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KAM Type-Theorem for Lower Dimensional Tori in Random Hamiltonian Systems*

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Abstract: In this paper, we study the persistence of lower dimensional tori for random Hamiltonian systems, which shows that majority of the unperturbed tori persist as Cantor fragments of lower dimensional ones under small perturbation. Using this result, we can describe the stability of the non-autonomous dynamic systems.
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1 Introduction

We consider the persistence of lower dimensional tori for a family of random real analytic Hamiltonian systems of the parameterized action-angle form

$$H = e + \langle \omega, y \rangle + \frac{1}{2}\langle z, Mz \rangle + \varepsilon P(x, y, z, \theta_t), \tag{1.1}$$

where $(x, y, z) \in \mathbf{T}^d \times \mathbf{R}^d \times \mathbf{R}^{2m}$ varies in a complex neighborhood $D(r, s) = \{(x, y, z) : |\mathrm{Im} x| < r, |y| < s^2, z < s\}$ of $\mathbf{T}^d \times \{0\} \times \{0\}$, $\omega \in \mathcal{O}$ (a bounded closed region in $\mathbf{R}^d$), ε is a small parameter, $\theta_t : \Omega \subset \mathbf{R}^d \to \Omega$, $t \in \mathbf{R}^1_+$, is a continuous stationary stochastic processes with $\theta_0 = \mathrm{id}$, and $(\Omega, P, \mathcal{F})$ is a stochastic basis. Hereafter, all θ_t dependence function are of class C^{l_0} for some $l_0 \geq d$, and P is a small perturbation.

This kind of systems describes dynamics of harmonic oscillator under perturbations such as white noise, or under some effects of some noise θ_t which are neither periodic, quasi-periodic nor almost periodic.

With the symplectic form

$$\sum_{i=1}^{d} \mathrm{d}x_i \wedge \mathrm{d}y_i + \sum_{j=1}^{m} \mathrm{d}z_j \wedge \mathrm{d}z_{d+j},$$

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the equation of motion of (1.1) reads

$$\begin{cases} \dot{x} = \omega + \varepsilon \dfrac{\partial P}{\partial y}, \\ \dot{y} = -\varepsilon \dfrac{\partial P}{\partial x}, \\ \dot{z} = JMz + J\dfrac{\partial P}{\partial z}, \end{cases}$$

where $M(\theta_t)$ is a $2m \times 2m$ real symmetric matrix for each $\theta_t \in \varOmega$, and J is the standard $2m \times 2m$ symplectic matrix. Hence the associated unperturbed motion of (1.1) is simply described by the equation

$$\begin{cases} \dot{x} = \omega, \\ \dot{y} = 0, \\ \dot{z} = JMz, \end{cases}$$

which implies that the unperturbed system admits a family of invariant tori $T_\omega = \mathbf{T}^d \times \{0\} \times \{0\}$ parameterized by the frequency vectors $\omega \in \mathcal{O}$.

Similar to the classical KAM theorem (see [1]–[3]), Melnikov[4],[5] posed the persistence problem of lower dimensional tori in the deterministic Hamiltonian systems, which concludes that under some appropriate non-degenerate and non-resonance conditions, there exits a Cantor set $\mathcal{O}_* \subset \mathcal{O}$, such that those lower dimensional invariant d-tori with the frequencies $\omega \in \mathcal{O}_*$ will persist as ε sufficiently small; moreover, in the sense of Lebesgue measure $\mathcal{O}_* \to \mathcal{O}$, as $\varepsilon \to 0$. Some achievements on Melnikov persistence problem can be found in [6]–[20].

However, what happens to the Melnikov persistence for random or non-periodical perturbed systems (1.1)? In this paper, we are concern with this problem. We prove that for most of frequencies $\omega \in \varOmega$, there exists a set of Cantor set $\varOmega_\gamma \subset \varOmega$ such that the associated unperturbed lower dimensional invariant torus T_ω, $\omega \in \varOmega_\gamma$, persists as a set of Cantor fragments of the invariant torus with the"random frequency" close to $\omega(\theta_t)$ for the perturbed system (1.1), provided ε is sufficiently small.

The persistence of lower dimensional tori problem can describe the stability of non-autonom-ous systems. Different from previous, we need not to assume that the perturbation P is periodic or not. Applying the results, we know that there is a Cantor set $\varOmega_\gamma$, such that when $\theta_t \in \varOmega_\gamma$, the lower dimensional invariant tours of unperturbed system persists, provided ε is sufficiently small.

The paper is organized as follows. In Section 2, we state our theorem for a general random Hamiltonian system and the corollary A of non-autonomous systems. Then, a parameter-depended iterative scheme is described in Section 3 for one cycle. In Section 4, we derive the proof of our result by deriving an iteration lemma and giving measure estimates.

2 Main Results

We consider the random parameter-dependent, real analytic Hamiltonian system

$$H = e(\theta_t) + \langle \omega(\theta_t), y\rangle + \frac{1}{2}\langle M(\theta_t)z, z\rangle + P(x, y, z, \theta_t), \tag{2.1}$$

where(x, y, z) lies in a complex neighborhood $D(r, s) = \{(x, y, z) : \ |\mathrm{Im}x| < r, \ |y| < s, \ |z| < s\}$ of $\mathbf{T}^d \times \{0\} \times \{0\} \subset \mathbf{T}^d \times \mathbf{R}^d \times \mathbf{R}^{2m}$. As above, $\theta_t : \Omega \to \Omega$ is a continuous stationary stochastic processes with stochastic basis $(\Omega, P, \mathcal{F})$, where $\Omega \subset \mathbf{R}^d$ is a bounded closed region. Also, all θ_t dependence are of class C^{l_0} for some $l_0 \geq d$. Then, the motion of associated unperturbed system is simply described as

$$\begin{cases} \dot{x}(t) = \omega(\theta_t), \\ \dot{y}(t) = 0, \\ \dot{z}(t) = JM(\theta_t)z. \end{cases}$$

Definition 2.1 (simi-torus) *Let* $g : \ \mathbf{T}^d \times \mathcal{O} \times \Omega \to \mathbf{R}^d$ *be continuous and*

$$L = \{x(t) \in \mathbf{T}^d : \ \dot{x} = \omega(\theta_t) + g(x, \theta_t)\}.$$

We call $L \times \{0\}$ *a simi-torus with the frequency* $\omega(\theta_t) + g(x, \theta_t)$.

