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New Jacobi Elliptic Function Solutions for the Generalized Nizhnik-Novikov-Veselov Equation*

HONG BAO-JIAN
(*Department of Basic Courses, Nanjing Institute of Technology, Nanjing, 211167*)

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Abstract: In this paper, a new generalized Jacobi elliptic function expansion method based upon four new Jacobi elliptic functions is described and abundant solutions of new Jacobi elliptic functions for the generalized Nizhnik-Novikov-Veselov equations are obtained. It is shown that the new method is much more powerful in finding new exact solutions to various kinds of nonlinear evolution equations in mathematical physics.

Key words: generalized Jacobi elliptic function expansion method, Jacobi elliptic function solution, exact solution, generalized Nizhnik-Novikov-Veselov equation
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1 Introduction

In recent years, due to the wide applications of soliton theory in natural science, searching for exact soliton solutions of nonlinear evolution equations plays an important and significant role in the study on the dynamics of those phenomena (see [1]). Particularly, various powerful methods have been presented, such as inverse scattering transformation, Cole-Hopf transformation, Hirota bilinear method, homogeneous balance method, Backlund transformation, Darboux transformation, projective Riccati equations method and so on. In this paper, we discuss a generalized Nizhnik-Novikov-Veselov (GNNV) equation by our generalized Jacobi elliptic function expansion method (see [2]) proposed recently. As a result, more new exact solutions are obtained. The character feature of our method is that, without much extra effort, we can get series of exact solutions by using a uniform way. Another advantage of our method is that it also applies to general higher-dimensional nonlinear partial differential equations.

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We consider the following GNNV equations (see [3–6]):

$$\begin{cases} u_t + au_{xxx} + bu_{yyy} + cu_x + du_y - 3a(uv)_x - 3b(uw)_y = 0, \\ u_x - v_y = 0, \\ u_y - w_x = 0, \end{cases} \tag{1.1}$$

where a, b, c and d are arbitrary constants. For

$$c = d = 0,$$

the GNNV equations (1.1) are degenerated to the usual two-dimensional NNV equations (see [7–8]), which is an isotropic Lax extension of the classical (1+1)-dimensional shallow water-wave KdV model. When

$$a = 1, \qquad b = c = d = 0,$$

we get the asymmetric NNV equation, which may be considered as a model for an incompressible fluid. Some types of exact solutions of the GNNV equations have been studied in recent years (see [9–13]).

2 Summary of the New Generalized Jacobi Elliptic Functions Expansion Method

Given a partial differential equation with three variables x, y and t

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, u_{xt}, u_{yt}, u_{xy}, \cdots) = 0, \tag{2.1}$$

we seek the following formal solutions of the given system by a new intermediate transformation:

$$u(\xi) = \sum_{i=0}^{n} A_i F^i(\xi) + \sum_{\substack{i,j=1 \\ i \le j \le n}}^{n} [B_i F^{j-i}(\xi) E^i(\xi) + C_i F^{j-i}(\xi) G^i(\xi) + D_i F^{j-i}(\xi) H^i(\xi)], \tag{2.2}$$

where A_0, A_i, B_i, C_i, D_i $(i = 1, 2, \cdots, n)$ are constants to be determined later, $\xi = \xi(x, y, t)$ is an arbitrary function with the variables x, y and t, the parameter n can be determined by balancing the highest order derivative terms with the nonlinear terms in (2.1), and $E(\xi)$, $F(\xi)$, $G(\xi)$, $H(\xi)$ are the arbitrary arrays of the four functions

$$e = e(\xi), \quad f = f(\xi), \quad g = g(\xi), \quad h = h(\xi)$$

respectively. The selection obeys the principle which makes the calculation more simple. We ansatz

