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Quasi-periodic Solutions of the General Nonlinear Beam Equations*

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Abstract: In this paper, one-dimensional (1D) nonlinear beam equations of the form

$$u_{tt} - u_{xx} + u_{xxxx} + mu = f(u)$$

with Dirichlet boundary conditions are considered, where the nonlinearity f is an analytic, odd function and $f(u) = O(u^3)$. It is proved that for all $m \in (0, M^*] \subset \mathbf{R}$ (M^* is a fixed large number), but a set of small Lebesgue measure, the above equations admit small-amplitude quasi-periodic solutions corresponding to finite dimensional invariant tori for an associated infinite dimensional dynamical system. The proof is based on an infinite dimensional KAM theory and a partial Birkhoff normal form technique.

Key words: beam equation, KAM theorem, quasi-periodic solution, partial Birkhoff normal form

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1 Introduction and Main Result

Consider the general nonlinear beam equations of the form

$$u_{tt} - u_{xx} + u_{xxxx} + mu = f(u) \tag{1.1}$$

on the finite x-interval $[0, \pi]$ with Dirichlet boundary conditions

$$u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0, \tag{1.2}$$

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where the parameter $m \in (0, M^*] \subset \mathbf{R}$, the nonlinearity f is assumed to be real analytic in u and of the form

$$f(u) = au^3 + \sum_{n \geq 5} f_n u^n, \qquad a \neq 0. \tag{1.3}$$

We study the equations of the form (1.1) as a Hamiltonian system on

$$\mathcal{P} = H_0^1([0, \pi]) \times L^2([0, \pi])$$

with coordinates u and $v = u_t$. Then the Hamiltonian is

$$H = \frac{1}{2}\langle v,\, v\rangle + \frac{1}{2}\langle Au,\, u\rangle + \int_0^\pi g(u)\mathrm{d}x, \tag{1.4}$$

where

$$A = \frac{\mathrm{d}^4}{\mathrm{d}x^4} - \frac{\mathrm{d}^2}{\mathrm{d}x^2} + m, \qquad g = \int_0 -f(s)\mathrm{d}s, \tag{1.5}$$

and $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the usual scalar product in L^2. Then (1.1) can be written in the form

$$u_t = \frac{\partial H}{\partial v} = v, \qquad v_t = -\frac{\partial H}{\partial u} = -Au - f(u). \tag{1.6}$$

Let

$$\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin jx, \quad \lambda_j = \sqrt{j^4 + j^2 + m}, \qquad j = 1, 2, \cdots$$

be the basic modes and frequencies of the linear equation

$$u_{tt} - u_{xx} + u_{xxxx} + mu = 0$$

with Dirichlet boundary conditions (1.2). Then every solution of the linear equation is the superposition of their harmonic oscillations and of the form

$$u(t, x) = \sum_{j \geq 1} q_j(t)\phi_j(x), \qquad q_j(t) = \sqrt{I_j}\cos(\lambda_j t + \theta_j),$$

with amplitudes $I_j \geq 0$ and initial phases θ_j. The motions are periodic or quasi-periodic, respectively, depending on whether one or finitely many eigenfunctions are excited. In particular, for every choice

$$J = \{j_1 < j_2 < \cdots < j_n\} \subset \mathbf{N}$$

of finitely many modes there exists an invariant $2n$-dimensional linear subspace E_J which is completely foliated into rotational tori with frequencies $\lambda_{j_1}, \cdots, \lambda_{j_n}$:

$$E_J = \{(u, v) = (q_1\phi_{j_1} + \cdots + q_n\phi_{j_n},\, p_1\phi_{j_1} + \cdots + p_n\phi_{j_n})\} = \bigcup_{I \in \overline{P^n}} \mathcal{T}_J(I),$$

where

$$P^n = \{I \in \mathbf{R}^n : I_j > 0,\ 1 \leq j \leq n\}$$

is the positive quadrant in $\mathbf{R}^n$ and

$$\mathcal{T}_J(I) = \{(u, v) : q_j^2 + \lambda_j^{-2} p_j^2 = I_j,\ 1 \leq j \leq n\},$$

by using the above representations of u and v. In addition, such a torus is linearly stable, and all solutions have zero Lyapunov exponents.

Upon restoration of the nonlinearity f, we show that there exist a Cantor set $\mathcal{O} \subset P^n$, a family of n-tori

$$\mathcal{T}_J[\mathcal{O}] = \bigcup_{I \in \mathcal{O}} \mathcal{T}_J(I) \subset E_J \qquad \text{over } \mathcal{O},$$

and a Whitney smooth embedding $\Phi : \mathcal{T}_J[\mathcal{O}] \to \mathcal{E}_J \subset \mathcal{P}$, such that the restriction of Φ to each $\mathcal{T}_J(I)$ in the family is an embedding of a rotational n-torus for the nonlinear equations. The image $\mathcal{E}$ of $\mathcal{T}_J[\mathcal{O}]$ is called the Cantor manifold of rotational d-tori in [1].