Definition 2.2 (Cantor fragment) *For given Cantor* $\Omega_\gamma \subset \Omega$, *to make the definition clearly, we first denote*

$$T_\gamma = \{t : t \in [0, \infty), \ \theta_t \in \Omega_\gamma\}.$$

Then we call the set $F_\gamma \subset \mathbf{T}^d$ *the Cantor fragment of* $\mathbf{T}^d$, *if*

$$F_\gamma = \{x(t) \in \mathbf{T}^d : \dot{x} = \omega(\theta_t) + g(x, \theta_t), \ t \in T_\gamma\}.$$

Consider (2.1) and let $\lambda_1(\theta_t), \cdots, \lambda_{2m}(\theta_t)$ be the eigenvalues of $JM(\theta_t)$. We assume the weak form of Melnikov's second non-resonance condition, i.e.,

A1) The set

$$\{\theta_t \in \Omega : \ \sqrt{-1}\langle k, \omega(\theta_t)\rangle - \lambda_i(\theta_t) - \lambda_j(\theta_t) \neq 0, \ \forall \ k \in \mathbf{Z}^d \setminus \{0\}, \ 1 \leq i, j \leq 2m\}$$

admits full Lebesgue measure relative to Ω;

A2) θ_t is ergodic on Ω;

A3) $M(\theta_t)$ is nonsingular for each $\theta_t \in \Omega$.

The main result of the present paper is the following.

Theorem A *Consider* (2.1). *Let* $\tau > d(d-1) - 1$ *be fixed, and* $d_* = \max\{d_0, d\}$.

1) *Assume* A1), A2), A3). *Then there exists a sufficiently small* $\mu = \mu(r, s, m, l_0, \tau) > 0$ *such that if*

$$|\partial^l_{\theta_t} P|_{D(r,s)\times\Omega} \leq \gamma^{(|l_0|+1|)4m^2\tau} s^2\mu, \qquad |l| \leq l_0, \tag{2.2}$$

then there exist a Cantor set $\Omega_\gamma \subset \Omega$ *with* $|\Omega \setminus \Omega_\gamma| = O(\gamma^{\frac{1}{d_*-1}})$ *and a* C^{l_0-1} *Whitney smooth family of* C^2 *symplectic transformations*

$$\Psi_{\theta_t} : D\Big(\frac{r}{2}, \frac{s}{2}\Big) \to D(r, s), \qquad \theta_t \in \Omega_\gamma,$$

which is real analytic in x *and* C^2 *uniformly close to the identity such that*

$$H \circ \Psi_{\theta_t}(x, y) = e_*(\theta_t) + \langle \omega_*(\theta_t), y\rangle + \frac{1}{2}\langle M(\theta_t)z, z\rangle + P_*(x, y, \theta_t),$$

where, for all $\theta_t \in \Omega_\gamma$ *and* $(x,y) \in D\Big(\dfrac{r}{2}, \dfrac{s}{2}\Big)$, *we have*

$$\partial_y^j \partial_z^k P_*|_{(y,z)=(0,0)} = 0,$$

for $|j|+|k| \le 2$, *and* $\omega_*(\theta_t)-\omega(\theta_t)$, $M_*-M = O(\mu)$. *Thus, for each* $\theta_t \in \Omega_\gamma$, *corresponding to the unperturbed torus* T_{θ_t} *of* (2.1), *the associated perturbed invariant torus of Hamiltonian* (2.1) *can be described as*

$$\begin{cases} \dot{x}(t) = \omega_*, \\ \dot{y}(t) = 0, \\ \dot{z}(t) = JM_*(\theta_t)z. \end{cases}$$

Namely, the Cantor fragments of the unperturbed torus T_{θ_t} *associated to the toral frequency* $\omega(\theta_t)$ *as* $\theta_t \in \Omega_\gamma$, *persists and gives rise to a Cantor fragments* F_γ^* *of an analytic, Diophantine, simi-torus with the toral frequency* $\omega_*(\theta_t)$, *where*

$$F_\gamma^* = \{x(t) \in \mathbf{T}^d, \quad \dot{x} = \omega_*(\theta_t), \quad t \in T_\gamma\}.$$

Moreover, these perturbed tori form a C^{l_0-1} *Whitney smooth family.*

2) *There holds that*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Omega_*(\theta_s(p))\mathrm{d}s = \int_\Omega \Omega_*(q)\mathrm{d}q \qquad a.e.\ \Omega,$$

and

$$\lim_{t\to\infty} \frac{1}{t}|\{s \in [0,t] : \theta_s(p) \in \Omega_\gamma\}| = |\Omega_\gamma| \qquad a.e.\ \Omega.$$

3 KAM Step

In this section, we show a quasi-linear iterative scheme for the Hamiltonian (2.1) in one KAM cycle, say, from a νth KAM step to the $(\nu+1)$th-step. Then, one can find that the new perturbation get smaller, and the frequencies $\omega_{\nu+1}$ and matrix $M_{\nu+1}$ are of small deformation. For simplicity, we set $l_0 = d$.

Set

$$\begin{aligned} &r_0 = r, \quad \gamma_0 = \gamma, \quad \Lambda_0 = \Omega, \\ &H_0 = H, \quad e_0 = e, \quad \omega_0 = \omega, \\ &M_0 = M, \quad P_0 = P, \\ &N_0 = e_0(\theta_t) + \langle \omega_0(\theta_t), y\rangle + \frac{1}{2}\langle M_0(\theta_t)z, z\rangle. \end{aligned}$$

Without loss of generality, let $0 < r_0, \gamma_0 \le 1$. Then, for μ small, (2.2) becomes

$$|\partial_{\theta_t}^l P_0|_{D(r_0,s_0)} \le \gamma_0^a s_0^2 \mu_0, \qquad |l| \le d, \tag{3.1}$$

where

$$a = (l_0+1)4m^2\tau.$$

Now, suppose that after a νth-step, we have arrived at the following real analytic Hamiltonian:

$$\begin{aligned} &H_\nu = N_\nu + P_\nu, \qquad (3.2)\\ &N_\nu = e_\nu(\theta_t) + \langle \omega_\nu(\theta_t), y\rangle + \frac{1}{2}\langle M_\nu(\theta_t)z, z\rangle, \end{aligned}$$

which is defined on a phase domain $D(r_\nu, s_\nu)$ and depends smoothly on $\theta_t \in \Lambda_\nu$. In addition, $M_\nu(\theta_t)$ is non-singular and symmetry for each $\theta_t \in \Lambda_\nu$, and,

$$P_\nu = P_\nu(x, y, \theta_t)$$

satisfies

$$|\partial_{\theta_t}^l P_\nu|_{D(r_\nu, s_\nu)} \le \gamma_\nu^a s_\nu^2 \mu_\nu, \qquad |l| \le d, \tag{3.3}$$

for some $0 < \mu_\nu \le \mu_0$, $0 < \gamma_\nu \le \gamma_0$. We try to find a symplectic transformation $\Phi_{\nu+1}$ on a small phase domain $D(r_{\nu+1}, s_{\nu+1})$ and a smaller parameter domain $\Lambda_{\nu+1}$. It transforms the Hamiltonian (3.2) into the Hamiltonian of the next KAM cycle, i.e.,

$$\begin{aligned} &H_{\nu+1} = H \circ \Phi_{\nu+1} = N_{\nu+1} + P_{\nu+1}, \\ &N_{\nu+1} = e_{\nu+1}(\theta_t) + \langle \omega_{\nu+1}(\theta_t), y\rangle + \frac{1}{2}\langle M_{\nu+1}(\theta_t) z, z\rangle, \\ &|\partial_{\theta_t}^l P_{\nu+1}|_{D(r_{\nu+1}, s_{\nu+1})} \le \gamma_{\nu+1}^a s_{\nu+1}^2 \mu_{\nu+1}, \qquad |l| \le d. \end{aligned} \tag{3.4}$$

Also, $M_{\nu+1}$ is non-singular and symmetric for each $\theta_t \in \Lambda_{\nu+1}$.