$$\begin{cases} e = \dfrac{1}{p + q\mathrm{sn}\xi + r\mathrm{cn}\xi + l\mathrm{dn}\xi}, \\ f = \dfrac{\mathrm{sn}\xi}{p + q\mathrm{sn}\xi + r\mathrm{cn}\xi + l\mathrm{dn}\xi}, \\ g = \dfrac{\mathrm{cn}\xi}{p + q\mathrm{sn}\xi + r\mathrm{cn}\xi + l\mathrm{dn}\xi}, \\ h = \dfrac{\mathrm{dn}\xi}{p + q\mathrm{sn}\xi + r\mathrm{cn}\xi + l\mathrm{dn}\xi}, \end{cases} \tag{2.3}$$

where p, q, r, l are arbitrary constants which ensure denominator unequal to zero, so do the following situations. The four functions e, f, g, h satisfy the following restricted relations:

$$\begin{cases} e' = -qgh + rfh + lm^2fg, \\ f' = pgh + reh + leg, \\ g' = -pfh - qeh + l(m^2-1)ef, \\ h' = -m^2pfg - r(m^2-1)ef - qeg, \end{cases} \tag{2.4}$$

where "$'$" denotes $\frac{\mathrm{d}}{\mathrm{d}\xi}$, m $(0 \le m \le 1)$ is the modulus of the Jacobi elliptic function, and e, f, g, h satisfy one of the following relations at the same time.

Family 1: When $p = 0$, we can select $F(\xi) = f(\xi)$ or $F(\xi) = g(\xi)$, by using the relations

$$\begin{cases} lh = 1 - qf - rg, \\ e^2 = f^2 + g^2, \\ (l^2 - r^2)g^2 = 1 - 2(qf + rg - qrfg) + (l^2m^2 - l^2 + q^2)f^2. \end{cases} \tag{2.5a}$$

Family 2: When $q = 0$, we can select $F(\xi) = g(\xi)$ or $F(\xi) = h(\xi)$, by using the relations

$$\begin{cases} pe = 1 - rg - lh, \\ (m^2-1)f^2 = g^2 - h^2, \\ (l^2(m^2-1) + p^2)h^2 = (1-m^2)(1 - 2(lh + rg - rlgh) + r^2g^2) + m^2p^2g^2. \end{cases} \tag{2.5b}$$

Family 3: When $r = 0$, we can select $F(\xi) = h(\xi)$ or $F(\xi) = e(\xi)$, by using the relations

$$\begin{cases} qf = 1 - pe - lh, \\ m^2g^2 = h^2 + (m^2-1)e^2, \\ (q^2 - m^2p^2)e^2 = m^2 - 2m^2(lh + pe - pleh) + (l^2m^2 + q^2)h^2. \end{cases} \tag{2.5c}$$

Family 4: When $l = 0$, we can select $F(\xi) = e(\xi)$ or $F(\xi) = f(\xi)$, by using the relations

$$\begin{cases} rg = 1 - pe - qf, \\ h^2 = e^2 - m^2f^2, \\ (q^2 + r^2)f^2 = -1 + 2(pe + qf - pqef) + (r^2 - p^2)e^2. \end{cases} \tag{2.5d}$$

Substituting (2.4) along with (2.5a)–(2.5d) into (2.1), respectively, yields four families of polynomial equations for $E(\xi)$, $F(\xi)$, $G(\xi)$, $H(\xi)$.

Setting the coefficients of $F^i(\xi)E^{j_1}(\xi)^{j_2}G(\xi)^{j_3}H(\xi)^{j_4}$ $(i = 0, 1, 2, \cdots; j_1, j_2, j_3, j_4{=}0, 1;$ $j_1j_2j_3j_4 = 0)$ to be zero yields a set of over-determined differential equations in A_0, A_i, B_i, C_i, D_i $(i = 1, 2, \cdots, n)$ and $\xi(x, y, t)$. Solving the over-determined differential equations by Mathematica and Wu elimination, we obtain many exact solutions of (2.1) accroding to (2.2) and (2.3).

Obviously, if we choose the special values of p, q, r, l, m in (2.3), then we can get the results in [13–16], which has been discussed in [2].

