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The Third Initial-boundary Value Problem for a Class of Parabolic Monge-Ampère Equations*

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Abstract: For the more general parabolic Monge-Ampère equations defined by the operator $F(D^2u + \sigma(x))$, the existence and uniqueness of the admissible solution to the third initial-boundary value problem for the equation are established. A new structure condition which is used to get a priori estimate is established.

Key words: parabolic Monge-Ampère equation, admissible solution, the third initial-boundary value problem

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1 Introduction and Statement of the Main Results

In this paper, we discuss the third initial-boundary value problem for parabolic Monge-Ampère equations

$$-\frac{\partial u}{\partial t} + F(D^2u + \sigma(x)) = f(x,t) \qquad \text{in } Q_T, \tag{1.1}$$

$$\alpha(x)\frac{\partial u}{\partial \nu} + u = \phi(x,t) \qquad \text{on } \partial\Omega \times [0,T], \tag{1.2}$$

$$u = \psi(x,0) \qquad \text{on } \Omega \times \{t=0\}, \tag{1.3}$$

where Ω is a bounded uniformly convex domain in $\mathbf{R}^n$,

$$Q_T = \Omega \times (0,T],$$
$$\partial_p Q_T = \partial\Omega \times (0,T] \cup \bar{\Omega} \times \{t=0\},$$
$$F(D^2u + \sigma(x)) = \det{}^{\frac{1}{n}}(D^2u + \sigma(x)),$$

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and

$$D^2u = (D_{ij}u)$$

is the Hessian of u with respect to the variable x, ν is the unit exterior normal at $(x,t) \in \partial\Omega \times [0,T]$ to $\partial\Omega$, which has been extended on $\bar{Q}_T$ to be a properly smooth vector field independent of t, $\alpha(x) > 0$ is properly smooth for all $x \in \bar{\Omega}$, $\sigma(x) = (\sigma_{ij}(x))$ is an $n \times n$ symmetric matrix with smooth components, $f(x,t)$, $\phi(x,t)$, $\psi(x,t)$ are given properly smooth functions and satisfy some necessary compatibility conditions.

The first initial-boundary value problem for a class of elliptic Monge-Ampère equations

$$\begin{cases} \det(D^2u(x) + \sigma(x)) = f(x) & \text{in } \Omega, \\ u = \phi(x) & \text{on } \partial\Omega \end{cases}$$

was firstly discussed by Caffarelli *et al.*[1]

Ivochkina and Ladyzhenskaya[2] studied the following first initial-boundary value problem for parabolic Monge-Ampère equations

$$-\frac{\partial u}{\partial t} + \det^{\frac{1}{n}}(D^2u) = f(x,t) \qquad \text{in } Q_T, \tag{1.1*}$$

$$u = \phi(x,t) \qquad \text{on } \partial_p Q_T. \tag{1.2*}$$

They derived two structure conditions as follows:

$$\begin{cases} \min\limits_{\bar{Q}_T} f(x,t) + \min\limits_{\partial_p Q_T} \dfrac{\partial}{\partial t}\phi(x,0) - \dfrac{1}{2}ad^2 > 0, \text{ in which } d \text{ is} \\ \qquad \text{the radius of the minimal ball } B_d(x_0) \text{ containing } \Omega, \\ \qquad a = \max\left\{0,\ \max\limits_{\bar{Q}_T} \dfrac{\partial}{\partial t} f(x,t)\right\}, \end{cases} \tag{C$_2$)*}$$

$$\begin{cases} \min\limits_{\partial_p Q_T} \left(f(x,t) + \dfrac{\partial}{\partial t}\phi(x,t)\right) > 0, \\ D^2 f(x,t),\ D^2(\det^{\frac{1}{n}} D^2\phi(x,0)) \text{ are nonpositive definite.} \end{cases} \tag{C$'_2$)*}$$

By $(\mathrm{C}_2)^*$ or $(\mathrm{C}'_2)^*$, they obtained the existence and uniqueness of the solution. The third initial-boundary value problem for equation $(1.1)^*$ was studied by Zhou and Lian[3]. They also got two structure conditions similar to $(\mathrm{C}_2)^*$ and $(\mathrm{C}'_2)^*$ in [2].

Therefore, it is natural for us to consider the problem (1.1)–(1.3) as an extension of the result of [2–3].

