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Uniquely Strongly Clean Group Rings*

Wang Xiu-lan
(*Department of Mathematics, Harbin Institute of Technology, Harbin, 150001*)

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Abstract: A ring R is called clean if every element is the sum of an idempotent and a unit, and R is called uniquely strongly clean (USC for short) if every element is uniquely the sum of an idempotent and a unit that commute. In this article, some conditions on a ring R and a group G such that RG is clean are given. It is also shown that if G is a locally finite group, then the group ring RG is USC if and only if R is USC, and G is a 2-group. The left uniquely exchange group ring, as a middle ring of the uniquely clean ring and the USC ring, does not possess this property, and so does the uniquely exchange group ring.

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1 Introduction

In this paper, R is an associative ring with identity 1. A ring R is called clean if every element is the sum of an idempotent and a unit. This definition first appeared in the paper by Nicholson[1] in 1977, in which it was also proved that clean rings are exchange rings, i.e., a ring R is exchange if and only if for any $x \in R$, there exists $e^2 = e \in R$ such that $e \in Rx$ and $1 - e \in R(1 - x)$. And the two concepts are equivalent for rings with all idempotents central. A ring R is called uniquely clean if each element has a unique representation as the sum of an idempotent and a unit. For instance, every boolean ring is uniquely clean, and a homomorphic image of a uniquely clean ring is uniquely clean. Uniquely clean rings were discussed in [2–4]. Nicholson and Zhou[3] proved that a ring R is uniquely clean if and only if R modulo its Jacobson radical $J(R)$ is boolean, idempotents lift modulo $J(R)$, and idempotents in R are central if and only if for every $a \in R$ there exists a unique idempotent $e \in R$ such that $e - a \in J(R)$. A ring R is called strongly clean if every element of R is the sum of an idempotent and a unit that commute. Strongly clean rings were introduced by Nicholson[5]. Recently, Chen *et al.*[6] raised a new concept about uniquenly strongly clean

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(USC for short) ring. They called a ring R USC if every element is uniquely the sum of an idempotent and a unit that commute. They also gave the equivalent condition for USC ring, that is, a ring R is USC if and only if for all $a \in R$ there exists a unique idempotent $e \in R$ such that $ea = ae$ and $e - a \in J(R)$. Nicholson and Zhou[3], Chen *et al.*[6] proved the following results which we can use in this paper:

(1) If R is uniquely clean , then $R/J(R)$ is boolean, and $2 \in J(R)$;

(2) If R is USC, then $R/J(R)$ is boolean, and $2 \in J(R)$.

We denote by RG the group ring of G over R. The augmentation mapping

$$\varepsilon : RG \to R$$

is given by

$$\varepsilon\Big(\sum_{g\in G} a_g g\Big) = \sum_{g\in G} a_g$$

and its kernel, denoted by $\Delta(G)$ (or by Δ_{RG}), is an ideal generated by $\{1 - g, g \in G\}$. If H is a subgroup of G, then εH denotes the right ideal of RG generated by $\{1 - h, h \in H\}$. If H is a normal subgroup of G, then εH is an ideal and $RG/\varepsilon H \cong R(G/H)$. If I is a right ideal of R, then IG denotes the elements of RG with coefficients in I, when I is an ideal so is IG, and $RG/IG \cong (R/I)G$. For further details see [7].

Three years ago, Chen *et al.*[8] raised a question: if R is a ring and G is a group, when is the group ring RG clean? Wang[9] studied the cleanness of group rings for a class of Abelian p-groups. But we know that $Z_{(3)}S_3$ is a clean group ring, where S_3 is not Abelian. This motivates us to look at the cleanness of group rings of Abelian or non-Abelian groups. In Section 2, some conditions on a ring R and a group G such that RG is clean are given. Moreover, in Sections 3 and 4 it is shown that if G is a locally finite group, then the group ring RG is USC if and only if R is USC, and G is a 2-group. The left uniquely exchange group ring, as a middle ring of the uniquely clean ring and the USC ring, does not possess this property, and so does the uniquely exchange group ring. We give an example to indicate this.

Throughout this paper, R denotes an associative ring with identity 1. As usual $J(R)$ denotes the Jacobson radical of the ring R and $U(R)$ the group of units in R. We write $T_n(R)$ for the ring of all upper triangular $n \times n$ matrices over the ring R. Let G denote a group. Then a group G is called a p-group if every element of G is a power of p, where p is a prime. Let S_n stand for the symmetric group of degree n. The ring of integers is denoted by Z, and we write Z_n for the ring of integers modulo n.