Theorem 1.1(Main Theorem) *Suppose that the nonlinearity f is real analytic and of the form* (1.3). *Then for each index set $J = \{j_1 < \cdots < j_n\}$, there exists, for all $m \in (0, M^*] \subset$ $\mathbf{R}$, but a set of small Lebesgue measure, a Cantor manifold $\mathcal{E}_J$ given by a Whitney smooth embedding $\Phi : \mathcal{T}_J[\mathcal{O}] \to \mathcal{E}_J$, which is a higher order perturbation of the inclusion map $\Phi_0 : E_J \to \mathcal{P}$ restricted to $\mathcal{T}_J[\mathcal{O}]$. Moreover, the Cantor manifold $\mathcal{E}_J$ is foliated by real analytic, linearly stable, n-dimensional invariant tori carrying quasi-periodic solutions.*

Their starting point is to take (1.1) as a perturbed sine-Gordon equation. This result is regained by Pöschel[1] by the infinite KAM theory and the normal form technique. Later, the existence of quasi-periodic solutions of the Hamiltonian partial differential equations have been studied in [2–8]. In this paper, by using the KAM approach originating from [9–11], we can obtain that (1.1) admits small-amplitude quasi-periodic solutions for all $m \in (0, M^*] \subset \mathbf{R}$ (M^* is a fixed large number), but a set of small Lebesgue measure.

2 An Infinite-dimensional KAM Theory

We consider a small perturbation of infinitely dimensional Hamiltonian in the parameter dependent form

$$H = N + P = \sum_{1\le j\le n} \omega_j(\xi) y_j + \frac{1}{2}\sum_{j\ge 1} \Omega_j(u_j^2 + v_j^2) + P \tag{2.1}$$

in n dimensional angle-action coordinates (x, y) and infinite-dimensional Cartesian coordinate (u, v) with symplectic structure

$$\sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}y_j + \sum_{j=n+1} \mathrm{d}u_j \wedge \mathrm{d}v_j,$$

on the phase space

$$\mathcal{P}^{a,s} = T^n \times \mathbf{R}^n \times l^{a,s} \times l^{a,s} \ni (x, y, u, v),$$

where T^n is the usual n torus with $1 \le n < \infty$. The tangent frequencies $\omega = (\omega_1, \cdots, \omega_n)$ and the normal frequencies $\Omega = (\Omega_1, \Omega_2, \cdots)$ depend on n parameters $\xi \in \mathcal{O} \subset \mathbf{R}^n$. $\mathcal{O}$ is a closed bounded set of positive Lebesgue measure.

As in [2], we set

$$z = \frac{u + \mathrm{i}v}{\sqrt{2}}, \qquad \bar{z} = \frac{u - \mathrm{i}v}{\sqrt{2}},$$

and

$$D(\hat{s}, r) = \{(x, y, z, \bar{z}) \in \mathcal{P}^{a,s} : |\mathrm{Im}x| < \hat{s},\ |y| < r^2,\ \|z\|_{a,s} + \|\bar{z}\|_{a,s} < r\},$$

where $|\cdot|$ denotes the sup-norm for the complex vector and $\|\cdot\|_{a,s}$ is the norm in the space $l^{a,s}$, which are to be defined later. We define the weighted phase norm

$$|W|_r = |W|_{\bar{s},r} = |x| + \frac{1}{r^2}|y| + \frac{1}{r}\|z\|_{a,\bar{s}} + \frac{1}{r}\|\bar{z}\|_{a,\bar{s}}$$

for $W = (x, y, z, \bar{z}) \in \mathcal{P}^{a,\bar{s}}$ with $\bar{s} = s+1$. Denote by $\mathcal{O}$ the parameter set $[1,2]^n$. For a map $F:\ D(s,r) \to \mathcal{P}^{a,\bar{s}}$, define its Lipschitz semi-norm $|F|_r^L$ as follows:

$$|F|_r^L = \sup_{\xi\neq\zeta} \frac{|\Delta_{\xi,\zeta} F|_r}{|\xi - \zeta|} = \sup_{\xi\neq\zeta} \frac{|F(\,\cdot\,,\xi) - F(\,\cdot\,,\zeta)|_r}{|\xi-\zeta|},$$

where the supremum is taken over $\mathcal{O}$.

For each $\xi \in \mathcal{O}$, there is an n-torus

$$\mathcal{T}_0^n = T^n \times \{0, 0, 0\}$$

with frequencies

$$\omega(\xi) = (\omega_1(\xi), \cdots, \omega_n(\xi))$$

of the linear integrable Hamiltonian N. In its norm space, described by u-v coordinates, the origin is an elliptic fixed point with characteristic frequencies $\Omega(\xi)$. The KAM theorem by Pöschel[11] shows the existence of this linear stable rotational tori under a small perturbation P. In order to obtain the result we have to give some assumptions:

(A1) Non-degeneracy. The real map $\xi \to \omega(\xi)$ is Lipeomorphism between $\mathcal{O}$ and its image. Moreover, for all integer vectors $(k, l) \in \mathbf{Z}^n \times \mathbf{Z}^\infty$ with $1 \leq |\,l\,| \leq 2$,

$$\text{meas}\{\xi : \langle k,\ \omega(\xi)\rangle + \langle l,\ \Omega(\xi)\rangle = 0\} = 0, \tag{2.2}$$

and $\langle l,\ \Omega(\xi)\rangle \neq 0$ on $\mathcal{O}$, where $|\,l\,| = \sum\limits_j |\,l\,|_j$ for integer vectors, and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the usual scalar product.

(A2) Spectral Asymptotics. There exist $d \geq 1$ and $\delta < d-1$ such that

$$\Omega_j(\xi) = j^d + \cdots + O(j^\delta), \tag{2.3}$$

where the dots stand for fixed lower order terms in j, allowing also negative exponents. More precisely, there exists a fixed parameter independent sequence $\bar{\Omega}$ with $\bar{\Omega}_j = j^d + \cdots$ such that $\tilde{\Omega}_j = \Omega_j - \bar{\Omega}_j$ gives rise to a Lipschitz map $\tilde{\Omega} : \mathcal{O} \to l_\infty^{-\delta}$, where l_∞^p is the space of all real sequences with the finite norm $|\omega|_p = \sup\limits_j |\omega_j| j^p$.

