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Strong Converse Inequality for the Meyer-König and Zeller-Durrmeyer Operators*

QI QIU-LAN[1] AND LIU JUAN[2]

(*1. College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, 050016*)

(*2. No.1 Middle Shcool of Handan, Handan, Hebei, 056002)*

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Abstract: In this paper we give a strong converse inequality of type B in terms of unified K-functional $K_\lambda^\alpha(f, t^2)$ $(0 \le \lambda \le 1,\ 0 < \alpha < 2)$ for the Meyer-König and Zeller-Durrmeyer type operators.

Key words: Meyer-König and Zeller-Durrmeyer type operator, moduli of smoothness, K-functional, strong converse inequality, Hölder's inequality

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1 Introduction

The Meyer-König and Zeller operators were given by

$$\begin{aligned} M_n(f, x) &= \sum_{k=0}^{\infty} f\Big(\frac{k}{n+k}\Big) m_{n,k}(x), \qquad 0 \le x < 1, \\ M_n(f, 1) &= f(1), \\ m_{n,k}(x) &= \binom{n+k}{k} x^k (1-x)^{n+1}, \end{aligned}$$

which were the object of several investigations in approximation theory (see [1–3]). In recent years there are many results of strong converse inequalities for various operators (see [4–7]). Since the expression of the moment of the Meyer-König and Zeller type operators is very complicated (see [8–10]), we have not seen any result of strong converse inequality for Meyer-König and Zeller-Durrmeyer type operators. In this paper, we study the modification of Meyer-König and Zeller-Durrmeyer type operators $\tilde{M}_n(f, x)$:

$$\tilde{M}_n(f, x) = \sum_{k=0}^{\infty} \Phi_{n,k}(f) m_{n,k}(x), \qquad f \in C[0, 1],$$

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where

$$\begin{aligned}
&\Phi_{n,k}(f)=C_{n-2,k-1}^{-1}\int_0^1 f(t)m_{n-2,k-1}(t)\mathrm{d}t,\\
&m_{n,k}(x)=\binom{n+k}{k}x^k(1-x)^{n+1},\\
&m_{n,-1}(x):=0,\\
&C_{n,k}=\int_0^1 m_{n,k}(t)\mathrm{d}t=\frac{n+1}{(n+k+1)(n+k+2)},
\end{aligned}$$

and give a strong converse inequality of type B.

We recall that for $0\le\lambda\le1$, and $\varphi(x)=\sqrt{x}(1-x)$,

$$\omega_{\varphi^\lambda}^2(f,t)=\sup_{0<h\le t}\|\Delta_{h\varphi^\lambda}^2\|,$$

where

$$\|f\|:=\sup_{x\in[0,1)}|f(x)|,$$

$$\Delta_{h\varphi^\lambda}^2f(x)=\begin{cases} f(x+h\varphi^\lambda(x))-2f(x)+f(x-h\varphi^\lambda(x)), & \text{if } x\pm h\varphi^\lambda(x)\in[0,1);\\ 0, & \text{otherwise,}\end{cases}$$

and

$$K_{\varphi^\lambda}^2(f,t^2)=\inf_{g\in D}\{\|f-g\| \ +t^2\|\varphi^{2\lambda}g''\|\},$$

where

$$D=\{g\mid g'\in A.C._{\mathrm{loc}},\|\varphi^{2\lambda}g''\|<\infty\}.$$

In this paper we use the relation $\omega_{\varphi^\lambda}^2(f,t)\sim K_{\varphi^\lambda}^2(f,t^2)$ (see [11]), which means that, there exists a positive constant C such that

$$C^{-1}K_{\varphi^\lambda}^2(f,t^2)\le\omega_{\varphi^\lambda}^2(f,t)\le CK_{\varphi^\lambda}^2(f,t^2).$$

Before state our results, we give some new notations.

