) = h(\phi)^{\Rightarrow f(x)} = h(\psi)^{\Rightarrow f(x)} = h(\psi^{\Rightarrow x}).$$

So $\phi^{\Rightarrow x} \equiv \psi^{\Rightarrow x} (\text{mod } \theta)$.

2. If $\phi_1 \equiv \psi_1(\text{mod } \theta)$, $\phi_2 \equiv \psi_2(\text{mod } \theta)$, then

$$h(\phi_1 \otimes \phi_2) = h(\phi_1) \otimes h(\phi_2) = h(\psi_1) \otimes h(\psi_2) = h(\psi_1 \otimes \psi_2).$$

Thus $\phi_1 \otimes \phi_2 \equiv \psi_1 \otimes \psi_2(\text{mod } \theta)$.

By what we have proven above, θ is a congruence in the valuation algebra.

Now we can get the general homomorphism theorem of valuation algebras.

Theorem 4.2(Homomorphism Theorem) *Let (Φ, D) and (Ψ, E) be two domain-free valuation algebras. A mapping $h : (\Phi, D) \to (\Psi, E)$ is a surjective homomorphism with respect to a bijective mapping $f : D \to E$. Then, with respect to f, there exists an isomorphism $k : (\Phi/\theta, D) \to (\Psi, E)$ such that $k \circ p = h$, where θ is defined as in Lemma* 4.1*, and p is the projection associated with the congruence θ, i.e., $p : \Phi \to \Phi/\theta$, $p(\phi) = [\phi]_\theta$.*

Proof. Define a mapping k by $k([\phi]_\theta) = h(\phi)$. By the definition of θ, k is well-defined. Now we show that k is an isomorphism.

1. For $[\phi_1]_\theta, [\phi_2]_\theta \in \Phi/\theta$, we have

$$k([\phi_1]_\theta \otimes [\phi_2]_\theta) = k([\phi_1 \otimes \phi_2]_\theta) = h(\phi_1 \otimes \phi_2) = h(\phi_1) \otimes h(\phi_2) = k([\phi_1]_\theta) \otimes k([\phi_2]_\theta).$$

2. For $[\phi]_\theta \in \Phi/\theta$, $x \in D$,

$$k([\phi]_\theta^{\Rightarrow x}) = k([\phi^{\Rightarrow x}]_\theta) = h(\phi^{\Rightarrow x}) = h(\phi)^{\Rightarrow f(x)} = k([\phi]_\theta)^{\Rightarrow f(x)}.$$

3. It is obvious that $[e]_\theta$ is the neutral element of Φ/θ and $k([e]_\theta) = h(e)$. Since h is a homomorphism, $h(e)$ is the neutral element of Ψ, that is, k preserves the neutral element.

If $k([\phi]_\theta) = k([\psi]_\theta)$, then $h(\phi) = h(\psi)$. By the definition of θ, we have $\phi \equiv \psi(\text{mod } \theta)$, i.e., $[\phi]_\theta = [\psi]_\theta$. Thus k is injective. Since h is surjective, k is surjective too. Then k is bijective.

It is clear that, for any $\psi \in \Psi$, one has

$$k(p(\psi)) = k([\psi]_\theta) = h(\psi).$$

So $k \circ p = h$. Then the conclusion is correct now.

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