(A3) Regularity. The perturbation $P(x, y, u, v)$ is real analytic for a real argument $(x, y, u, v) \in D(r, s)$ for any given $r,\ s > 0$, and Lipschitz in the parameter $\xi \in \mathcal{O}$. For each $\xi \in \mathcal{O}$, its gradients with respect to u, v satisfy

$$\mathrm{i}P_z,\ -\mathrm{i}P_{\bar{z}} \in \mathcal{A}(l^{a,p},\ l^{a,\bar{p}}) \begin{cases} \bar{p} \geq p, & \text{for } d > 1; \\ \bar{p} > p, & \text{for } d = 1, \end{cases} \tag{2.4}$$

where $\mathcal{A}(l^{a,p},\ l^{a,\bar{p}})$ denotes the class of the maps from some neighborhoods of the origin in $l^{a,p}$ into $l^{a,\bar{p}}$, which is real analytic in the real and imaginary parts of the complex coordinates. To state Pöschel's theorem we assume that

$$|\omega|_{\mathcal{O}}^L + |\Omega|_{-\delta,\mathcal{O}}^L \leq M < \infty, \qquad |\omega^{-1}|_{\omega(\mathcal{O})}^L \leq L < \infty. \tag{2.5}$$

Moreover, we introduce the notations

$$\langle\, l\,\rangle_d = \max\Big\{1,\ \Big|\sum_j j^d l^j\Big|\Big\}, \qquad A_k = 1 + |k|^\tau,$$

where $\tau > n+1$ will be fixed later. Finally, let

$$z = \{(k,\ l) \neq 0,\ |\,l\,| \leq 2\} \subset \mathbf{Z}^n \times \mathbf{Z}^\infty.$$

We now state the basic KAM Theorem which is recited from [11].

Theorem 2.1 *Suppose that $H = N + P$ satisfies* (A1)–(A3), *and*

$$\epsilon = \sup_{D(s,r)\times\mathcal{O}} |X_P|_r + \sup_{D(s,r)\times\mathcal{O}} |X_P|_r^L \le \gamma\alpha, \tag{2.6}$$

where $0 < \alpha \le 1$ is another parameter, and γ depends on n, τ and s. Then there exist a Cantor set $\mathcal{O}^\alpha \subset \mathcal{O}$ with

$$\mathrm{meas}(\mathcal{O}_\alpha/\mathcal{O}) \to 0 \qquad \text{as } \alpha \to 0,$$

a Lipschitz continuous family of torus embedding $\Phi : T^n \times \mathcal{O}_\alpha \to \mathcal{P}^{a,\bar{p}}$, and a Lipschitz continuous map $\tilde{\omega} : \mathcal{O}_\alpha \to \mathbf{R}^n$, such that for each $\xi \in \mathcal{O}_\alpha$, the map Φ restricted $T^n \times \{\xi\}$ is a real analytic embedding of rotational torus with frequencies $\tilde{\omega}(\xi)$ for the Hamiltonian H at ξ.

Each embedding is analytic on $|\mathrm{Im}x| < \frac{s}{2}$, and

$$|\Phi - \Phi_0|_r + \frac{\alpha}{M}|\Phi - \Phi_0|_r^L \le \frac{c\epsilon}{\alpha}, \tag{2.7}$$

$$|\tilde{\omega} - \omega| + \frac{\alpha}{M}|\tilde{\omega} - \omega|^L \le c\epsilon, \tag{2.8}$$

uniformly on that domain and $\mathcal{O}_\alpha$, where $\Phi_0 : T^n \times \mathcal{O} \to \mathcal{T}_0^n$ is the trivial embedding, and $c \le \gamma^{-1}$ depends on the same parameters as γ.

Moreover, there exist Lipschitz maps ω_ν and Ω_ν on $\mathcal{O}$ for $\nu \ge 1$ satisfying

$$\omega_0 = \omega, \qquad \Omega_0 = \Omega$$

and

$$|\omega_\nu - \omega| + \frac{\alpha}{M}|\omega_\nu - \omega|^\alpha \le c\epsilon, \tag{2.9}$$

$$|\Omega_\nu - \Omega|_{-\delta} + \frac{\alpha}{M}|\Omega_\nu - \Omega|_{-\delta}^L \le c\epsilon, \tag{2.10}$$

such that

$$\mathrm{meas}(\mathcal{O}/\mathcal{O}_\alpha) \subset \bigcup \mathcal{R}_{k,l}^j(\alpha),$$

where

$$\mathcal{R}_{k,l}^j(\alpha) = \Big\{\xi \in \mathcal{O} : |\langle k,\ \omega_j(\xi)\rangle + \langle l,\ \Omega_j(\xi)\rangle| < \alpha\frac{\langle l\rangle_d}{A_k}\Big\}, \tag{2.11}$$

and the union is taken over all $j \ge 0$ and $(k,l) \in \mathbf{Z}^n \times \mathbf{Z}^\infty$ such that $|k| > K_0 2^{j-1}$ for $j \ge 1$ with a constant $K_0 \ge 1$ depending on n and τ.

Concerning the measure of the bad frequency set $\mathcal{O}/\mathcal{O}_\alpha$, we have the following theorem.