For $0\le\lambda\le1$, $0<\alpha<2$, and $\varphi(x)=\sqrt{x}(1-x)$,

$$\begin{aligned}
&C_0=\{f\in C[0,1],\ f(0)=f(1)=0\}, && \|f\|_0=\sup_{x\in(0,1)}|\varphi^{\alpha(\lambda-1)}(x)f(x)|,\\
&C_{\lambda,\alpha}^0=\{f\in C_0,\ \|f\|_0<\infty\}, && \|f\|_2=\sup_{x\in(0,1)}|\varphi^{2+\alpha(\lambda-1)}(x)f''(x)|,\\
&C_{\lambda,\alpha}^2=\{f\in C_0,\ \|f\|_2<\infty, f'\in A.C._{\mathrm{loc}}\},\\
&K_\lambda^\alpha(f,t^2)=\inf_{g\in C_{\lambda,\alpha}^2}\{\|f-g\|_0+t^2\|g\|_2\}, \qquad f\in C_0.
\end{aligned}$$

The main results of this paper can be stated as follows.

Theorem 1.1 *Suppose* $0\le\lambda\le1$, $0<\alpha<2$, *and* $f\in C_{\lambda,\alpha}^0$. *Then there exists a constant* $K>1$ *such that for* $l\ge Kn$ *we have*

$$K_\lambda^\alpha\Big(f,\frac{1}{n}\Big)\le C\frac{l}{n}(\|\tilde{M}_nf-f\|_0+\|\tilde{M}_lf-f\|_0).$$

Throughout this paper, C denotes a positive constant independent of n and x, which are not necessarily the same at each occurrence.

2 Lemmas

In order to prove our main result, we need the following fundamental lemmas.

Lemma 2.1[10] *Let $\varphi(x)=\sqrt{x}(1-x)$, and $A_{n,2p}(x)=M_n((t-x)^{2p},x)$, $p\in \mathbf{N}$. Then for $n>2p$ we have the estimates*

$$A_{n,2p}(x)\le \begin{cases} C\dfrac{\varphi^{2p}(x)}{n^p}, & x\ge \dfrac{1}{n};\\[2mm] C\dfrac{\varphi^2(x)(1-x)^{2p-2}}{n^{2p-1}}, & x<\dfrac{1}{n}.\end{cases}$$

By simple calculations, we can obtain

Lemma 2.2 *For $x\in[0,1)$, it holds that*

$$\tilde{M}_n(t-x,x)=0, \tag{2.1}$$

$$\frac{1}{2n}\varphi^2(x)\le \tilde{M}_n((t-x)^2,\ x)\le \frac{4}{n}\varphi^2(x),\qquad n\ge 2, \tag{2.2}$$

$$\tilde{M}_n((t-x)^4,\ x)\le C\frac{\varphi^4(x)}{n^2}. \tag{2.3}$$

Lemma 2.3 *For $k\ge 1$, one has*

$$C_{n,k+1}^{-1}\int_0^1(1-t)^{-4}m_{n,k+1}(t)\mathrm{d}t\le C\Big(\frac{n-1}{n+k-1}\Big)^{-4},\qquad n\ge 3. \tag{2.4}$$

$$C_{n,k+1}^{-1}\int_0^1 t^{-2}m_{n,k+1}(t)\mathrm{d}t\le C\Big(\frac{k}{n+k-1}\Big)^{-2}, \tag{2.5}$$

$$C_{n,k+1}^{-1}\int_0^1(1-t)^{-6}m_{n,k+1}(t)\mathrm{d}t\le C\Big(\frac{n}{n+k}\Big)^{-6},\qquad n\ge 7. \tag{2.6}$$

$$C_{n,k+1}^{-1}\int_0^1 \varphi^{-2}(t)m_{n,k+1}(t)\mathrm{d}t\le C\Big(\frac{k}{n+k-1}\Big)^{-2}\Big(\frac{n-1}{n+k-1}\Big)^{-4},\qquad n\ge 3. \tag{2.7}$$

Proof. Using Hölder's inequality, (2.4) and (2.5), we can get (2.7). The methods to estimate (2.4), (2.5) and (2.6) are similar, so we only give the proof of (2.4).

First, for $k\ge 1$, $n=3$, by simple calculations, we can get (2.4).