3 Exact Solutions to the Generalized Nizhnik-Novikov-Veselov Equation

To seek the traveling wave solutions of (1.1), we make the gauge transformation

$$\xi = kx + \tau y - \omega t + \xi_0, \tag{3.1}$$

where k, τ, ω are constants to be determined later, and ξ_0 is an arbitrary constant.

Substituting (3.1) into (1.1) yields the ordinary differential equations (ODEs) of $u(\xi)$, $v(\xi)$, $w(\xi)$ and integrating these ODEs makes the equations (1.1) to become

$$\begin{cases} u'' - \dfrac{\omega - ck - d\tau + 3akC_2 + 3b\tau C_1}{ak^3 + b\tau^3}u - \dfrac{3}{k\tau}u^2 = 0, & (3.2a) \\ v = \dfrac{k}{\tau}u + C_2, & (3.2b) \\ w = \dfrac{\tau}{k}u + C_1, & (3.2c) \end{cases}$$

where C_1 and C_2 are integral constants. By balancing the highest-order of the linear term u'' and the nonlinear term u^2 in (3.2a), we obtain $n = 2$. Thus we assume that (3.2a) has the following solutions:

$$\begin{aligned} u = {} & c_0 + c_1 e + c_2 f + c_3 g + c_4 h + d_1 e^2 + d_2 f^2 + d_3 g^2 \\ & + d_4 h^2 + d_5 fg + d_6 fh + d_7 gh + d_8 ef + d_9 eg + d_{10} eh, \end{aligned} \tag{3.3}$$

where

$$u = u(x, y, t) = u(\xi),$$

and

$$e = e(\xi), \quad f = f(\xi), \quad g = g(\xi), \quad h = h(\xi)$$

satisfy (2.4) and (2.5a)–(2.5d). Substituting (2.4) and (2.5a)–(2.5d) along with (3.3) into (3.2a), respectively, and setting the coefficients of $F^i(\xi)E^{j_1}(\xi)^{j_2}G(\xi)^{j_3}H(\xi)^{j_4}$ $(i = 0, 1, 2, \cdots;$ $j_1, j_2, j_3, j_4 = 0, 1;\ j_1 j_2 j_3 j_4 = 0)$ to be zero yield an ODEs with respect to the unknowns c_i $(i = 0, \cdots, 4)$, d_j $(j = 1, \cdots, 10)$, ω, k, τ, p, q, r, l. After solving the ODEs by Mathematica and Wu elimination, we determine the following solutions:

Family 1: For $p = 0$, we have

Case 1.

$$\begin{cases} r = l = 1, \\ q = \pm 1, \\ c_2 = \mp k\tau(m^2 - 2), \\ c_4 = -2k\tau(m^2 - 1), \\ d_2 = -\dfrac{k\tau(m^2 - 2)^2}{2}, \\ d_4 = 2k\tau, \\ \omega = \Delta - (ak^3 + b\tau^3)(7 - 8m^2) \end{cases}$$

with

$$\Delta = ck - 3aC_2k - 3bC_1\tau + d\tau - \frac{6c_0(ak^3 + b\tau^3)}{k\tau},$$

where k, τ, ξ_0, c_0, C_1 are arbitrary constants. c_i $(i = 1, \cdots, 4)$ and d_j $(j = 1, \cdots, 10)$ not mentioned here are zero, so do the following situations.

Therefore, from (2.3), (3.1), (3.3) and Case 1, we obtain the following solutions to the

GNNV equations (1.1):

$$\begin{cases} u_1(\xi_1) = c_0 + \dfrac{\mp k\tau(m^2-2)\mathrm{sn}\xi_1 - 2k\tau(m^2-1)\mathrm{dn}\xi_1}{\pm\mathrm{sn}\xi_1 + \mathrm{cn}\xi_1 + \mathrm{dn}\xi_1} - \dfrac{\frac{k\tau(m^2-2)^2}{2}\mathrm{sn}^2\xi_1 - 2k\tau\mathrm{dn}^2\xi_1}{(\pm\mathrm{sn}\xi_1 + \mathrm{cn}\xi_1 + \mathrm{dn}\xi_1)^2}, \\ v_1(\xi_1) = \dfrac{k}{\tau}u_1(\xi_1) + C_2, \\ w_1(\xi_1) = \dfrac{\tau}{k}u_1(\xi_1) + C_1, \\ \xi_1 = kx + \tau y - (\Delta - (ak^3 + b\tau^3)(7 - 8m^2))t + \xi_0. \end{cases}$$