Theorem 2.2 ([11], Theorem D) *Suppose that in Theorem* 2.1 *the unperturbed frequencies are affine functions of the parameters. Then there is a constant $\tilde{c}$ such that*

$$\mathrm{meas}(\mathcal{O}/\mathcal{O})_\alpha \le \tilde{c}(\mathrm{diam}\mathcal{O})^{n-1}\alpha^\mu, \tag{2.12}$$

where

$$\mu = \begin{cases} 1, & d > 1; \\ \dfrac{\kappa}{\kappa + 1 - \iota/4}, & d = 1, \end{cases}$$

for all sufficiently small α, and ι is any number with $0 \le \iota < \min\{\bar{p} - p,\ 1\}$. In the case $d = 1$, κ is a positive constant such that

$$\frac{\Omega_i - \Omega_j}{i - j} = 1 + O(j^{-\kappa}) \tag{2.13}$$

uniformly on $\mathcal{O}$.

3 The Hamiltonian for the General Beam Equations

We recall that the Hamiltonian of our nonlinear beam equation is

$$H = \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle Au, u\rangle + \int_0^\pi g(u)\mathrm{d}x. \tag{3.1}$$

As in [1], we introduce coordinates $q = (q_1, q_2, \cdots)$, $p = (p_1, p_2, \cdots)$ through the relations

$$u = \sum_{j\ge 1} \frac{q_j}{\sqrt{\lambda_j}}\phi_j, \qquad v = \sum_{j\ge 1} \sqrt{\lambda_j} p_j \phi_j, \tag{3.2}$$

where

$$\phi_j = \sqrt{\frac{2}{\pi}}\sin jx, \qquad j = 1, 2, \cdots$$

are the normalized Dirichlet eigenfunctions of the operator A with eigenvalues

$$\lambda_j^2 = j^4 + j^2 + m,$$

and the coordinates q and p are taken from the Hilbert space $l^{a,s}$. We obtain the Hamiltonian

$$H = \Lambda + G = \frac{1}{2}\sum_{j\ge 1}\lambda_j(p_j^2 + q_j^2) + \int_0^\pi g\Big(\sum_{j\ge 1}\frac{q_j}{\sqrt{\lambda_j}}\phi_j\Big)\mathrm{d}x \tag{3.3}$$

with the lattice Hamiltonian equations

$$\dot q_j = \frac{\partial H}{\partial p_j} = \lambda_j p_j, \qquad \dot p_j = -\frac{\partial H}{\partial q_j} = -\lambda_j q_j - \frac{\partial G}{\partial q_j}. \tag{3.4}$$

Instead of discussing its validity, we just take the latter Hamiltonian as our new starting point and make the following simple observation.

Lemma 3.1 *Let $a \ge 0$, $s > 0$, I be an interval, and $t \in I \to (q(t), p(t))$ be a real analytic solution of* (3.4) *such that*

$$\sup_{t\in I}\sum_{j\ge 1}(|q_j(t)|^2 + |p_j(t)|^2) j^{2s}\mathrm{e}^{2ja} < \infty.$$

Then

$$u = \sum_{j\ge 1}\frac{q_j}{\sqrt{\lambda_j}}\phi_j$$

is an analytic solution of (1.1).

Next we consider the regularity of the vector field of G. Let l^2 be the Hilbert space of bi-infinite square summable sequences with complex coefficients. For $a \ge 0$ and $s > 0$, let the subspace $l^{a,s} \subset l^2$ consist of, by definition, all bi-infinite sequences with the finite norm

$$\|q\|_{a,s}^2 = |q_0|^2 + \sum_j |q_j|^2 |j|^{2s}\mathrm{e}^{2|j|a}.$$

Let

$$\mathcal{F} : l^{a,s} \to L^2, \qquad q \mapsto \mathcal{F}q = \frac{1}{\sqrt{2\pi}}\sum_j q_j \mathrm{e}^{\mathrm{i}jx}$$

be the inverse discrete Fourier transform, which defines an isometry between the two spaces, where L^2 is all square-integrable complex valued functions on $[-\pi, \pi]$. Through $\mathcal{F}$ we can define subspaces $W^{a,s} \subset L^2$ that are normed by setting

$$\|\mathcal{F}q\|_{a,s} = \|q\|_{a,s}.$$

Lemma 3.2 *For $a \geq 0$ and $s > \frac{1}{2}$, the space $l^{a,s}$ is a Hilbert algebra with respect to convolution of sequences and*

$$\|q * p\|_{a,s} \leq C\|q\|_{a,s}\|p\|_{a,s}$$

with a constant C depending on s. Consequently, $W^{a,s}$ is a Hilbert algebra with respect to multiplication of functions.

Lemma 3.3 *For $a \geq 0$ and $s > 0$, the vector field X_G is a map from some neighborhoods of the origin in $l^{a,s}$ into $l^{a,s+2}$, with*

$$\|X_G\|_{a,s+2} = O(\|q\|^3_{a,s}).$$

Proof. In a sufficient small neighborhood of the origin, we can consider the nonlinearity $f = u^3$. Due to

$$G = \frac{1}{4}\int_0^\pi |u(x)|^4 \mathrm{d}x = \frac{1}{4}\sum_{i,j,r,l} G_{ijrl} q_i q_j q_r q_l,$$

we have

$$\frac{\partial G}{\partial q_l} = \sum_{i,j,r} G_{ijrl} q_i q_j q_r.$$

Hence

$$\begin{aligned}
\|G_q\|^2_{a,s+2} &= \sum_{l\geq 1} |G_{q_l}|^2 l^{2(s+2)} \mathrm{e}^{2al} \\
&\leq c\sum_{l\geq 1} \sum_{\pm i\pm j\pm r=l} \Big(\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_r\lambda_l}}|q_i q_j q_r|\Big)^2 l^{2(s+2)}\mathrm{e}^{2al} \\
&\leq c\sum_{l\geq 1}\Big(\frac{1}{l}\Big)^2 \sum_{\pm i\pm j\pm r=l}\Big(\frac{|q_i q_j q_r|}{|i||j||r|}\Big)^2 l^{2(s+2)}\mathrm{e}^{2al} \\
&\leq c\sum_{l\geq 1}\frac{1}{l^k}(\tilde{q} * \tilde{q} * \tilde{q})^2 l^{2(s+2)}\mathrm{e}^{2al} \\
&\leq c\sum_{l\geq 1}(\tilde{q} * \tilde{q} * \tilde{q})^2 l^{2(s+1)}\mathrm{e}^{2al} \\
&\leq c\|\tilde{q} * \tilde{q} * \tilde{q}\|^2_{a,s+1} \\
&\leq c(\|\tilde{q}\|^2_{a,s+1})^3 \\
&\leq c(\|q\|^2_{a,s})^3
\end{aligned}$$