For simplicity, we shall omit index for all quantities of the present KAM step (the νth-step) and index all quantities (Hamiltonian, normal form, perturbation, transformation, and domains, etc.) in the next KAM step (the $(\nu+1)$-th step) by "+". All constants c_i, c below are positive and independent of the iteration process. To simplify the notations, we shall suspend the θ_t dependence in most terms of this section.

Define

$$\begin{aligned} r_+ &= \frac{r}{2} + \frac{r_0}{4}, \quad \gamma_+ = \frac{\gamma}{2} + \frac{\gamma_0}{4}, \\ s_+ &= \frac{1}{8}\alpha s, \quad \alpha = \mu^{\frac{1}{3}}, \quad \mu = s^{\frac{1}{2}}, \\ \beta_+ &= \frac{\beta}{2} + \frac{\beta_0}{4}, \quad K_+ = \Big(\Big[\log\frac{1}{\mu}\Big] + 1\Big)^{3\eta}, \\ D_{i\alpha} &= D(r_+ + \frac{i-1}{8}(r - r_+), i\alpha s), \quad i = 1, 2, \cdots, 8, \\ D_+ &= D_\alpha = D(r_+, s_+), \quad \tilde{D}_+ = D\Big(r_+ + \frac{3}{4}(r - r_+), \beta_+\Big), \\ D(\xi) &= \{y \in C^d : |y| < \xi\}, \ \ \hat{D}(\xi) = D\Big(r_+ + \frac{7}{8}(r - r_+), \xi\Big), \quad \xi > 0, \\ \Gamma(r - r_+) &= \sum_{0<|k|\le K_+} |k|^{2|l_0| + (|l_0|+1)4m^2\tau} \mathrm{e}^{-|k|\frac{r - r_+}{8}}. \end{aligned}$$

3.1 Truncation

First, we write P in the Taylor-Fourier series and let R be the truncation, i.e.,

$$\begin{aligned} P &= \sum_{k \in \mathbf{Z}^d, \jmath \in \mathbf{Z}_+^d} p_{k\imath\jmath} y^\imath z^\jmath \mathrm{e}^{\sqrt{-1}\langle k, x\rangle}, \\ R &= \sum_{|k| \le K_+} (p_{k00} + \langle p_{k10}, y\rangle + \langle p_{k01}, z\rangle + \langle z, p_{k02} z\rangle)\mathrm{e}^{\sqrt{-1}\langle k, x\rangle}, \end{aligned} \tag{3.5}$$

where K_+ will be specified below.

Lemma 3.1 *Assume that*

H1) $\int_{K_+}^{\infty} t^{d+3}\mathrm{e}^{-t\frac{r-r_+}{16}}\mathrm{d}t \le \mu.$

Then, there is a constant c_1 *such that*

$$|\partial^l_{\theta_t}(P-R)|_{D_{7\alpha}} \le c_1\gamma^a s^2\mu^2, \quad |\partial^l_{\theta_t}R|_{D_{7\alpha}} \le c_1\gamma^a s^2\mu, \qquad \forall\ |l|\le d, \theta_t \in \Lambda_0.$$

Proof. See [20] for details.

3.2 Linearized Equations

In this subsection, we construct the time 1 map generating by the Hamiltonian F to eliminating resonant terms R. The construction of F is as follows:

$$F = \sum_{0<|k|\le K_+} (f_{k00} + \langle f_{k10,y}\rangle + \langle f_{k01}, z\rangle + \langle z, f_{k02}z\rangle)\mathrm{e}^{\sqrt{-1}\langle k,x\rangle} + \langle f_{001}, z\rangle, \tag{3.6}$$

where $f_{kȷ}$ are (matrix valued) functions of θ_t. Let

$$[R] = \frac{1}{(2\pi)^d}\int_{\mathbf{T}^d} R(x,\cdot)\mathrm{d}x.$$

If F matches the equation

$$\{N, F\} + R - [R] + \langle p_{001}, z\rangle = 0, \tag{3.7}$$

then

$$\begin{aligned} H\circ\phi^1_F &= (N+[R])\circ\phi^1_F + (P-R)\circ\phi^1_F, \\ &= N + [R] - \langle p_{001}, z\rangle + \int_0^1 \{R_t, F\}\circ\phi^t_F\mathrm{d}t + (P-R)\circ\phi^1_F, \end{aligned}$$

where

$$R_t = (1-t)([R] - R - \langle p_{001}, z\rangle) + R,$$

and

$$N_+ = N + [R] - \langle p_{001}, z\rangle, \qquad P_+ = \int_0^1 \{R_t, F\}\circ\phi^t_F\mathrm{d}t + (P-R)\circ\phi^1_F.$$

Substituting (3.5) and (3.6) into (3.7) yields

$$\begin{aligned} &-\sum_{0<|k|\le K_+} \sqrt{-1}\langle k, \omega(\theta_t)\rangle(f_{k00} + \langle f_{k10,y}\rangle + \langle f_{k01}, z\rangle + \langle z, f_{02}z\rangle)\mathrm{e}^{\sqrt{-1}\langle k,x\rangle} \\ &+\sum_{0<|k|\le K_+} (\langle M(\theta_t)z, Jf_{k01}\rangle + 2\langle M(\theta_t)z, Jf_{k02}z\rangle)\mathrm{e}^{\sqrt{-1}\langle k,x\rangle} \\ = &-\sum_{0<|k|\le K_+} (p_{k00} + \langle p_{k10}, y\rangle + \langle p_{k01}, z\rangle + \langle z, p_{k02}z\rangle)\mathrm{e}^{\sqrt{-1}\langle k,x\rangle} - \langle P_{001}, z\rangle. \end{aligned}$$

Comparing the coefficients above, and assuming f_{k02} is symmetric, we deduce the following linear equations for all $0<|k|\le K_+$:

$$\sqrt{-1}\langle k, \omega(\theta_t)\rangle f_{k00} = P_{k00}, \qquad \sqrt{-1}\langle k, \omega(\theta_t)\rangle f_{k10} = P_{k10}, \tag{3.8}$$

$$-\sqrt{-1}\langle k, \omega(\theta_t)\rangle f_{k01} + MJf_{k01} = -P_{k01}, \tag{3.9}$$

$$-\sqrt{-1}\langle k, \omega(\theta_t)\rangle f_{k02} + MJf_{k02} - f_{k02}JM = -P_{k02}, \qquad Mf_{001} = -P_{001}. \tag{3.10}$$

Denote

$$L_{0k} = \sqrt{-1}\langle k, \omega(\theta_t)\rangle,$$
$$L_{1k} = \sqrt{-1}\langle k, \omega(\theta_t)\rangle I_{2m} - MJ,$$
$$L_{2k} = \sqrt{-1}\langle k, \omega(\theta_t)\rangle I_{4m^2} + (MJ)\otimes I_{2m} + I_{2m}\otimes (JM).$$

The linear equations (3.8)–(3.10) are equivalent to

$$L_{0k} f_{kj0} = P_{kj0}, \qquad j = 0, 1,$$
$$L_{1k} f_{k01} = P_{k01}, \qquad L_{2k} f_{k02} = P_{k02},$$

for $0 < |k| \le K_+$. Obviously, the above equations are solvable if L_{0k}, L_{1k}, L_{2k} are invertible.