With the same process we derive the other three families of new exact solutions of (1.1), where

$$u_i = u_i(\xi_i), \quad v_i(\xi_i) = \frac{k}{\tau}u_i(\xi_i) + C_2, \quad w_i(\xi_i) = \frac{\tau}{k}u_i(\xi_i) + C_1, \qquad i = 2, 3, 4, \cdots.$$

Family 2: For $q = 0$, we have

Case 2.

$$\begin{cases} p = 0, \\ l = 1, \\ r = \mp\sqrt{m}, \\ d_7 = \mp 2k\tau\sqrt{m}(1-m)^2, \\ \omega = \Delta - (ak^3 + b\tau^3)(1 - 18m + m^2). \end{cases}$$

Case 3.

$$\begin{cases} r = \pm(\sqrt{1-m^2} - \varepsilon), \\ \varepsilon = \pm 1, \\ c_3 = \mp k\tau((m^2-1)\varepsilon + \sqrt{1-m^2}), \\ p = \sqrt{1-m^2}, \\ l = 1, \\ \omega = \Delta - (ak^3 + b\tau^3)(m^2 - 2 + 3\varepsilon\sqrt{1-m^2}). \end{cases}$$

Case 4.

$$\begin{cases} p = \sqrt{1-m^2}, \\ l = 1, \\ r = \mp m, \\ \varepsilon = \pm 1, \\ c_1 = -2k\tau m^2\sqrt{1-m^2}, \\ d_1 = -2k\tau m^2(1-m^2), \\ \omega = \Delta - (ak^3 + b\tau^3)(m^2 - 2 + 3\varepsilon\sqrt{1-m^2}). \end{cases}$$

We obtain the following solutions of (1.1):

$$\begin{cases} u_2 = c_0 \mp \dfrac{2k\tau\sqrt{m}(1-m)^2\mathrm{cn}\xi_2\mathrm{dn}\xi_2}{(\mp\sqrt{m}\mathrm{cn}\xi_2 + \mathrm{dn}\xi_2)^2}, \\ \xi_2 = kx + \tau y - (\Delta - (ak^3 + b\tau^3)(1 - 18m + m^2))t + \xi_0; \end{cases}$$

$$\begin{cases} u_3 = c_0 \mp \dfrac{k\tau((m^2-1)\varepsilon+\sqrt{1-m^2})\mathrm{cn}\xi_3}{\sqrt{1-m^2}\pm(\sqrt{1-m^2}-\varepsilon)\mathrm{cn}\xi_3+\mathrm{dn}\xi_3}, \\ \xi_3 = kx+\tau y-(\Delta-(ak^3+b\tau^3)(m^2-2+3\varepsilon\sqrt{1-m^2}))t+\xi_0; \end{cases}$$

$$\begin{cases} u_4 = c_0 - \dfrac{2k\tau m^2\sqrt{1-m^2}}{\sqrt{1-m^2}\mp m\mathrm{cn}\xi_4+\mathrm{dn}\xi_4} - \dfrac{2k\tau m^2(1-m^2)}{(\sqrt{1-m^2}\mp m\mathrm{cn}\xi_4+\mathrm{dn}\xi_4)^2}, \\ \xi_4 = kx+\tau y-(\Delta-(ak^3+b\tau^3)(m^2-2+3\varepsilon\sqrt{1-m^2}))t+\xi_0. \end{cases}$$

Family 3: For $r=0$, we have

Case 5.