with

$$\tilde{q}_j = \frac{|q_j|}{j},$$

where the constant c may be different at each appearance. Hence

$$\|G_q\|_{a,s+2} \leq c(\|q\|_{a,s})^3.$$

The regularity of X_G follows from the regularity of its components.

For the nonlinearity u^3 we find

$$G = \frac{1}{4}\int_0^\pi |u(x)|^4 \mathrm{d}x = \frac{1}{4}\sum_{i,j,r,l} G_{ijrl} q_i q_j q_r q_l \tag{3.5}$$

with

$$G_{ijrl} = \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_r \lambda_l}} \int_0^\pi \phi_i \phi_j \phi_r \phi_l \mathrm{d}x.$$

It is not difficult to verify that $G_{ijrl} = 0$ unless $\pm i \pm j \pm r \pm l = 0$ for some combination of plus and minus signs. Particularly, we have

$$G_{iijj} = \frac{1}{2\pi} \cdot \frac{2 + \delta_i^j}{\lambda_i \lambda_j} \tag{3.6}$$

by the elementary calculation. In the following, we focus on the nonlinearity u^3, since a non-zero coefficient in front of u^3 and all terms of order five or more make no difference.

Next we transform the Hamiltonian (3.3) into some partial Birkhoff form of order four so that it may serve as a small perturbation of some nonlinear integrable system in a sufficiently small neighborhood of the origin. we introduce the complex coordinates

$$z_j = \frac{1}{\sqrt{2}}(q_j + \mathrm{i}p_j), \qquad \bar{z}_j = \frac{1}{\sqrt{2}}(q_j - \mathrm{i}p_j).$$

Then the Hamiltonian is given by

$$H = \Lambda + G = \sum_j \lambda_j |z_j|^2 + \int_0^\pi g\Big(\sum_j \frac{z_j + \bar{z}_j}{\sqrt{2\lambda_j}} \phi_j\Big) \mathrm{d}x \tag{3.7}$$

with symplectic structure $\mathrm{i}\sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$.

Lemma 3.4 *If $\{i, j, r, l\}$ are nonzero integers such that $i \pm j \pm r \pm l = 0$, but $(i, j, r, l) \neq (p, -p, q, -q)$, then for all $m \in (0, M^*] \subset \mathbf{R}$, but a set of small Lebesgue measure, we have $|\lambda_i \pm \lambda_j \pm \lambda_r \pm \lambda_l| \geq c$, where c is a constant depending on m.*

Proof. Without loss of generality, we may assume that $i \leq j \leq r \leq l$. The condition $i \pm j \pm r \pm l = 0$ then reduces to two possibilities, either $i - j - r + l = 0$ or $i + j + k - l = 0$. We have to study divisors of the form

$$\delta = \pm\lambda_i \pm \lambda_j \pm \lambda_r \pm \lambda_l$$

for all possible combinations of plus and minus signs. To this end, we distinguish them according to their number of minus signs. To shorten notation we let, for example,

$$\delta_{++-+} = \lambda_i + \lambda_j - \lambda_r + \lambda_l,$$

and similarly, for all other combinations of plus and minus signs.

Case 0. No minus sign. This is trivial.

Case 1. One minus sign. Obviously,

$$\delta_{-+++} > \delta_{+-++} > \delta_{++-+} > \sqrt{i^4 + i^2 + m} + \sqrt{j^4 + j^2 + m} > 1,$$

so it suffices to study $\delta = \delta_{+++-}$. We consider δ as a function of m and notice that

$$\delta^{(n)}(m) = (-1)^{n-1} \frac{(2n-1)!!}{2^n} (\lambda_i^{\frac{1}{2}-n} + \lambda_j^{\frac{1}{2}-n} + \lambda_r^{\frac{1}{2}-n} - \lambda_l^{\frac{1}{2}-n}) > d > 0.$$

According to Lemma 5.1 in the Appendix,

$$I_c = \{m : |f(m)| \leq c,\, m \in I,\, c > 0\} < \tilde{c}c,$$

so after excising a set of small measure, we obtain that $\delta(m) > c$.

Case 2. Two minus signs. Here we have $\delta_{-+-+}, \delta_{--++} > \delta_{+--+}$, and all other cases reduce to these ones by inverting the signs. So it suffices to study $\delta(m) = \delta_{+--+}$. Let

$$f(t) = \sqrt{t^4 + t^2 + m}.$$

It is easy to verify that for $t \geq 1$,

$$f'(t) = \frac{t(2t^2+1)}{\sqrt{t^4+t^2+m}} > 0$$

and

$$f''(t) = \frac{(t^3-1)^2+(t-1)^2+5t^6+7t^4+m(6t^2+1)-3/4}{(\sqrt{t^4+t^2+m})^3} > 0,$$

so f is increasing and convex for $t \geq 1$. Hence we have

$$\lambda_l - \lambda_r \geq \lambda_{l-p} - \lambda_{r-p}, \qquad 1 \leq p \leq r.$$