Secondly, for $k\ge 1$, $n\ge 4$, one has

$$\int_0^1 t^{k-1}(1-t)^{n-3}\mathrm{d}t=\frac{(n-3)!}{(k+2)(k+3)\cdots(n+k-1)}.$$

By

$$\begin{aligned}&\frac{(n+k+2)(n+k+3)}{n+1}\cdot\frac{(n+k+1)!}{n!(k+1)!}\cdot\frac{(n-3)!}{(k+2)(k+3)\cdots(n+k-1)}\Big(\frac{n-1}{n+k-1}\Big)^4\\ \le&\frac{(n+k)(n+k+1)(n+k+2)(n+k+3)}{(n+k-1)^4}\\ \le&\ 16,\end{aligned}$$

we can obtain (2.4).

Lemma 2.4[11] *For $l \in \mathbf{N}$, $m \in \mathbf{Z}$, one has*

$$\sum_{k=1}^{\infty} m_{n,k}(x)\Big(\frac{k}{n+k}\Big)^{-l} \leq Mx^{-l},$$

$$\sum_{k=1}^{\infty} m_{n,k}(x)\Big(\frac{n}{n+k}\Big)^{-m} \leq M(1-x)^{-m}.$$

Remark 2.1 For $x \in E_n = \Big[\frac{1}{n},\ 1\Big)$, one has

$$\sum_{k=1}^{\infty} m_{n,k}(x)\Big(\frac{k}{n+k}\Big)^{2} \leq Mx^{2}.$$

Lemma 2.5 *For $n \in \mathbf{N},\ n \geq 3,\ 0 \leq \lambda \leq 1,\ 0 < \alpha < 2,\ f \in C^2_{\lambda,\alpha}$, we have*

$$\|\varphi^3 \tilde{M}_n''' f\|_0 \leq C\sqrt{n}\|f\|_2.$$

Proof. First by direct computations (see [1]) we have

$$(\tilde{M}_n f)''(x) = (1-x)^{-2}\sum_{k=0}^{\infty} \Delta_{n,k}(f) m_{n,k}(x), \qquad x \in (0,1), \tag{2.8}$$

where

$$\Delta_{n,k}(f) = (n+k+1)[(n+k+2)\varPhi_{n,k+2}(f) - 2(n+k+1)\varPhi_{n,k+1}(f) + (n+k)\varPhi_{n,k}(f)],$$

and

$$\begin{aligned} m'_{n,k}(x) &= \frac{n+1}{(1-x)^2}[m_{n+1,k-1}(x) - m_{n+1,k}(x)], \\ m''_{n,k}(x) &= \frac{1}{(1-x)^2}[(n+k)(n+k+1)m_{n,k}(x) - 2(n+k)^2 m_{n,k-1}(x) \\ &\quad + (n+k)(n+k-1)m_{n,k-2}(x)], \end{aligned}$$

so

$$\Delta_{n,k}(f) = \int_0^1 f(t) n m''_{n,k+1}(t)\mathrm{d}t = n\int_0^1 f''(t) m_{n,k+1}(t)\mathrm{d}t.$$

Furthermore, we have

$$(\tilde{M}_n f)''(x) = (1-x)^{-2}\sum_{k=0}^{\infty} n\int_0^1 f''(t) m_{n,k+1}(t)\mathrm{d}t m_{n,k}(x).$$

Noticing that for $x \in (0,1)$, $n \geq 3$, by direct computations one has

$$\begin{aligned} (1-x)^{-2} m_{n,k}(x) &= \frac{(n+k)(n+k-1)}{n(n-1)} m_{n-2,k}(x), \\ m'_{n-2,k}(x) &= \frac{n-1}{\varphi^2(x)}\Big(\frac{k}{n+k-1} - x\Big) m_{n-1,k}(x), \end{aligned}$$

we obtain

$$\begin{aligned} &|\varphi^{3+\alpha(\lambda-1)}(x)(\tilde{M}_n f)'''(x)| \\ &\leq 2n\|f\|_2 \varphi^{1+\alpha(\lambda-1)}(x) \\ &\quad \cdot \Big(\sum_{k=1}^{\infty} \Big|\frac{k}{n+k-1} - x\Big| c^{-1}_{n,k+1}\int_0^1 \varphi^{-(2+\alpha(\lambda-1))}(t) m_{n,k+1}(t)\mathrm{d}t m_{n-1,k}(x)\Big) \\ &:= 2n\|f\|_2 \varphi^{1+\alpha(\lambda-1)}(x) G_1. \end{aligned} \tag{2.9}$$