$$\begin{cases} q=\pm\varepsilon(1+\varepsilon\sqrt{1-m^2}), \\ \varepsilon=\pm1, \\ c_2=\mp k\tau(\varepsilon(1-m^2)+\sqrt{1-m^2}), \\ p=l=1, \\ \omega=\Delta-(ak^3+b\tau^3)(m^2-2-3\varepsilon\sqrt{1-m^2}). \end{cases}$$

We obtain the following solutions of (1.1):

$$\begin{cases} u_5 = c_0 \mp \dfrac{k\tau(\varepsilon(1-m^2)+\sqrt{1-m^2})\mathrm{sn}\xi_5}{1\pm\varepsilon(1+\varepsilon\sqrt{1-m^2})\mathrm{sn}\xi_5+\mathrm{dn}\xi_5}, \\ \xi_5 = kx+\tau y-(\Delta-(ak^3+b\tau^3)(m^2-2-3\varepsilon\sqrt{1-m^2}))t+\xi_0. \end{cases}$$

Family 4: For $l=0$, we have

Case 6.

$$\begin{cases} d_2=2k\tau m^2p^2, \\ q=r=0, \\ p\neq 0, \\ \omega=\Delta-4(ak^3+b\tau^3)(1+m^2); \end{cases}$$

Case 7.

$$\begin{cases} d_5=-2kr^2\tau\sqrt[4]{1-m^2}(m^2-2\sqrt{1-m^2}-2), \\ q=\pm r\sqrt[4]{1-m^2}, \\ p=0, \\ \omega=\Delta-(m^2-18\sqrt{1-m^2}-2)(ak^3+b\tau^3); \end{cases}$$

Case 8.

$$\begin{cases} p^2=1, \\ q^2=1, \\ r=\pm1, \\ c_3=\mp 2k\tau, \\ d_3=2k\tau, \\ \omega=\Delta-(ak^3+b\tau^3)(4m^2-5). \end{cases}$$

We obtain the following solutions of (1.1):

$$\begin{cases} u_6=c_0+2k\tau m^2\mathrm{sn}^2\xi_6, \\ \xi_6=kx+\tau y-(\Delta-4(ak^3+b\tau^3)(1+m^2))t+\xi_0; \end{cases}$$

$$\begin{cases} u_7 = c_0 - \dfrac{2k\tau\sqrt[4]{1-m^2}(m^2-2\sqrt{1-m^2}-2)\mathrm{sn}\xi_7\mathrm{cn}\xi_7}{(\pm\sqrt[4]{1-m^2}\mathrm{sn}\xi_7+\mathrm{cn}\xi_7)^2}, \\ \xi_7 = kx+\tau y-(\Delta-(ak^3+b\tau^3)(m^2-18\sqrt{1-m^2}-2))t+\xi_0; \end{cases}$$

$$\begin{cases} u_8 = c_0 \mp \dfrac{2k\tau\mathrm{cn}\xi_8}{p+q\mathrm{sn}\xi_8\pm\mathrm{cn}\xi_8} + \dfrac{2k\tau\mathrm{cn}^2\xi_8}{(p+q\mathrm{sn}\xi_8\pm\mathrm{cn}\xi_8)^2}, \\ \xi_8 = kx+\tau y-(\Delta-(ak^3+b\tau^3)(4m^2-5))t+\xi_0. \end{cases}$$

Remark 3.1 Solutions u_1, u_6, u_7, u_8 degenerate to solitary solutions when the modulus $m \to 1$, and solutions u_1, u_3, u_5, u_7, u_8 degenerate to triangular function solutions when the modulus $m \to 0$. Here u_6 is just the solutions u_1, u_2, u_3 in [1]. The other seven types of explicit solutions to (1.1) we obtained are not shown in the previous literature to our knowledge.

4 Conclusion

In this paper, we propose an approach for finding the new exact solutions for the nonlinear evolution equations by constructing the four new types of Jacobi elliptic functions (2.3). By using this method and computerized symbolic computation, we have found abundant new exact solutions of (1.1). More importantly, our method is much simple and powerful for finding new solutions to various kinds of nonlinear evolution equations. We believe that this method should play an important role in finding the exact solutions in mathematical physics.

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