In the case $l = i + j + r$, we thus obtain

$$\lambda_l - \lambda_r \geq \lambda_{l-(r-i)} - \lambda_i = \lambda_{j+2i} - \lambda_i.$$

Hence

$$\delta \geq \lambda_{j+2i} - \lambda_j \geq 2if'(j) = \frac{2ij(2j^2+1)}{\sqrt{j^4+j^2+m}} \geq \frac{i}{\sqrt{1+m}},$$

by using the mean value theorem and the monotonicity of f'. With the other alternative, we have

$$i - j = r - l \neq 0.$$

Hence

$$\lambda_l - \lambda_r \geq \lambda_{j+1} - \lambda_{i+1},$$

and

$$\lambda_{j+1} - \lambda_j \geq \lambda_{i+2} - \lambda_{i+1}.$$

So we obtain that

$$\delta \geq \lambda_{j+1} - \lambda_{i+1} - \lambda_j + \lambda_i \geq \lambda_{i+2} - 2\lambda_{i+1} + \lambda_i \geq f''(i) \geq \frac{1}{(\sqrt{i^4+i^2+m})^3}.$$

Cases 3 and 4. Three and four minus signs. These ones can be reduced to Cases 1 and 0, respectively.

Proposition 3.1 *For any index set $J = \{j_1 < \cdots < j_n\}$, and all $m \in (0, M^*] \subset \mathbf{R}$, but a set of small Lebesgue measure, there exists a change of coordinates Γ in a neighborhood of the origin in $l^{a,s}$ such that the Hamiltonian*

$$H = \Lambda + G$$

with the nonlinearity (3.5) *is changed into*

$$H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K,$$

where $X_{\bar{G}},\ X_{\hat{G}},\ X_K : l^{a,s} \to l^{a,s+2}$, and

$$\bar{G} = \frac{1}{2} \sum_{\text{one of } \{i,j\} \in J} \bar{G}_{ij} z_i^2 z_j^2$$

with coefficients

$$\bar{G}_{ij} = \frac{6}{\pi} \cdot \frac{4 - \delta_{ij}}{\lambda_i \lambda_j},$$

and

$$|\hat{G}| = O(\|\hat{z}\|_{a,s}^4), \quad |K| = O(\|z\|_{a,s}^6), \qquad \hat{z} = \{z_j\}_{j \notin J}.$$

Moreover, the dependence of Γ on m is real analytic for almost all compact m-interval in $(0, +\infty)$.

Proof. It is convenient to introduce coordinates $(\cdots, w_{-2}, w_{-1}, w_1, w_2, \cdots)$ in $l^{a,s}$ by setting

$$z_j = w_j, \qquad \bar{z}_j = w_{-j}.$$

Let

$$\lambda_i' = (\mathrm{sgn}\ i)\lambda_{|i|}.$$

The Hamiltonian under consideration then reads as

$$H = \sum_n \lambda_n w_n w_{-n} + \sum_{i,j,r,l} G_{ijrl} w_i w_j w_r w_l. \tag{3.8}$$

Consider a Hamiltonian function

$$F = \sum_{i,j,r,l} F_{ijrl} w_i w_j w_r w_l$$

with coefficients

$$\mathrm{i}F_{ijrl} = \begin{cases} \dfrac{G_{ijrl}}{\lambda_i' + \lambda_j' + \lambda_r' + \lambda_l'}, & \{|i|, |j|, |r|, |l|\} \in \mathcal{L}_n \backslash \mathcal{N}_n; \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\mathcal{L}_n = \{(i, j, r, l) \in \mathbf{Z}^4 : \text{one of } \{|i|, |j|, |r|, |l|\} \in J = \{j_1, \cdots, j_n\}\},$$
$$\mathcal{N}_n = \{(i, j, r, l) \in \mathbf{Z}^4 : (i, j, r, l) = (p, -p, q, -q)\} \subset \mathcal{L}_n.$$

Let Γ be the time-1 map of the flow of the Hamiltonian vector field F. Expanding at $t = 0$ and by Taylor's formula, we obtain

$$\begin{aligned} H \circ \Gamma &= H + \{H, F\} + \int_0^1 (1 - t)\{\{H, F\}, F\} \circ X_F^t \mathrm{d}t \\ &= \Lambda + \{\Lambda, F\} + G + \{G, F\} + \int_0^1 (1 - t)\{\{H, F\}, F\} \circ X_F^t \mathrm{d}t, \end{aligned}$$

where

$$\{\Lambda, F\} = -\mathrm{i} \sum_{i,j,r,l} (\lambda_i' + \lambda_j' + \lambda_r' + \lambda_l')\ F_{ijrl} w_i w_j w_r w_l.$$

Hence

$$G + \{\Lambda, F\} = \sum_{(i,j,r,l)\in\mathcal{N}_n} + \sum_{(i,j,r,l)\notin\mathcal{L}_n} G_{ijrl} w_i w_j w_r w_l = \bar{G} + \hat{G}.$$

Returning to the notations z_j, $\bar{z}_j$, we have

$$\bar{G} = \frac{1}{2} \sum_{\text{one of } \{i,j\}\in J} \bar{G}_{ij} |z_i|^2 |z_j|^2$$

with

$$\bar{G}_{ij} = \begin{cases} 24G_{iijj} = \dfrac{24}{\pi} \cdot \dfrac{1}{\lambda_i \lambda_j}, & i \neq j; \\[2ex] 12G_{iijj} = \dfrac{18}{\pi} \cdot \dfrac{1}{\lambda_i \lambda_j}, & i = j \end{cases}$$

by (3.6), where $\hat{G}$ is independent of $\{z_j\}_{j\notin J}$. Hence, formally we have

$$H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K$$

as claimed.