Using Hölder's inequality, Jensen's inequality and Lemmas 2.1–2.3, we get

$$\begin{aligned} G_1 &= \sum_{k=1}^{\infty} \Big|\frac{k}{n+k-1} - x\Big| c_{n,k+1}^{-1} \int_0^1 \varphi^{-(2+\alpha(\lambda-1))}(t) m_{n,k+1}(t)\mathrm{d}t m_{n-1,k}(x) \\ &\le Cn^{-\frac{1}{2}} \varphi^{-(1+\alpha(\lambda-1))}(x), \qquad n \ge 2. \end{aligned} \tag{2.10}$$

From (2.9) and (2.10), we have

$$\|\varphi^3 \tilde{M}_n''' f\|_0 \le C\sqrt{n}\|f\|_2.$$

Lemma 2.6 *For $n \in \mathbf{N}$, $n \ge 7$, $0 \le \lambda \le 1$, $0 < \alpha < 2$, $f^{(i)} \in C^0_{\lambda,\alpha}(i = 0, 1, 2, 3)$, and $\varphi^3 f''' \in C^0_{\lambda,\alpha}$, we have*

$$\Big\|\tilde{M}_n f - f - \frac{1}{2} f''(x)\tilde{M}_n((t-x)^2, x)\Big\|_0 \le Cn^{-\frac{3}{2}} \|\varphi^3 f'''\|_0.$$

Proof. We expand $f(t)$ by the Taylor expansion and use Lemma 2.2 to obtain

$$\tilde{M}_n(f, x) - f(x) - \frac{1}{2} f''(x)\tilde{M}_n((t-x)^2, x) = \tilde{M}_n\Big(\frac{1}{2}\int_x^t (t-v)^2 f'''(v)\mathrm{d}v,\ x\Big),$$

so it is sufficient to show that

$$\Big\|\tilde{M}_n\Big(\int_x^t (t-v)^2 f'''(v)\mathrm{d}v,\ x\Big)\Big\|_0 \le Cn^{-\frac{3}{2}} \|\varphi^3 f'''\|_0. \tag{2.11}$$

For $x \in (0, 1)$, $t \in (0, 1)$, by simple calculations, we have

$$\Big|\int_x^t \frac{(t-v)^2}{\varphi^{3+\alpha(\lambda-1)}(v)}\mathrm{d}v\Big| \le |t-x|^3\Big(\varphi^{-(3+\alpha(\lambda-1))}(x) + (x(1-x))^{-\frac{3+\alpha(\lambda-1)}{2}}(1-t)^{-\frac{3+\alpha(\lambda-1)}{2}}\Big).$$

Combining the above inequality with Hölder's inequality and Lemmas 2.2–2.4, we can get (2.11). We have thus completed the proof of Lemma 2.6.

Lemma 2.7 *Let $n \in \mathbf{N}$, $n \ge 2$, $0 \le \lambda \le 1$, $0 < \alpha < 2$, $f \in C^0_{\lambda,\alpha}$. Then*

$$\|\tilde{M}_n f\|_2 \le Cn\|f\|_0.$$

Proof. Suppose that $E_n = \Big[\frac{1}{n},\ 1\Big)$. We now prove Lemma 2.7 in E_n and E_n^c repectively.