2 Clean Group Rings

A group G is called locally finite if every finitely generated subgroup is finite.

Lemma 2.1 *Let R be a ring, G a group, and $\triangle(G) \subseteq J(RG)$. Then*

$$J(RG) = \{\gamma \in RG \mid \varepsilon(\gamma) \in J(R)\}.$$

Proof. Write $B = \{\gamma \in RG \mid \varepsilon(\gamma) \in J(R)\}$.

"$\subseteq$". It is obvious that $\varepsilon(J(RG)) \subseteq J(R)$ since ε is an epimorphism, and thus $J(RG) \subseteq B$ is clear.

"$\supseteq$". Since B is an ideal of RG, it suffices to show that $1+\gamma$ has a right inverse in RG wherever $\gamma \in B$. $1+\varepsilon(\gamma)$ is invertible in R, since $\gamma \in B$ implies $\varepsilon(\gamma) \in J(R)$. It follows that there is a $\beta \in R$ such that

$$(1+\varepsilon(\gamma))(1+\beta) = 1.$$

Since ε is epimorphic, there also exists a $\beta' \in RG$ such that $\varepsilon(\beta') = \beta$. So we have

$$\varepsilon[(1+\gamma)(1+\beta')] = (1+\varepsilon(\gamma))(1+\beta) = 1,$$

and

$$(1+\gamma)(1+\beta') - 1 \in \ker\varepsilon = \triangle(G) \subseteq J(RG).$$

Denote

$$t = (1+\gamma)(1+\beta') - 1 \in J(RG).$$

Then we get that

$$(1+\gamma)(1+\beta') = 1+t$$

is invertible, so

$$(1+\gamma)(1+\beta')(1+t)^{-1} = 1.$$

We say that idempotents lift modulo an ideal A of a ring R if whenever $\alpha^2 - \alpha \in A$, there exists $e^2 = e \in R$ such that $e - \alpha \in A$.

Lemma 2.2 *If R is a clean (exchange) ring, G is a locally finite group, and $\triangle(G) \subseteq J(RG)$, then RG is clean (exchange).*

Proof. We know that R is a clean (exchange) ring if and only if $R/J(R)$ is clean (exchange), and idempotents can be lifted modulo $J(R)$.

In this spirit, we first prove $RG/J(RG)$ is clean (exchange). Since G is a locally finite group, it implies

$$J(R) \subseteq J(RG),$$

and

$$J(RG)/\triangle(G) \cong J(R).$$

Then

$$RG/J(RG) \cong \frac{RG/\triangle(G)}{J(RG)/\triangle(G)} \cong R/J(R)$$

is clean as R is clean.

Next we prove idempotents can be lifted modulo $J(RG)$.

For any $\alpha \in RG$ such that

$$\alpha^2 - \alpha \in J(RG),$$

we have

$$\varepsilon(\alpha)^2 - \varepsilon(\alpha) \in J(R).$$

Since R is clean (exchange), it follows that there is $e^2 = e \in R \subseteq RG$ such that

$$e - \varepsilon(\alpha) \in J(R),$$

that is,

$$\varepsilon(e - \alpha) = e - \varepsilon(\alpha) \in J(R).$$

Then $e - \alpha \in J(RG)$ by Lemma 2.1.

Write $\overline{R} = R/J(R)$, and denote $\overline{r} = r + J(R) \in \overline{R}$. Let

$$\pi_1 : RG \to \overline{R}G$$

be the epimorphism given by

$$\sum r_g g \mapsto \sum \overline{r_g} g.$$

If $J(R)G \subseteq J(RG)$, we let

$$\pi_2 : \overline{R}G \to \overline{RG}$$

given by

$$\sum \overline{r_g} g \mapsto \sum \overline{r_g g}.$$

Lemma 2.3 *Let R be a ring and G a locally finite group. Then*

$$\pi_1^{-1}(J(\overline{R}G)) = J(RG).$$

Proof. "$\subseteq$". It is clear that $\pi_1^{-1}(J(\overline{R}G))$ is an ideal of RG. It remains to show that for each $\gamma \in \pi_1^{-1}(J(\overline{R}G))$, $1+\gamma$ has a right inverse in RG. Assume that $\gamma \in \pi_1^{-1}(J(\overline{R}G))$. Then there is $\alpha \in J(\overline{R}G)$ such that $\pi_1(\gamma) = \alpha$. So $\pi_1(1+\gamma) = \overline{1} + \alpha$ is invertible in $\overline{R}G$ since $\alpha \in J(\overline{R}G)$, and there exists $\beta \in \overline{R}G$ such that