To prove analyticity and regularity of the preceding transformation we first show $X_F : l^{a,s} \to l^{a,s+2}$. Indeed, by Lemma 3.4 and (3.5) with

$$\tilde{w}_j = \frac{1}{|j|}(|w_j| + |w_{-j}|),$$

we have

$$\begin{aligned}\Big|\frac{\partial F}{\partial w_l}\Big| &\le \sum_{\pm i \pm j \pm r = l} |F_{ijrl}||w_i w_j w_r| \\ &\le \frac{c}{|\,l\,|} \sum_{\pm i \pm j \pm r = l} \frac{|w_i w_j w_r|}{|ijr|} \\ &\le \frac{c}{|\,l\,|} \sum_{\pm i \pm j \pm r = l} \tilde{w}_i \tilde{w}_j \tilde{w}_r \\ &= \frac{c}{|\,l\,|} (\tilde{w} * \tilde{w} * \tilde{w})_l.\end{aligned}$$

By Lemma 3.2, we have

$$\|F_w\|_{a,s+2} \le c\|\tilde{w} * \tilde{w} * \tilde{w}\|_{a,s+1} \le \|w\|^3_{a,s}. \tag{3.9}$$

The analyticity of F_w follows from the analyticity of each component functions and its local boundedness. Hence in a sufficiently small neighborhood of the origin in $l^{a,s}$ the time-1-map Γ is well defined with the estimates

$$\|\Gamma - id\|_{a,s+2} = O(\|w\|^3_{a,s}), \qquad \|D\Gamma - I\|_{a,s+2,s} = O(\|w\|^2_{a,s}),$$

where the operator norm $\|\cdot\|_{a,\bar{r},s}$ is defined by

$$\|A\|_{a,\bar{r},s} = \sup_{w \ne 0} \frac{\|Aw\|_{a,\bar{r}}}{\|w\|_{a,s}}.$$

Obviously,

$$\|D\Gamma - I\|_{a,s+2,s+2} \le \|D\Gamma - I\|_{a,s+2,s},$$

while in a sufficiently small neighborhood of the origin, $D\Gamma$ defines an isomorphism of $l^{a,s+2}$. Since $X_H : l^{a,s} \to l^{a,s+2}$, we have

$$\Gamma^* X_H = D\Gamma^{-1} X_H \circ \Gamma = X_{H\circ\Gamma} : l^{a,s} \to l^{a,s+2}.$$

The same holds for the Lie bracket: the boundedness of $\|DX_F\|_{a,s+2,s}$ implies that

$$[X_F, X_H] = X_{\{H,F\}} : l^{a,s} \to l^{a,s+2}.$$

These two facts show that $X_K : l^{a,s} \to l^{a,s+2}$. The analogous claims for $X_{\bar{G}}$ and $X_{\hat{G}}$ are obvious.

4 Proof of the Main Theorem

We now prove Theorem 1.1 by applying Theorems 2.1 and 2.2. In Section 3 we see that there exists a real analytic, symplectic change of coordinates Γ , which takes H into

$$\tilde{H} = H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K$$

with the notation of the previous section:

$$\Lambda = \langle \alpha, I\rangle + \langle \beta, Z\rangle, \quad \bar{G} = \frac{1}{2}\langle AI, I\rangle + \langle BI, Z\rangle, \quad |\hat{G}| = O(\|\hat{z}\|^4_{a,s}), \quad |K| = O(\|z\|^6_{a,s}),$$

where

$$\alpha = (\lambda_j)_{j\in J}, \qquad \beta = (\beta_j)_{j\notin J}, \qquad A = (\bar{G}_{ij})_{i,j\in J},$$
$$B = (\bar{G}_{ij})_{j\in J, i\notin J}, \qquad I = (|z_j|^2)_{j\in J}, \qquad Z = (|z_j|^2)_{j\notin J}.$$

Moreover, the regularity of the nonlinear vector field is preserved. We introduce symplectic polar and real coordinates by setting

$$z_j = \begin{cases} \sqrt{\xi_j + y_j}\mathrm{e}^{\mathrm{i}x_j}, & j \in J; \\ \dfrac{1}{\sqrt{2}}(u_j + \mathrm{i}v_j), & j \notin J, \end{cases}$$

where the parameter $\xi \in \mathcal{O} = [0, 1]^n$. Then the Hamiltonian $H \circ \Gamma$ can be read as

$$\tilde{H} = N + P = \langle \omega(\xi), y\rangle + \langle \Omega(\xi), u^2 + v^2\rangle + \tilde{G} + \hat{G} + K$$

with the frequency

$$\omega(\xi) = \alpha + A\xi, \qquad \Omega(\xi) = \beta + B\xi$$

and perturbation

$$P = \tilde{G} + \hat{G} + K = O(|y|^2) + O(|y|\|u^2 + v^2\|) + \hat{G} + K.$$

Now, we only have to verify (A1)–(A3) of the infinite KAM Theory for the above Hamiltonian. Since

$$\lambda_j = \sqrt{j^4 + j^2 + m} = j^2 + \cdots + \frac{m}{2j} + O\Big(\frac{1}{j^3}\Big),$$

one has

$$\Omega_{j-n} = (\beta + BI)_{j-n} = \lambda_j + \frac{\langle \nu, I\rangle}{\lambda_j}$$

with

$$\nu = \frac{24}{\pi}(\lambda_1^{-1}, \cdots, \lambda_n^{-1}).$$

This gives the asymptotic expansion

$$\Omega_{j-n} = j^2 + \cdots + \frac{m}{2j} + \frac{\langle \nu, I\rangle}{j} + O\Big(\frac{1}{j^3}\Big) = j^2 + \cdots + \frac{m_I}{j} + O\Big(\frac{1}{j^3}\Big)$$

with

$$m_I = \frac{1}{2}m + \langle \nu, I\rangle.$$

So (A2) is satisfied with

$$d = 2, \qquad \delta = -1, \qquad \bar{\Omega} = \beta,$$

and

$$\tilde{\Omega}_j = \Omega_j - \bar{\Omega}_j,$$

which is a Lipschitz map from $\mathcal{O}$ to H^1_∞.