(1) For $f \in E_n$, in view of

$$(\tilde{M}_n f)''(x) = \sum_{k=0}^{\infty} \varPhi_{n,k}(f) r_{n,k} m_{n,k}(x), \qquad x \in (0, 1), \tag{2.12}$$

where

$$r_{n,k} := \frac{1}{x^2}\Big[\Big(k - \frac{(n+1)x}{1-x}\Big)^2 - \Big(k - \frac{(n+1)x}{1-x}\Big)\Big] - \frac{n+1}{x(1-x)^2},$$

we have

$$\begin{aligned} &|\varphi^{2+\alpha(\lambda-1)}(\tilde{M}_n f)''(x)| \\ \le &\Big|\varphi^{\alpha(\lambda-1)}(x)\sum_{k=0}^{\infty}\frac{(1-x)^2}{x}\Big(k - \frac{(n+1)x}{1-x}\Big)^2 \varPhi_{n,k}(f) m_{n,k}(x)\Big| \\ &+ \Big|\varphi^{\alpha(\lambda-1)}(x)\sum_{k=0}^{\infty}\frac{(1-x)^2}{x}\Big(k - \frac{(n+1)x}{1-x}\Big) \varPhi_{n,k}(f) m_{n,k}(x)\Big| \\ &+ \Big|\varphi^{\alpha(\lambda-1)}(x)\sum_{k=0}^{\infty}(n+1)\varPhi_{n,k}(f) m_{n,k}(x)\Big| \\ := &T_1 + T_2 + T_3. \end{aligned}$$

Now we estimate T_1, T_2 and T_3. Using the similar method of estimating (2.7), we get

$$C_{n-2,k-1}^{-1}\int_0^1\varphi^{\alpha(1-\lambda)}(t)m_{n-2,k-1}(t)\mathrm{d}t\le C\Big(\frac{k}{n+k}\Big)^{\frac{\alpha(1-\lambda)}{2}}\Big(\frac{n}{n+k}\Big)^{\alpha(1-\lambda)}. \tag{2.13}$$

Using (2.13), Jensen's inequality, Hölder's inequality and Lemma 2.4, one has

$$\begin{aligned}T_3&\le Cn\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}C_{n-2,k-1}^{-1}\int_0^1\varphi^{\alpha(1-\lambda)}(t)m_{n-2,k-1}(t)\mathrm{d}t m_{n,k}(x)\\&\le Cn\|f\|_0.\end{aligned}$$

Noticing that

$$\frac{(1-x)^2}{x}\Big|k-\frac{(n+1)x}{1-x}\Big|m_{n,k}(x)\le(n+2)(m_{n+2,k-1}(x)+m_{n,k}(x))$$

and

$$\begin{aligned}\sum_{k=0}^{\infty}\Big(\frac{n}{n+k+1}\Big)^4m_{n+2,k}(x)&=\sum_{k=1}^{\infty}\Big(\frac{n}{n+k+1}\Big)^4m_{n+2,k}(x)+\Big(\frac{n}{n+1}\Big)^4(1-x)^{n+3}\\&\le C(1-x)^4,\end{aligned}$$

one has

$$\begin{aligned}T_2&\le C\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}(n+2)(m_{n+2,k-1}(x)+m_{n,k}(x))\Big(\frac{k}{n+k}\Big)^{\frac{\alpha(1-\lambda)}{2}}\Big(\frac{n}{n+k}\Big)^{\alpha(1-\lambda)}\\&\le Cn\|f\|_0\end{aligned}$$

and

$$\begin{aligned}T_1&\le C\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\Big[\sum_{k=1}^{\infty}\frac{1}{x}[k-(n+k)x]^2\Big(\frac{k}{n+k}\Big)^{\frac{\alpha(1-\lambda)}{2}}\Big(\frac{n}{n+k}\Big)^{\alpha(1-\lambda)}m_{n,k}(x)\\&\qquad+x\sum_{k=1}^{\infty}\Big(\frac{k}{n+k}\Big)^{\frac{\alpha(1-\lambda)}{2}}\Big(\frac{n}{n+k}\Big)^{\alpha(1-\lambda)}m_{n,k}(x)\Big]\\&\le Cn\|f\|_0.\end{aligned}$$

From the estimates of T_1, T_2 and T_3, we obtain the result.