Consider the set

$$\Lambda_+ = \{\theta_t \in \Lambda:\ |L_{0k}| > \frac{\gamma}{|k|^\tau},\ |\mathrm{det} L_{1k}| > \frac{\gamma^{2m}}{|k|^{2\tau m}},\ |\mathrm{det} L_{2k}| > \frac{\gamma^{4m^2}}{|k|^{4\tau m^2}},\ 0 < |k| \le K_+\}.$$

Since M is non-singular, the above linear equations (3.8)–(3.10) are uniquely solvable. Also, the norm of F is controlled.

In the usual manner, we have the following:

Lemma 3.2 *Assume that*

H2)

$$|\partial^l_{\theta_t} M - \partial^l_{\theta_t} M_0|,\ |\partial^l_{\theta_t}\omega - \partial^l_{\theta_t}\omega_0| \le \mu_*, \qquad 0 \le |l| \le d, \tag{3.11}$$

where μ_* will be specified below. Then, there exits a constant c_2 such that the following hold:

(1) On Λ_+,

$$|\partial^l_{\theta_t} f_{k00}| \le c_2|k|^{|l|+(|l|+1)\tau} s^2\mu \mathrm{e}^{-|k|^\tau},$$
$$|\partial^l_{\theta_t} f_{k10}| \le c_2|k|^{|l|+(|l|+1)2m\tau} s\mu \mathrm{e}^{-|k|^\tau},$$
$$|\partial^l_{\theta_t} f_{k01}| \le c_2|k|^{|l|+(|l|+1)2m\tau} s\mu \mathrm{e}^{-|k|^\tau},$$
$$|\partial^l_{\theta_t} f_{k02}| \le c_2|k|^{|l|+(|l|+1)4m^2\tau} \mu \mathrm{e}^{-|k|^\tau}, \qquad |\partial^l_{\theta_t} f_{001}| \le c_2 s\mu,$$

for all $0 < |k| \le K_+$;

(2) On $D_* \times \Lambda_+$,

$$|F|,\ |F_x|,\ s|F_y|,\ s|F_z| \le c_2 s^2\mu\Gamma(r - r_+) + c_2 s^2\mu,$$

and on $\tilde{D}\times\Lambda_+$,

$$|\partial^l_{\theta_t}\partial^i_x\partial^{(p,q)}_{(y,z)} F| \le c_2\mu\Gamma(r - r_+) + c_2\mu,$$

for all $0 \le |l|$, $|i| \le d$, $|p| \le 1$, $|q| \le 2$.

Lemma 3.3 *Assume*

H3) $c_2\mu\Gamma(r - r_+) + c_2\mu < \frac{1}{8}(r - r_+)$;

H4) $c_2 s\mu\Gamma(r - r_+) + c_2 s\mu < s_+$.

Let ϕ^t_F be the flow generated by F. We have that

1) *For all $0 \le t \le 1$,*

$$\phi^t_F : D_3 \to D_4$$

are well defined, real analytic and depend smoothly on $\theta_t \in \Lambda_+$;

2) *Let $\Phi_+ = \phi^1_F$. Then for all $\theta_t \in \Lambda_+$,*

$$\Phi_+ : D_+ \to D;$$

3) *There is a constant* c_3 *such that*

$$|\partial_{\theta_t}^l(\phi_F^t - id)|_{D(s)\times\Lambda_+} \le c_3 s\mu\Gamma(r-r_+),$$

$$|\partial_{\theta_t}^l D^i(\Phi_+ - id)|_{\tilde{D}_+\times\Lambda_+} \le c_3\mu\Gamma(r-r_+),$$

for all $|l|\le d$, $0\le t\le 1$, $i=0,1$, *where*

$$D=\partial_{(x,y,z)}.$$

3.3 New Normal Form

For the new normal form N_+, we have

Lemma 3.4 *There is a constant* c_4 *such that for all* $0\le|l|\le d$ *the following hold*:

$$|\partial_{\theta_t}^l(e_+-e)|_{\Lambda_+} \le c_4\gamma^a s^2\mu,$$

$$|\partial_{\theta_t}^l(\omega_+-\omega)|_{\Lambda_+} \le c_4\gamma^a s\mu,$$

$$|\partial_{\theta_t}^l(M^+-M)|_{\Lambda_+} \le c_4\gamma^a\mu.$$

3.4 Melnikov's Conditions

Lemma 3.5 *Assume that*

H5) $c_4\mu K_+^{\tau+1} \le \dfrac{\gamma-\gamma_+}{\gamma_0^a}$;

H6) $c_4\mu K_+^{2m\tau+2m} \le \dfrac{\gamma^{2m}-\gamma_+^{2m}}{\gamma_0^{2am}}$;

H7) $c_4\mu K_+^{4m^2\tau+4m^2} \le \dfrac{\gamma^{4m^2}-\gamma_+^{4m^2}}{\gamma_0^{4am^2}}$.

Then, for all $0<|k|\le K_+$, *and* $\theta_t\in\Lambda_+$, *the following hold*:

$$|L_{0k}^+| > \frac{\gamma_+}{|k|^\tau},$$

$$|\mathrm{det}L_{1k}^+| > \frac{\gamma_+^{2m}}{|k|^{2m\tau}},$$

$$|\mathrm{det}L_{2k}^+| > \frac{\gamma_+^{4m^2}}{|k|^{\tau^{4m^2\tau}}}.$$

3.5 New Perturbation P_+

Now we estimate the new perturbation P_+:

Lemma 3.6 *On* $D_+\times\Lambda_+$, *there exits a constant* c_5, *such that*

$$|\partial_{\theta_t}^l P_+| \le c_5\gamma^a(s^3\mu^2\Gamma(r-r_+)+s^3\mu^2+s^2\mu^2), \qquad |l|\le d.$$

Let $c_0=\max\{1,c_1,\cdots,c_5\}$ and assume that

H9) $c_0\gamma^a(s^3\mu^2\Gamma(r-r_+)+s^3\mu^2+s^2\mu^2)\le\gamma_+^a s_+^2\mu_+$.

Then, on $D_+\times\Lambda_+$,

$$|\partial_{\theta_t}^l P_+| \le \gamma_+^a s_+^2\mu_+, \qquad |l|\le d.$$

This completes one cycle of KAM steps.