$$(\overline{1} + \alpha)(\overline{1} + \beta) = \overline{1}.$$

As π_1 is an onto ring morphism, there also exists $\beta' \in RG$ such that $\pi_1(\beta') = \beta$. So we have

$$\pi_1[(1+\gamma)(1+\beta')] = (\overline{1} + \alpha)(\overline{1} + \beta) = \overline{1},$$

that is,

$$(1+\gamma)(1+\beta') - 1 \in \ker \pi_1 = J(R)G \subseteq J(RG).$$

Write

$$t = (1+\gamma)(1+\beta') - 1 \in J(RG).$$

Then

$$(1+\gamma)(1+\beta')(1+t)^{-1} = 1.$$

"$\supseteq$". To prove $\pi_1^{-1}(J(\overline{R}G)) \supseteq J(RG)$, it suffices to prove $J(\overline{R}G) \supseteq \pi_1(J(RG))$, which is obvious since π_1 is an onto ring morphism.

Theorem 2.1 *Let R be a ring with* $\mathrm{char}\overline{R} = p > 0$ *and G a locally finite p-group. If R is clean, then so is RG.*

Proof. Since G is a locally finite p-group, and p is nilpotent in $\overline{R}$, we see that $\triangle_{\overline{R}G}$ is a nil ideal in $\overline{R}G$ by Proposition 16 in [7], and $\triangle_{\overline{R}G} \subseteq J(\overline{R}G)$. Then by Lemma 2.3, we have

$$\triangle(G) = \triangle_{RG} \subseteq \pi_1^{-1}(\triangle_{\overline{R}G}) \subseteq \pi_1^{-1}(J(\overline{R}G)) = J(RG).$$

So RG is clean by Lemma 2.2. The proof is completed.

More generally, we have the following results.

Theorem 2.2 *Let R be a ring with* $\mathrm{char}\overline{R} = p > 0$*, G a locally finite group, N the normal p-subgroup of G, and H any subgroup of G such that $NH = G$. If RH is clean, then so is RG.*

Proof. Assume that $g \in G$. By $G = NH$, there are $n \in N$ and $h \in H$ such that
$$g = nh = (n-1)h + h \in \varepsilon N + RH.$$
And then we have
$$RG = \varepsilon N + RH.$$

Let $\pi : RG \to \overline{RG}$ be the canonicial homomorphism. Since G is a locally finite group, we have $J(R)G \subseteq J(RG)$, so the following definitions of mappings are reasonable. Let
$$\pi_1 : RG \to \overline{R}G, \qquad \pi_2 : \overline{R}G \to \overline{RG},$$
where $\pi = \pi_2 \circ \pi_1$. As N is a locally finite p-group, and p is nilpotent in $\overline{R}$, $\triangle_{\overline{R}N}$ is a nil ideal of $\overline{R}N$ by Proposition 16 in [7], and $\triangle_{\overline{R}N} \subseteq J(\overline{R}N) \subseteq J(\overline{R}G)$. Then by Lemma 2.3, we have
$$\triangle_{RN} \subseteq \pi_1^{-1}(\triangle_{\overline{R}N}) \subseteq \pi_1^{-1}(J(\overline{R}G)) = J(RG).$$
So
$$\varepsilon N = \triangle_{RN} \subseteq J(RG),$$
and
$$RG = J(RG) + RH.$$

On one hand, by the definition of π, there is
$$\overline{RG} = \pi(RG) \cong \pi(RH) = \frac{RH}{RH \cap J(RG)}.$$
As
$$J\Big(\frac{RH}{RH \cap J(RG)}\Big) = J(\overline{RG}) = 0,$$
one has
$$RH \cap J(RG) \supseteq J(RH).$$

On the other hand, we also have $RH \cap J(RG) \subseteq J(RH)$ by Proposition 9 in [7]. Then
$$RH \cap J(RG) = J(RH),$$
and
$$\overline{RG} \cong \overline{RH}.$$
So $\overline{RG}$ is clean since RH is clean.