(2) For $x\in E_n^c=\Big(0,\ \dfrac{1}{n}\Big)$, the representation (2.8) shows that

$$\begin{aligned}&|\varphi^{2+\alpha(\lambda-1)}(x)(\tilde{M}_nf)''(x)|\\&\le\varphi^{\alpha(\lambda-1)}(x)x\Big[\Big|\sum_{k=0}^{\infty}(n+k+1)(n+k+2)\varPhi_{n,k+2}(f)m_{n,k}(x)\Big|\\&\qquad+\Big|\sum_{k=0}^{\infty}2(n+k+1)^2\varPhi_{n,k+1}(f)m_{n,k}(x)\Big|\\&\qquad+\Big|\sum_{k=0}^{\infty}(n+k+1)(n+k)\varPhi_{n,k}(f)m_{n,k}(x)\Big|\Big]\\&:=I_1+I_2+I_3.\end{aligned}$$

The methods of estimating $I_1,\ I_2,\ I_3$ are similar, so we estimate I_1 for an example. It is easy to see that

$$\begin{aligned} I_1 &\le \|f\|_0\varphi^{\alpha(\lambda-1)}(x)x\sum_{k=1}^{\infty}(n+k+1)(n+k+2)C_{n-2,k+1}^{-1}\int_0^1\frac{m_{n-2,k+1}(t)}{\varphi^{\alpha(\lambda-1)}(t)}\mathrm{d}tm_{n,k}(x)\\ &\le C\|f\|_0\varphi^{\alpha(\lambda-1)}(x)xn(n+1)\sum_{k=1}^{\infty}\Big(\frac{k}{n+k}\Big)^{\frac{\alpha(1-\lambda)}{2}}\Big(\frac{n}{n+k}\Big)^{\alpha(1-\lambda)-2}m_{n,k}(x). \end{aligned}$$

Using Hölder's inequality and Lemma 2.4, noticing that

$$x<\frac{1}{n},\qquad (1-x)^{-2}<\Big(1-\frac{1}{n}\Big)^{-2}\le 4$$

for $n\ge 2$, we have

$$\begin{aligned} I_1 &\le C\|f\|_0\varphi^{\alpha(\lambda-1)}(x)xn(n+1)\\ &\quad\cdot\Big(\sum_{k=1}^{\infty}\frac{k}{n+k}m_{n,k}(x)\Big)^{\frac{\alpha(1-\lambda)}{2}}\Big(\sum_{k=1}^{\infty}\Big(\frac{n}{n+k}\Big)^{-2}m_{n,k}(x)\Big)^{\frac{2-\alpha(1-\lambda)}{2}}\\ &\le C_1 n\|f\|_0. \end{aligned}$$

Lemma 2.8[1] *For $0\le\beta<1,\ 0<h\le\dfrac{1}{8}$, one has*

$$\int\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{\mathrm{d}s\mathrm{d}t}{\varphi^{2\beta}(x+s+t)}\le\frac{Mh^2}{\max\{\varphi(x\pm h),\ \varphi(x)\}^{2\beta}},\qquad x\in[h,1-h].$$

Lemma 2.9 *Let $n\in\mathbf{N}$, $n\ge 2$, $0\le\lambda\le 1$, $0<\alpha<2$, and $f\in C^0_{\lambda,\alpha}$. Then*

$$\|\tilde{M}_nf\|_0\le C\|f\|_0.$$

Proof. Using (2.13), Jensen's inequality and Lemma 2.4, we get

$$\begin{aligned} &|\varphi^{\alpha(\lambda-1)}(x)\tilde{M}_n(f,x)|\\ \le{}& \|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=0}^{\infty}C_{n-2,k-1}^{-1}\int_0^1\varphi^{-\alpha(\lambda-1)}(t)m_{n-2,k-1}(t)\mathrm{d}tm_{n,k}(x)\\ \le{}& C\|f\|_0,\qquad n\ge 2. \end{aligned}$$

3 Main Results

Theorem 3.1 *Suppose that $0\le\lambda\le 1$, $0<\alpha<2$, and $f\in C^0_{\lambda,\alpha}$. Then there exists a constant $K>1$ such that for $l\ge Kn$ we have*

$$K^{\alpha}_{\lambda}\Big(f,\frac{1}{n}\Big)\le C\frac{l}{n}(\|\tilde{M}_nf-f\|_0+\|\tilde{M}_lf-f\|_0).$$