4 Proof of Main Result

4.1 Iteration Lemma

Set

$$
\begin{aligned}
&H_\nu = N_\nu + P_\nu, \qquad N_\nu = e_\nu + \langle \omega_\nu, y\rangle + \frac{1}{2}\langle z, M_\nu z\rangle,\\
&r_\nu = r_0\Big(1-\sum_{i=1}^{\nu}\frac{1}{2^{i+1}}\Big), \qquad \gamma_\nu = \gamma_0\Big(1-\sum_{i=1}^{\nu}\frac{1}{2^{i+1}}\Big),\\
&\beta_\nu = \beta_0\Big(1-\sum_{i=1}^{\nu}\frac{1}{2^{i+1}}\Big), \qquad \eta_\nu = \mu_\nu^{\frac{1}{3}}, \qquad \mu_\nu = s_\nu^{\frac{1}{2}} = \mu_{\nu-1}^{\frac{7}{6}},\\
&s_\nu = \frac{1}{8}\alpha_{\nu-1}s_{\nu-1}, \quad \alpha_\nu = \mu_\nu^{\frac{1}{3}}, \quad K_\nu = \Big(\Big[\log\frac{1}{\mu_{\nu-1}}\Big]+1\Big)^{3\eta},\\
&D_{i\alpha} = D\Big(r_\nu + \frac{i-1}{8}(r_{\nu-1}-r_\nu), i\alpha_{\nu-1}s_{\nu-1}\Big), \qquad i=1,2,\cdots,8,\\
&D_\nu(\xi) = \{y\in C^d : |y|<\xi\}, \quad \hat{D}_\nu(\xi) = D(r_\nu + \frac{7}{8}(r_{\nu-1}-r_\nu),\xi), \qquad \xi>0,
\end{aligned}
$$

for all $\nu = 1,2,\cdots$.

Lemma 4.1 *If $\mu_0 = \mu_0(r_0,\beta_0,m,d,\tau)$, or equivalently, $\mu = \mu(r,s,m,d,\tau)$, is sufficiently small ,then the KAM step described in Section 3 is valid for all $\nu = 0,1,\cdots$. Consider the sequences*

$$\Lambda_\nu,\ H_\nu,\ N_\nu,\ e_\nu,\ \omega_\nu,\ M_\nu,\ P_\nu,\ \Phi_\nu, \qquad \nu=1,2,\cdots.$$

Then the following properties hold:

1) *$\Phi_\nu : D_\nu\times\Lambda_\nu \longrightarrow D_{\nu-1}$ is symplectic for each $\theta_t\in\Lambda_0$ or Λ_ν, and is of class $C^{2,d}$, and*

$$|\partial_{\theta_t}^l D^i(\Phi_\nu - id)|_{(\hat{D}_\nu\times\Lambda_0)} \le \frac{\mu_*^{\frac{1}{4}}}{2^\nu},$$

where $i=0,1$, $\mu_* = \mu_0^{1-\sigma}$, $\sigma\in\Big[\frac{3}{4},1\Big)$;

2) *On $\hat{D}_\nu\times\Lambda_\nu$,*

$$H_\nu = H_{\nu-1}\circ\Phi_\nu = N_\nu + P_\nu,$$

where

$$N_\nu = e_\nu + \langle\omega_\nu, y\rangle + \frac{1}{2}\langle z, M_\nu z\rangle.$$

For all $|l|\le d$,

$$
\begin{aligned}
&|\partial_{\theta_t}^l e_\nu - \partial_{\theta_t}^l e_{\nu-1}|_{\Lambda_\nu} \le \gamma_0^a\frac{\mu_*}{2^\nu},\\
&|\partial_{\theta_t}^l e_\nu - \partial_{\theta_t}^l e_0|_{\Lambda_\nu} \le \gamma_0^a\mu_*,\\
&|\partial_{\theta_t}^l \omega_\nu - \partial_{\theta_t}^l \omega_{\nu-1}|_{\Lambda_\nu} \le \gamma_0^a\frac{\mu_*}{2^\nu},\\
&|\partial_{\theta_t}^l \omega_\nu - \partial_{\theta_t}^l \omega_0|_{\Lambda_\nu} \le \gamma_0^a\mu_*,\\
&|\partial_{\theta_t}^l M_\nu - \partial_{\theta_t}^l M_{\nu-1}|_{\Lambda_\nu} \le \gamma_0^a\frac{\mu_*}{2^\nu},\\
&|\partial_{\theta_t}^l M_\nu - \partial_{\theta_t}^l M_0|_{\Lambda_\nu} \le \gamma_0^a\mu_*,\\
&|\partial_{\theta_t}^l P_\nu|_{D_\nu\times\Lambda_\nu} \le \gamma_\nu^a s_\nu^2\mu_\nu.
\end{aligned}
$$

Moreover, M_ν is real symmetric and non-singular on Λ_ν;

3)

$$\Lambda_\nu = \{\theta_t \in \Omega_{\nu-1} : |L_{0k}^\nu| > \frac{\gamma_\nu}{|k|^\tau},\ |\det L_{1k}^\nu| > \frac{\gamma_\nu^{2m}}{|k|^{2\tau m}},\ |\det L_{2k}^\nu| > \frac{\gamma_\nu^{4m^2}}{|k|^{4\tau m^2}},\ K_{\nu-1} \le |k| \le K_\nu\}.$$

Proof. We have to verify H1)–H9) for all ν to guarantee the KAM cycle in section 3. For simplicity, let $r_0 = \beta_0 = 1$, and s_0, μ_0 be sufficiently small. Note that

$$\mu_\nu = \mu_0^{(7/6)^\nu}, \qquad s_\nu = s_0^{(7/6)^\nu}. \tag{4.1}$$

It follows from (4.1) that

$$\begin{aligned} &\log(d+3)! + (\nu+6)(d+3)\log 2 + 3\eta d \log\Big(\Big[\log\frac{1}{\mu_\nu}\Big] + 1\Big) - \frac{K_{\nu+1}}{2^{\nu+2}} - \log\mu_\nu, \\ \le\ &\log(d+3)! + (\nu+6)(d+3)\log 2 + 3\eta d \log\Big(\Big[\log\frac{1}{\mu_\nu}\Big] + 2\Big) + (7/6)^\nu - \frac{(7/6)^{3\eta\nu}}{2^{\nu+2}} \\ \le\ &0, \end{aligned}$$

as μ_0 is small and $(7/6)^{3\eta\nu-1} \gg 2$, so

$$\int_{K_{\nu+1}}^\infty t^{d+2} \mathrm{e}^{-\frac{t}{2^{\nu+3}}}\,\mathrm{d}t \le (d+3)! 2^{(\nu+6)(d+2)} K_{\nu+1}^n \mathrm{e}^{-\frac{K_{\nu+1}}{2^{\nu+2}}} \le \mu_\nu, \tag{4.2}$$

i.e., H1) holds. Also, we have

$$\begin{aligned} \frac{\gamma_\nu^{4m^2} - \gamma_{\nu+1}^{4m^2}}{\gamma_0^{4m^2 a}} &\le c_* \gamma_\nu^{4m^2}, \\ c_0 c_* \mu_\nu {K_{\nu+1}}^{4m^2} &\le \frac{1}{2^{4m^2}}, \end{aligned}$$

so H7) holds. H6) and H5) hold similarly, and we omit the details. Note that

$$\begin{aligned} \Gamma_\nu &= \Gamma(r_\nu - r_{\nu+1}) \\ &\le \int_1^\infty \lambda^{2d+(d+1)4m^2\tau+1} \mathrm{e}^{-\frac{\lambda}{2^{\nu+6}}}\,\mathrm{d}\lambda \\ &\le (2d + [(d+1)4m^2\tau] + 2)! 2^{(\nu+6)(2d+(d+1)4m^2\tau+1)}. \end{aligned}$$