Moreover, since

$$A = (\bar{G}_{ij})_{j_1\le i,j\le j_n} = \frac{6}{\pi}\begin{pmatrix} \dfrac{3}{\lambda_{j_1}\lambda_{j_1}} & \dfrac{4}{\lambda_{j_1}\lambda_{j_2}} & \cdots & \dfrac{4}{\lambda_{j_1}\lambda_{j_n}} \\ \dfrac{4}{\lambda_{j_2}\lambda_{j_1}} & \dfrac{3}{\lambda_{j_2}\lambda_{j_2}} & \cdots & \dfrac{4}{\lambda_{j_2}\lambda_{j_n}} \\ \vdots & \vdots & & \vdots \\ \dfrac{4}{\lambda_{j_n}\lambda_{j_1}} & \dfrac{4}{\lambda_{j_n}\lambda_{j_2}} & \cdots & \dfrac{3}{\lambda_{j_n}\lambda_{j_n}} \end{pmatrix},$$

we have

$$\det\left(\frac{\pi}{6}A\right) = (4n-1)\prod_{1\le j\le n}\frac{1}{\lambda_j^2} \neq 0.$$

So the matrix A is non-degenerate and the map $\xi \to \omega(\xi)$ is a lipeomorphism of $\mathbf{R}^n$ onto itself. The measure condition is satisfied, since $\langle k,\, \omega(\xi)\rangle + \langle l,\, \Omega(\xi)\rangle$ is a non-trivial affine function of ξ which vanishes on a codimension 1 subspace. Finally, clearly $\langle l,\, \beta\rangle \neq 0$, for $1 \le |l| \le 2$, and $B\xi$ is small because of $|\xi|$ small and $B = (\bar{G}_{ij})_{j\in J,\, i\notin J}$. Then we have $\langle l,\, \Omega(\xi)\rangle \neq 0$ on $\mathcal{O}$. So (A1) is satisfied.

Since

$$\tilde{G} = O(|y|^2) + O(|y|\|u^2+v^2\|), \qquad |R| = \hat{G} + K = O(|\hat{z}|^4) + O(|z|^6),$$

we have

$$X_P = X_{\tilde{G}+R} : l^{a,s} \to l^{a,s+2},$$

and thus, (A3) holds true with

$$p = s, \qquad \bar{p} = s+2.$$

Moreover, since the frequency

$$\omega(\xi) = \alpha + A\xi$$

with the matrix A is invertible, we find that the condition (2.5) is satisfied.

Finally, as in [7], we can chose γ, α such that

$$c_1 r^2 \le \gamma\alpha \le c_2 r^{4/3},$$

where c_1, c_2 are constants. The Hamiltonian $\tilde{H}$ is well defined on the phase space domain

$$D(\hat{s}, r) = \{(x,y,u,v) : |\mathrm{Im} x| < \hat{s},\ |y| < r^2,\ \|u\|_{a,s} + \|v\|_{a,s} < r\}$$

and the parameter domain

$$\mathcal{O}_{\alpha,r} = U_\alpha \mathcal{O}_r, \qquad \mathcal{O}_r = \{\xi : 0 < |\xi| < r^{4/3}\},$$

where $U_\alpha\mathcal{O}_r$ is the subset of all points in $\mathcal{O}_r$ with boundary distance greater than α. On these domains, we have

$$|\tilde{G}| = O(r^4), \qquad |\hat{G}| = O(r^4), \qquad |K| = O(r^6).$$

Using Cauchy estimates, we obtain

$$\|X_{\tilde{G}}\|_{r/2, D(\hat{s}/2, r/2)} + \|X_{\hat{G}}\|_{r/2, D(\hat{s}/2, r/2)} + \|X_K\|_{r/2, D(\hat{s}/2, r/2)} = O(r^2).$$

Similarly, with respect to ξ on $\mathcal{O}_{\alpha,r}$, we have

$$\|X_{\tilde{G}}\|^L_{r/2} + \|X_{\hat{G}}\|^L_{r/2} + \|X_K\|^L_{r/2} = O(r^2/\alpha).$$

According to the equality above, we obtain

$$\|X_P\|_{r/2, D(\hat{s}/2, r/2)} + \alpha\|X_P\|^L_{r/2, D(\hat{s}/2, r/2)} = O(r^2) \le \alpha r.$$

Thus the equation (2.6) holds true.

Thus, all the conditions of Theorems 2.1 and 2.2 are satisfied, and we finish the proof of the main theorem.

5 Appendix

Lemma 5.1 *Suppose that $f(m)$ is an n-th differentiable function on the closure $\bar{I}$ of I , where $I \in \mathbf{R}$ is an interval. Let*

$$I_c = \{m : |f(m)| \le c,\ m \in I,\ c > 0\}.$$

If $|f^n(m)| > d > 0$ *on* I*, where* d *is a constant, then* $I_c < \tilde{c}c^{\frac{1}{n}}$ *with*

$$\tilde{c} = 2(2 + 3 + \cdots + n + d^{-1}).$$

The proof can be found in [12].

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