Proof. By the definition of $K^{\alpha}_{\lambda}\Big(f,\dfrac{1}{n}\Big)$, for $\tilde{M}^2_n(f,x)\in C^2_{\lambda,\alpha}$, using Lemma 2.9, we have

$$K^{\alpha}_{\lambda}\Big(f,\frac{1}{n}\Big)\le\|f-\tilde{M}^2_nf\|_0+\frac{1}{n}\|\tilde{M}^2_nf\|_2\le C\|f-\tilde{M}_nf\|_0+\frac{1}{n}\|\tilde{M}^2_nf\|_2.\tag{3.1}$$

From Lemma 2.6, we have

$$\Big\|\tilde{M}_l(\tilde{M}^2_nf)-\tilde{M}^2_nf-\frac{1}{2}(\tilde{M}^2_nf)''\tilde{M}_l((t-x)^2,\ x)\Big\|_0\le Cl^{-\frac{3}{2}}\|\varphi^3(\tilde{M}^2_nf)'''\|_0.$$

Therefore, combining Lemmas 2.2, 2.5, 2.7 and 2.9, we get

$$\begin{aligned}\frac{1}{4l}\|\tilde{M}_n^2 f\|_2 \le\ & \|\tilde{M}_l(\tilde{M}_n^2 f - \tilde{M}_n f)\|_0 + \|\tilde{M}_l(\tilde{M}_n f - f)\|_0 + \|\tilde{M}_l f - f\|_0 + \|f - \tilde{M}_n f\|_0 \\ & + \|\tilde{M}_n f - \tilde{M}_n^2 f\|_0 + C_1 l^{-\frac{3}{2}}\sqrt{n}(\|\tilde{M}_n f - \tilde{M}_n^2 f\|_2 + \|\tilde{M}_n^2 f\|_2) \\ \le\ & C_2\|\tilde{M}_n f - f\|_0 + \|\tilde{M}_l f - f\|_0 + C_1 l^{-\frac{3}{2}}\sqrt{n}(C_3 n\|f - \tilde{M}_n f\|_0 + \|\tilde{M}_n^2 f\|_2) \\ \le\ & C_2\|\tilde{M}_n f - f\|_0 + \|\tilde{M}_l f - f\|_0 + C_4 l^{-\frac{3}{2}} n^{\frac{3}{2}}\|f - \tilde{M}_n f\|_0 + C_1 l^{-\frac{3}{2}}\sqrt{n}\|\tilde{M}_n^2 f\|_2.\end{aligned}$$

For $l \ge Kn$, we can choose $K > 1$, such that $C_1 l^{-\frac{3}{2}}\sqrt{n} \le \frac{1}{8l}$. Then

$$\frac{1}{8l}\|\tilde{M}_n^2 f\|_2 \le C_5\|\tilde{M}_n f - f\|_0 + \|\tilde{M}_l f - f\|_0. \tag{3.2}$$

In view of (3.1) and (3.2), we get

$$\begin{aligned}K_\lambda^\alpha\Big(f, \frac{1}{n}\Big) &\le C\|\tilde{M}_n f - f\|_0 + \frac{l}{n}(8C_5\|\tilde{M}_n f - f\|_0 + 8\|\tilde{M}_l f - f\|_0) \\ &\le C_6\frac{l}{n}(\|\tilde{M}_n f - f\|_0 + \|\tilde{M}_l f - f\|_0).\end{aligned}$$

Corollary 3.1 *Let $\lambda = 1$, and $f \in C[0,1)$. Then there exist a constant $K > 1$ such that*

$$\omega_\varphi^2\Big(f, \frac{1}{\sqrt{n}}\Big) \le C(\|\tilde{M}_n f - f\| + \|\tilde{M}_{Kn} f - f\|). \tag{3.3}$$

Proof. For $\lambda = 1$, $K_\lambda^\alpha(f, t^2)$ is the usual K-functional (see [11])

$$K_\varphi^2(f, t^2) = \inf_g\{\|f - g\| + t^2\|\varphi^2 g''\|,\ g' \in A.C._{\text{loc}}\},$$

which is equivalent to $\omega_\varphi^2(f, t)$ (see [11]). One immediately obtains (3.3) from Theorem 3.1.