Now, it suffices to prove that idempotents can be lifted modulo $J(RG)$. Let
$$\omega : RG \to R(G/N).$$
For any $\alpha \in RG$ such that $\alpha^2 - \alpha \in J(RG)$, there holds that
$$\omega(\alpha)^2 - \omega(\alpha) \in J(R(G/N))$$
since ω is an epimorphism. Since $RH \cong R(G/N)$ and RH is clean, there is
$$e^2 = e \in R(G/N)$$

such that

$$e - \omega(\alpha) \in J(R(G/N)),$$

that is,

$$\omega(e - \alpha) = e - \omega(\alpha) \in J(R(G/N)).$$

By the following lemma, we have

$$e - \alpha \in J(RG).$$

Lemma 2.4 *Let R be a ring, G a group, N a normal subgroup of G, and $\varepsilon N \subseteq J(RG)$. Let*

$$\omega : RG \to R(G/N).$$

Then

$$J(RG) = \{\gamma \in RG \mid \omega(\gamma) \in J(R(G/N))\}.$$

Proof. Write $B = \{\gamma \in RG \mid \omega(\gamma) \in J(R(G/N))\}$.

"$\subseteq$". It is obvious since $\omega(J(RG)) \subseteq J(R(G/N))$.

"$\supseteq$". Since B is an ideal of RG, for each $\gamma \in B$, it remains to prove that $1+\gamma$ has a right inverse in RG. The rest of the proof is similar to that of Lemma 2.1.

Example 2.1 Let

$$R = Z_{(p)} = \left\{ \frac{m}{n} \,\middle|\, m, n \in Z,\ \gcd(p, n) = 1 \right\},$$

and

$$G = D_p = \{x, y \mid x^p = 1,\ y^2 = 1,\ yxy = x^{-1}\}$$

be the dihedral groups of order $2p$. Then the group ring RG is clean.

Proof. By Theorem 2.2, it remains to prove that $Z_{(p)}C_2$ is clean.

If $p = 2$, then $Z_{(p)}C_2$ is a local ring, and so it is clean.

If $p \neq 2$, then $2 \in U(Z_{(p)})$, and $Z_{(p)}C_2 \cong Z_{(p)} \times Z_{(p)}$ is also clean.

3 USC Group Rings

Lemma 3.1 *If $RG/J(RG)$ is boolean, then $\triangle(G) \subseteq J(RG)$.*

Proof. Suppose that $RG/J(RG)$ is a boolean ring. It implies that for any $g \in G$, $(1-g)^2 - (1-g)$ is in $J(RG)$, that is, $g^2 - g \in J(RG)$, and hence

$$1 - g \in J(RG).$$

So

$$\triangle(G) \subseteq J(RG).$$

Corollary 3.1([6], Proposition 24) *If the group ring RG is USC, then R is USC and G is a 2-group.*

Proof. Suppose that RG is USC. R as an image of RG is strongly clean. Moreover, R is a subring of RG, which implies R is USC and $2 \in J(R)$. As RG is USC, it follows that $RG/J(RG)$ is boolean. Then $\triangle(G) \subseteq J(RG)$ by Lemma 3.1, which implies G is a p-group and $p \in J(R)$ by Proposition 15 in [7]. If $p = 2k+1$ is an odd prime, by $p, 2 \in J(R)$, we have

$$1 = p - 2k \in J(R),$$

which is impossible since 1 is a unit of R. Then p must be equal to 2.

Lemma 3.2 *Let R be a USC ring and G be a group. If $\triangle(G) \subseteq J(RG)$, then RG is USC.*

Proof. Assume that R is a USC ring. Then for any $\alpha \in RG$, $\varepsilon(\alpha) \in R$ and there exists a unique idempotent $e \in R \subseteq RG$ such that

$$e\varepsilon(\alpha) = \varepsilon(\alpha)e$$

and

$$e - \varepsilon(\alpha) \in J(R).$$

As $e\varepsilon(\alpha) = \varepsilon(\alpha)e$, it follows that

$$e\alpha - \alpha e \in \ker\varepsilon = \triangle(G),$$

and so

$$e\alpha - e\alpha e = e(e\alpha - \alpha e) \in \triangle(G).$$

By calculation, we have that

$$(e + e\alpha - e\alpha e)^2 = e + e\alpha - e\alpha e$$

is an idempotent of RG.

Since

$$RG/\triangle(G) \cong R,$$

the idempotent of RG has the form $e + \triangle(G)$, where e is an idempotent of R. Let $j \in \triangle(G)$. If $e + j \in RG$ is an idempotent, then we have

$$(e+j)^2 = e + ej + je + j^2 = e + j. \tag{$*$}$$

Multiplying the equation $(*)$ by e from the left hand (right hand) side, we have $eje + ej^2 = 0$ ($eje + j^2e = 0$). Hence

$$j \in eRGe, \qquad j^2 - j \in eRGe.$$

Since

$$j^2 - j = ej^2 - ej = -eje - ej = -2j \in eRGe,$$

we obtain that $j^2 = -j \in eRGe$, and $j^4 = j^2 \in \triangle(G) \subseteq J(RG)$ is an idempotent. It follows that $-j = j^2 = 0$ since there is no non-zero idempotent in $J(RG)$. Then the idempotents of RG are all in R.