Let

$$a^* = (2d + [(d+1)4m^2\tau] + 2)! 64^{(2d+(d+1)4m^2\tau+1)}, \tag{4.3}$$

$$b^* = 2d + (d+1)4m^2\tau + 1. \tag{4.4}$$

Then

$$\Gamma_\nu \le a^*(2^{b^*})^\nu. \tag{4.5}$$

It is clear that H3) and H4) are equivalent to

$$\begin{aligned} \frac{8c_0\mu_\nu\Gamma_\nu}{r_\nu - r_{\nu+1}} &\le 16 a^* \mu_\nu 2^{(b^*+1)\nu}, \\ c_0 s\mu\Gamma_\nu &\le 8a^* \mu_\nu^{\frac{2}{3}} 2^{b^*\nu}. \end{aligned} \tag{4.6}$$

Observe that H9) and (4.1) imply

$$\begin{aligned} \frac{c_0\gamma_\nu^a(s_\nu^3\mu_\nu^2\Gamma_\nu + s_\nu^3\mu_\nu^2 + s^2\mu_\nu^2)}{\gamma_{\nu+1}^a s_{\nu+1}^2 \mu_{\nu+1}} &\le 2^a c_0 \Big(\frac{s_\nu^3\mu_\nu^2\Gamma_\nu}{s_{\nu+1}^2\mu_{\nu+1}} + \frac{s_\nu^3\mu_\nu^2}{s_{\nu+1}^2\mu_{\nu+1}} + \frac{s^2\mu_\nu^2}{s_{\nu+1}^2\mu_{\nu+1}}\Big) \\ &\le 2^{a+3} c_0 (s^{\frac{5}{6}}\mu^{\frac{4}{5}} 2^{b^{*\nu}} + s^{\frac{5}{6}}\mu^{\frac{4}{5}} + \mu^{\frac{2}{15}}). \end{aligned} \tag{4.7}$$

Then, we only to show that

$$c^*\mu_\nu^{(1/3)}2^{(b^*+1)\nu} \le \frac{1}{2}, \tag{4.8}$$

where $c^* = 2^{a+3}c_0a^*$. Since

$$\mu_\nu^{(1/3)}2^{(b^*+1)\nu} \le (\mu_0^{\frac{1}{3}})^{(1+(\nu/5))}2^{(b^*+1)\nu} \le (\mu_0^{\frac{1}{3}})(\mu_0^{\frac{1}{15}}2^{b^*+1})^\nu,$$

(4.10) holds. Then H3), H4), H9) are verified, as μ_0 is sufficiently small. From (4.8), we have

$$\mu_\nu\Gamma_\nu \le \frac{1}{2}\mu_\nu^{(3/4)} \le \frac{\mu_0^{\frac{1}{4}}}{2^{\nu+1}},$$

where $\mu_* = \mu_0^{1-\sigma}$, $\sigma \ge \dfrac{3}{4}$, so

$$c_0\mu_\nu\Gamma_\nu \le \frac{\mu_*}{2^{\nu+1}}, \qquad c_0\gamma^a\mu_\nu\Gamma_\nu \le \frac{\gamma^a\mu_*}{2^{\nu+1}}, \tag{4.9}$$

for all $\nu = 0, 1\cdots$.

In the following, we are to prove H2). For $\nu_* = 0$, Lemma 3.4 automatically holds. For $\nu_* \ge 1$, assume that Lemma 3.4 holds. Then we have

$$\begin{aligned}|\partial_{\theta_t}^l(M_{\nu_*+1} - M_0)|_{D_{\nu_*+1}\times\Lambda_{\nu_*+1}} &\le \sum_{\nu=0}^{\nu_*}|\partial_{\theta_t}^l(M_{\nu+1} - M_\nu)|_{D_{\nu+1}\times\Lambda_{\nu+1}}\\ &\le \sum_{\nu=0}^{\nu_*}c_0\gamma_0^a\mu_\nu\\ &\le \gamma_0^a\sum_{\nu=0}^{\nu_*}\frac{\mu_*}{2^{\nu+1}}\\ &\le \gamma_0^a\mu_0^{\frac{1}{4}}\\ &< \mu_*.\end{aligned}$$

The case for ω can be handled similarly. Then H2) holds for ν_*+1. Therefore all assumptions in Section 3 hold for all ν. Moreover, $\mu_* < 1$ and

$$|M_{\nu+1}|_{\Lambda_{\nu+1}} \le \frac{|(M_0)^{-1}|_{\Lambda_0}}{1 - \mu^{(1/4)}|(M_0)^{-1}|_{\Lambda_0}} \le 2|(M_0)^{-1}|_{\Lambda_0},$$

so $M_{\nu+1}$ is invertible.

3) is obvious.

4.2 Convergence

In this section, we prove the convergence of the sequences from Section 3. Let

$$\begin{aligned}&\Psi^\nu = \Phi_0\circ\Phi_1\circ\cdots\circ\Phi_\nu : D_{\nu+1}\times\Lambda_{\nu+1} \to D_0,\\ &H\circ\Psi^\nu = H_\nu = N_\nu + P_\nu,\\ &N_\nu = e(\theta_t)_\nu + \langle\omega_\nu(\theta_t), y\rangle + \frac{1}{2}\langle z, M_\nu(\theta_t)z\rangle,\end{aligned}$$

$\nu = 0, 1, \cdots$, which satisfy all properties described in Lemma 3.1. By the iteration lemma, it is easy to verify that Ψ^ν converges to a function $\Psi^\infty \in C^{2,d-1}\Big(D\Big(\dfrac{r_0}{2}, \dfrac{\beta_0}{2}\Big)\times\Lambda_0\Big)$, in

$C^{2,d-1}\Big(D\Big(\frac{r_0}{2},\frac{\beta_0}{2}\Big)\times\lambda_0\Big)$ norm, and each Ψ_{θ_t}, $\theta_t\in\Lambda_0$, is symplectic and C^2. Let

$$\Lambda^*=\bigcap_{\nu=0}^{\infty}\Lambda_\nu,\qquad G_*=D\Big(\frac{r_0}{2},\frac{\beta_0}{2}\Big)\times\Lambda^*.$$

Then Λ^* is a Cantor-like set, and $\{\Psi_{\theta_t}:\theta_t\in\Lambda_*\}$ is a C^{d-1} Whitney smooth family of analytic symplectic transformations on $D\Big(\frac{r_0}{2},\frac{s_0}{2}\Big)$.