Corollary 3.2 *For $0 < \alpha < 2$, $0 \le \lambda \le 1$, and $f \in C[0,1)$, we have*

$$|\tilde{M}_n(f, x) - f(x)| = O(n^{-\frac{\alpha}{2}}\varphi^{\alpha(1-\lambda)}(x)) \Rightarrow \omega_{\varphi^\lambda}^2(f, t) = O(t^\alpha).$$

Proof. From the condition

$$|\tilde{M}_n(f, x) - f(x)| = O(n^{-\frac{\alpha}{2}}\varphi^{\alpha(1-\lambda)}(x)),$$

one has

$$\|\tilde{M}_n f - f\|_0 \le Cn^{-\frac{\alpha}{2}}.$$

By using Theorem 3.1, there exists a constant $K > 1$ such that for $l \ge Kn$ we have

$$\begin{aligned}K_\lambda^\alpha\Big(f, \frac{1}{n}\Big) &\le C\frac{l}{n}(\|\tilde{M}_n f - f\|_0 + \|\tilde{M}_l f - f\|_0) \\ &\le C\frac{l}{n}(C_1 n^{-\frac{\alpha}{2}} + C_2 l^{-\frac{\alpha}{2}}) \\ &\le C_3 n^{-\frac{\alpha}{2}}.\end{aligned}$$

For $0 < t < 1$, we can choose $n \in \mathbf{N}$ such that $\frac{1}{\sqrt{n+1}} < t \le \frac{1}{\sqrt{n}}$. Then

$$K_\lambda^\alpha(f, t^2) \le K_\lambda^\alpha\Big(f, \frac{1}{n}\Big) \le C_3 n^{-\frac{\alpha}{2}} \le C_4 t^\alpha. \tag{3.4}$$

By the definition of $K_\lambda^\alpha(f, t^2)$, we can choose $g \in C_{\lambda,\alpha}^2$ such that

$$\|f - g\|_0 + n^{-1}\|g\|_2 \le 2K_\lambda^\alpha(f, n^{-1}). \tag{3.5}$$

Now we estimate $|\Delta_{h\varphi^\lambda}^2 f(x)|$.

(i) For fixed $h \in \left(0, \dfrac{1}{8}\right)$, $x \in [h, 1-h]$ and $f \in C^0_{\lambda,\alpha}$, one has

$$\begin{aligned}|\Delta^2_{h\varphi^\lambda} f(x)| &\le |f(x + h\varphi^\lambda(x))| + 2|f(x)| + |f(x - h\varphi^\lambda(x))| \\ &\le 4\|f\|_0 m(x, h\varphi^\lambda)^{\alpha(1-\lambda)},\end{aligned}$$

where

$$m(x, h\varphi^\lambda) := \max\{|\varphi(x + h\varphi^\lambda(x))|,\ |\varphi(x)|,\ |\varphi(x - h\varphi^\lambda(x))|\}.$$

(ii) Using Lemma 2.8, for any $g \in C^2_{\lambda,\alpha}$, one has

$$\begin{aligned}|\Delta^2_{h\varphi^\lambda} g(x)| &\le \|g\|_2 \int\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} |\varphi^{-2+\alpha(1-\lambda)}(x + s + t)| \mathrm{d}s\mathrm{d}t \\ &\le C\|g\|_2 h^2 m(x, h\varphi^\lambda)^{(\alpha-2)(1-\lambda)}.\end{aligned}$$

By (i), (ii), (3.4) and (3.5), one has

$$|\Delta^2_{h\varphi^\lambda} f(x)| \le Ch^\alpha,$$

which implies

$$\omega^2_{\varphi^\lambda}(f, t) = O(t^\alpha).$$

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