From the former description, we have $e\alpha - e\alpha e = 0$, which implies $e\alpha = e\alpha e$. Similarly, we also have $\alpha e = e\alpha e$. Then $\alpha e = e\alpha$.

Now we see $\varepsilon(e - \alpha) = e - \varepsilon(\alpha) \in J(R)$. By Lemma 2.1, we have $e - \alpha \in J(RG)$.

Next, we prove e is unique. Assume that there is another idempotent $f \in RG$ (then in R) such that $f - \alpha \in J(RG)$. Then

$$f - \varepsilon(\alpha) = \varepsilon(f - \alpha) \in J(R).$$

We have $e = f$ since R is a USC ring.

Theorem 3.1 *Let R be a ring, and G a locally finite group. Then RG is USC (uniquely clean) if and only if R is USC (uniquely clean) and G is a 2-group.*

Proof. Necessity can be proved by Corollary 3.1.

Sufficiency. Suppose that R is a USC (uniquely clean) ring. Then $R/J(R)$ is boolean and $\mathrm{char}\overline{R} = 2$. By using Theorem 2.1, we have $\triangle(G) \subseteq J(RG)$. Then RG is USC (uniquely clean) by Lemma 3.2 (see Corollary 3.5 of [9]).

Remark 3.1 This is why $T_2(Z_2)D_4$ is USC in Example 26 of [6], since $T_2(Z_2)$ is a USC ring, and D_4 is a 2-group as a dihedral group of order 8.

Corollary 3.2 *If R is a ring and G a locally finite group, then the following statements are equivalent:*

(1) *RG is a USC group ring;*

(2) *R is USC, and $\triangle(G) \subseteq J(RG)$;*

(3) *R is USC, and $J(RG)/\triangle(G) \cong J(R)$;*

(4) *R is USC, and $RG/J(RG) \cong R/J(R)$;*

(5) *R is USC, and G is a 2-group.*

Proof. (1) ⇒ (2) is clear by Lemma 3.1 and Corollary 3.1.

(2) ⇒ (1) follows from Lemma 3.2.

(2) ⇒ (3) ⇒ (4) ⇒ (2) are clear since G is a locally finite group implies $J(R)G \subseteq J(RG)$.

(1) ⇔ (5) can be proved by Theorem 3.1.

4 Uniquely Exchange Group Rings

In PH.D dissertation "clean rings and regular local rings" (see [10]), Ying[10] studied the (left) uniquely exchange ring. It was proved that an element of a ring R is called left uniquely exchange, if for any $x \in R$ there exists a unique idempotent $e \in Rx$ such that $1 - e \in R(1 - x)$. R is called left uniquely exchange ring if every element of R is left uniquely exchange. The right uniquely exchange element and right uniquely exchange ring can be defined accordingly. We call an element or a ring uniquely exchange if it is both left and right uniquely exchange. We know that left exchange ring and right exchange ring are equivalent. But for both left and right uniquely exchange rings, this is not true. In that article, Ying[10] proved that R is a left uniquely exchange ring, then $R/J(R)$ is boolean and $2 \in J(R)$, and also gave the following relations:

$$R \text{ is uniquely clean} \Rightarrow R \text{ is (left) uniquely exchange} \Rightarrow R \text{ is USC}.$$

For the uniquely clean group rings and the USC group rings, we obtain the parallel result, i.e., Theorem 3.1. Naturally, we consider the (left) uniquely exchange group rings. As a middle ring, does the uniquely exchange group ring also possess this property?

Next, we give an example to indicate that this property is not fit for the left uniquely exchange group rings, and so is the uniquely exchange group rings.

Theorem 4.1 *Let R be a boolean ring. Then $T_2(R)C_2$ is not uniquely exchange while $T_2(R)$ is uniquely exchange.*

Proof. Assume that R is a boolean ring. Then, we see that $T_2(R)$ is uniquely exchange since $T_2(R)$ is uniquely exchange if and only if R is boolean by Corollary 2.2.3 of [8]. We have known that $T_2(R)C_2 \cong T_2(RC_2)$. So, it remains to prove RC_2 is not a boolean ring. As RC_2 is uniquely clean, and all its idempotents are in R. Moreover, $R \subset RG$, i.e., RG contain elements which are not idempotent. Thus, RC_2 is not boolean.

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