By Lemma 3.1 2), it is clear that e_ν, ω_ν and M_ν converge uniformly on Λ^*. We denote e_∞, ω_∞ and M_∞ as their limits, respectively. It follows from the Whitney's extension theorem (see [21]) that these limits are also Hölder continuous in θ_t. Moreover, by Lemma 3.1 1), we have that

$$|e_\infty-e_0|_{\Lambda_*}=O(\gamma_0^a\mu_*),$$
$$|\omega_\infty-\omega_0|_{\Lambda_*}=O(\gamma_0^a\mu_*),$$
$$|M_\infty-M_0|_{\Lambda_*}=O(\gamma_0^a\mu_*).$$

Thus, on G_*, N_ν converges uniformly to

$$N_\infty=e_\infty+\langle\omega_\infty,y\rangle+\frac{1}{2}\langle z,M_\infty z\rangle,$$

and the perturbation P_ν converges uniformly to

$$P_\infty=H\circ\Psi^\infty-N_\infty.$$

Clearly, these limits above are uniformly continuous in $\theta_t\in\Lambda^*$ and analytic in $(x,y,z)\in D\Big(\frac{r_0}{2},\frac{\beta_0}{2}\Big)$.

Note that

$$|P_\nu|_{D_\nu}\le\gamma_\nu^a s_\nu^2\mu_\nu.$$

It follows from Cauchy's estimate that, for any $\theta_t\in\Lambda^*$, $j\in\mathbf{Z}_+^d$, $k\in\mathbf{Z}_+^{2m}$ with $|j|+|k|\le 2$,

$$|\partial_y^j\partial_z^k P_\nu|_{D(r_{\nu+i},\frac{1}{2}s_\nu)}\le\gamma_\nu^a\mu_\nu.$$

Since, by (4.8), the right hand side of the above converges to 0 as $\nu\to 0$, we have

$$\partial_y^j\partial_z^k P_\infty|_{(y,z)=0}=0$$

for all $x\in\mathbf{T}^n$, $\theta_t\in\Lambda^*$, $j\in\mathbf{Z}_+^d$, $k\in\mathbf{Z}_+^{2m}$ with $|j|+|k|\le 2$.

Next, we prove Theorem A-2). Since θ_t is stationary and ergodic, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t\Omega_*(\theta_s(p))\mathrm{d}s&=\int_\Omega\Omega_*(\theta_0(q))\mathrm{d}q\\&=\int_\Omega\Omega_*(q)\mathrm{d}q\qquad a.e.\ \Omega.\end{aligned}$$

According to the ergodic theorem, we choose the characteristic function of Ω_γ, i.e.,

$$\chi_{(\Omega_\gamma)}(x)=\begin{cases}1, & x\in\Omega_\gamma;\\0, & \text{others},\end{cases}$$

and have

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}|\{s\in[0,t]:\theta_s(p)\in\Omega_\gamma\}|&=\lim_{t\to\infty}\frac{1}{t}\int_0^t\chi_{(\Omega_\gamma)}(\theta_s(p))\mathrm{d}s\\&=\int_\Omega\chi_{(\Omega_\gamma)}(q)\mathrm{d}q\\&=|\Omega_\gamma|\qquad a.e.\ \Omega.\end{aligned}$$

4.3 Measure Estimate

The measure estimate can be completed by applying the following lemmas. See Section 5.2 of [12] for the details.

Lemma 4.2 *Let* A1) *hold and* $\lambda_i(\theta_t)$ $(i = 1, 2, \cdots m)$ *be the eigenvalues of* $JM_0(\theta_t)$. *Then the following hold*:

1) *For all* $k \in \mathbf{Z}^d$,

$$\det L_{1k}^0 = \prod_{i=1}^{2m}(\sqrt{-1}\langle k, \omega(\theta_t)\rangle - \lambda_i(\theta_t)),$$

$$\det L_{2k}^0 = \prod_{i,j=1}^{2m}(\sqrt{-1}\langle k, \omega(\theta_t)\rangle - \lambda_i(\theta_t) - \lambda_j(\theta_t)).$$

2)

$$\{\theta_t \in \Lambda_0 : \ \langle k, \omega(\theta_t)\rangle \neq 0, \ \det L_{1k}^0 \neq 0, \ \det L_{2k}^0 \neq 0, \ \forall k \in \mathbf{Z}^d \setminus 0\}$$

admits full Lebesgue measure relative to Λ_0.

Lemma 4.3 *Let* $\Lambda \subset \mathbf{R}^d$, $d > 1$, *be a bounded closed region and* $g : \Lambda \to \mathbf{R}^d$ *be such that*

$$\text{rank}\Big\{\frac{\partial^\alpha g}{\partial \lambda^\alpha} : |\alpha| \le d-1\Big\} = d.$$

Then, for a fixed $\tau > d(d-1) - 1$,

$$\Big|\{\lambda \in \Lambda : |\langle g(\lambda), k\rangle| \le \frac{\gamma}{|k|^\tau}\}\Big| \le c(\Lambda, d, \tau)\left(\frac{\gamma}{|k|^{\tau+1}}\right)^{\frac{1}{d-1}}, \qquad k \in \mathbf{Z}^d \setminus \{0\}, \ \gamma > 0.$$

Proof. See [22, 23].

5 Application

Consider the system

$$H(y, x, \theta_t) = h(y) + \varepsilon P(y, x, \theta_t), \tag{5.1}$$

where (y, x) varies in a complex neighbourhood of $G \times \mathbf{T}^d$, $G \subset \mathbf{R}^d$ is a closed bounded region, θ_t is defined as Section 1; h and P are real analytic in (y, x) and C^{l_0} differentiable in parameter θ_t; h satisfies the standard nondegeneracy condition $\det\dfrac{\partial^2 h}{\partial y^2} \neq 0$ in G, P is a perturbation and $\varepsilon > 0$ is a small parameter.

We denote

$$\omega(y) = \frac{\partial h}{\partial y}(y) = (\omega_1(y), \cdots, \omega_2(y)).$$

$\omega(y)$ is called nonresonant if $\langle k, \omega(y)\rangle \neq 0$ for any $k \in \mathbf{Z}^d \setminus \{0\}$. Otherwise, $\omega(y)$ is resonant.

Let $a_{ij} = \dfrac{\omega_i}{\omega_j}$. If each a_{ij} is a rational number, then the vector ω is commensurable. It is obvious that there exits a rank $d-1$ subgroup G_{d-1} of $\mathbf{Z}^d$, such that $\langle k, \omega(y)\rangle = 0$ for any $k \in G_{d-1}$ and $\langle k, \omega(y)\rangle \neq 0$ for all $k \in \mathbf{Z}^d \setminus G_{d-1}$. Thus

$$O(G_{d-1}, G) = \{y \in G : \ \langle k, \omega(y)\rangle = 0, \ k \in G_{d-1}\}$$

is a one dimensional surface and we call it G_{d-1}-resonant surface.

By the group theory, there are independent integer vectors $\tau_1,\cdots,\tau_{d-1},\tau_d$ such that $\det(\tau_1,\cdots,\tau_{d-1},\tau_d)=1$, G_{d-1} is generated by $\tau_1,\cdots,\tau_{d-1}$ and $\mathbf{Z}^d$ is generated by $\tau_1,\cdots,$ τ_{d-1}, τ_d. We say that $h(y)$ is G_{d-1}-nondegenerate if $h(y)$ is nondegenerate and

$$\det(\tau_1,\cdots,\tau_{d-1})^T\frac{\partial^2 h}{\partial y^2}(y)(\tau_1,\cdots,\tau_{d-1})\neq 0,\qquad y\in O(G_{d-1},G).$$

For the general case, $G_{\bar d}$ is a given subgroup generated by independent integer vectors $\tau_1,\cdots,\tau_{\bar d}$, such that $\langle k,\omega(y)\rangle=0$ for any $k\in G_{\bar d}$ and $\langle k,\omega(y)\rangle\neq 0$ for all $k\in\mathbf{Z}^d\setminus G_{\bar d}$. Then

$$O(G_{\bar d},G)=\{y\in G:\ \langle k,\omega(y)\rangle=0,\ k\in G_{\bar d}\}$$

is called a $G_{\bar d}$-resonant surface, and its dimension is $\tilde d=d-\bar d$.

We set

$$\mathcal{K}=(\tau_1',\cdots,\tau_{\tilde d}',\tau_1,\cdots,\tau_{\bar d}),\ \tilde{\mathcal{K}}=(\tau_1',\cdots,\tau_{\tilde d}'),\ \bar{\mathcal{K}}=(\tau_1,\cdots,\tau_{\bar d}),$$

where $\mathcal{K}$, $\tilde{\mathcal{K}}$, $\bar{\mathcal{K}}$ are $d\times d$, $d\times\tilde d$, $d\times\bar d$ matrices respectively, $\mathcal{K}$ generates $\mathbf{Z}^d$, and $\det\mathcal{K}=1$.

$h(y)$ is called $G_{\bar d}$-nondegenerate if $h(y)$ is nondegenerate and

$$\det\Big(\bar{\mathcal{K}}^T\frac{\partial^2 h}{\partial y^2}(y)\bar{\mathcal{K}}\Big)\neq 0,\qquad \text{for all } y\in O(G_{\bar d},G).$$

Write $P(y,x,\theta_t)$ in its Fourier's expansion:

$$P(y,x,\theta_t)=\sum_{k\in\mathbf{Z}^d}P_k\mathrm{e}^{\sqrt{-1}\langle k,x\rangle}.$$

For the subgroup $G_{\bar d}\subset\mathbf{Z}^d$, let

$$\bar p(y,\varphi,\theta_t)=\sum_{k\in G_{\bar d}}p_k\mathrm{e}^{\sqrt{-1}\langle k,x\rangle}=\sum_{l\in\mathbf{Z}^{\bar d}}p_{\bar{\mathcal{K}}l}\mathrm{e}^{\sqrt{-1}\langle l,\varphi\rangle},\tag{5.2}$$

where $\varphi=\bar{\mathcal{K}}^Tx$. Clearly, $\bar p$ has at least $\bar d$ critical points on $\mathbf{T}^{\bar d}$. For the subgroup G_{n-1}, there are at least two critical points (see [24]).

We have the following Poincaré Theorem for the random Hamiltonian system (5.1).

Theorem A1 *Suppose that $H=h+\varepsilon P$ is analytic, ω is commensurable and all the critical points of $\bar p(\varphi,y)$ are nondegenerate. Then there exits an ε_0 (depending on h, $G_{d-1},\bar p$) sufficiently small such that for $0<\varepsilon<\varepsilon_0$ the system* (5.1) *has at least two periodic solutions.*

For the resonant group $G_{\bar d}$, we have the following resonant KAM theorem for (5.1).

Theorem A2 (General Case) *Suppose that $H=h+\varepsilon P$ is analytic, and $\bar p(\varphi,y)$ has an analytic family of nondegeneracy critical points for all $y\in O(G_{\bar d},G)$. Then there exists an ε_0 (depending on h, $G_{\bar d}$, $\bar p$) sufficiently small and a Cantor set $\Lambda_*\subset O(G_{\bar d},G)$ such that for $0<\varepsilon<\varepsilon_0$ the system* (5.1) *admits a set of Cantor fragments of an analytic, Diophantine $\tilde d$-dimensional invariant torus I_{y_0} parametrized by $y_0\in\Lambda_*$. Moreover, the measure of $|O(G_{d-1},G)\setminus\Lambda_*|\to 0$ as $\varepsilon\to 0$.*

For any $y_0\in O(G_{\bar d,G})$, we expand the Hamiltonian (5.1) into Taylor's series:

$$H(y,x,\theta_t)=\langle\omega(y_0),y-y_0\rangle+\frac{1}{2}\Big\langle\frac{\partial^2 h}{\partial y^2}(y_0),y-y_0\Big\rangle+\varepsilon P(y,x,\theta_t)+O(|y-y_0|^3).$$

And then we use the linear transformation.

$$y-y_0=\mathcal{K}p,\qquad q=\mathcal{K}^Tx,$$

and the symplectic canonical coordinate transformation

$$(p, q \bmod 2\pi) \to (Y, X \bmod 2\pi): \ p = \frac{\partial S(q,Y)}{\partial q}, \ X = \frac{\partial S(q,Y)}{\partial Y},$$

where

$$S = \langle Y, q\rangle + \varepsilon \sum_{k\in \mathbf{Z}^{\bar d}\setminus\{0\}} \frac{\sqrt{-1}h_k}{\langle k,\omega\rangle}(q'' \mathrm{e}^{\sqrt{-1}\langle k,q'\rangle}),$$

with

$$\begin{aligned} h_k &= \int \bar P(q,0)\mathrm{e}^{\sqrt{-1}\langle k,q'\rangle}\mathrm{d}q', \\ p' &= Y' + \sqrt{-1}\varepsilon \sum_{k\in\mathbf{Z}^{\bar d}} kS_k \mathrm{e}^{\sqrt{-1}\langle k,q'\rangle}), \\ S_k &= \frac{\sqrt{-1}h_k}{\langle k,\omega\rangle}, \\ p'' &= Y'' + O(\varepsilon), \\ X &= q. \end{aligned}$$

Hence, we get the desired normal form:

$$H(y,x,z,\theta_t) = \langle \omega, y\rangle + \frac{\delta}{2}\langle Mz, z\rangle + O(\varepsilon^2) + \varepsilon(O|y|^2 + |y||z| + |z|^3),$$

where δ is a small positive number and $(x,y,z) \in \mathbf{T}^d \times \mathbf{R}^d \times \mathbf{R}^{2\bar d}$ varies in a complex neighborhood $D(r,s) = \{(x,y,z) : |\mathrm{Im}x| < r, |y| < s^2, z < s\}$. By the Main Theorem, we can prove Theorems A1 and A2.

See [25] for details.

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