Definition 1.1 *We say that $u(x,t)$ is an admissible function of* (1.1)–(1.3) *if $u(x,t) \in K$, where*

$$K = \{v \in C^{2,1}(\bar{Q}_T) \mid (D^2v(x,t) + \sigma(x)) > 0,\ (x,t) \in \bar{Q}_T\}.$$

Definition 1.2 *We say that $u(x,t)$ is an admissible solution of* (1.1)–(1.3) *if an admissible function $u(x,t)$ satisfies* (1.1)–(1.3).

Obviously, the equation (1.1) is of parabolic type for any admissible function $u(x,t)$.

For any admissible solution, the following condition is necessary:

$$(D^2\psi(x,0) + \sigma(x)) > 0, \qquad x \in \bar{\Omega}. \tag{C$_1$}$$

Following the idea of [2], we derive two structure conditions as follows:

$$\begin{cases} \min\limits_{\bar{Q}_T} f(x,t) + \min\left\{ \min\limits_{\partial_p Q_T} \dfrac{\partial}{\partial t}\phi(x,0),\ \min\limits_{\partial_p Q_T} \dfrac{\partial}{\partial t}\psi(x,0) \right\} \\ \qquad -\dfrac{1}{2}ad^2 - a\max\limits_{\bar{\Omega}} |\alpha(x)(x-x_0)\cdot\nu| > 0, \text{ in which } d \\ \qquad \text{is the radius of the minimal ball } B_d(x_0) \text{ containing } \Omega, \\ \qquad a = \max\left\{0,\ \max\limits_{\bar{Q}_T} \dfrac{\partial}{\partial t} f(x,t)\right\}, \end{cases} \tag{C$_2$}$$

$$\begin{cases} \min\limits_{\partial\Omega\times[0,T]} \left(f(x,t) + \dfrac{\partial}{\partial t}\phi(x,t) + \alpha\dfrac{\partial}{\partial \nu} f(x,t) \right) > 0, \\ (D^2 f(x,t)),\ (D^2 \det^{\frac{1}{n}}(D^2\psi(x,0)+\sigma(x))) \text{ are nonpositive definite.} \end{cases} \tag{C$_2'$}$$

Especially, we drive a new type of structure condition

$$\begin{cases} -\min\{\min\limits_{\partial_p Q_T} \phi_t(x,t),\ \min\limits_{\partial_p Q_T} \psi_t(x,0)\} + \min\limits_{\bar{Q}_T} f(x,t) - aT > 0, \\ \qquad \text{in which } a = \max\left\{0,\ \max\limits_{\bar{Q}_T} \dfrac{\partial}{\partial t} f(x,t)\right\}. \end{cases} \tag{C$_2''$}$$

Our main result is as follows.

Theorem 1.1 *Assume that Ω is a bounded uniformly convex domain, and for some $\beta \in (0,1)$, $\partial\Omega \in C^{4+\beta}$, $f \in C^{2+\beta,1+\beta/2}(\bar{Q}_T)$, $\phi \in C^{4+\beta,2+\beta/2}(\bar{Q}_T)$, $\psi \in C^{4+\beta,2+\beta/2}(\bar{Q}_T)$, where ϕ, ψ satisfy the compatibility conditions up to the second order. If* (C$_1$) *and one of the structure conditions* (C$_2$), (C$_2'$) *and* (C$_2''$) *hold, then the problem* (1.1)–(1.3) *has a unique admissible solution $u \in C^{4+\beta,2+\beta/2}(\bar{Q}_T)$.*

Remark 1.1 When $\alpha(x) \equiv 0 \equiv \sigma(x)$, the structure conditions (C$_2$) and (C$_2'$) are just (C$_2$)* and (C$_2'$)* in [2].

To simplify the formulations, we assume that $\phi(x,t)$ and $\psi(x,t)$ have been smoothly extended on $\bar{Q}_T$, and

$$-\frac{\partial\psi(x,0)}{\partial t} + \det^{\frac{1}{n}}(D^2\psi(x,0)+\sigma(x)) = f(x,0), \qquad x \in \bar{\Omega}. \tag{C}$$

Similarly to the argument in [4], we use Weyl's theorem (see [5]) to overcome the difficulty coming from $\sigma = \sigma(x)$ in (1.1). However, if $\sigma = \sigma(x,t)$ in (1.1), then the difficulty in the process of deriving the structure conditions is so hard that we are not accomplished.

Lemma 1.1[5] (Weyl's Theorem) *Assume that A and B are all real symmetric matrices of order n. Denote the eigenvalues of A, B, $A+B$ respectively by $\lambda_i(A)$, $\lambda_i(B)$, $\lambda_i(A+B)$, $i = 1,\cdots,n$. Suppose that these eigenvalues are arranged in increasing order, i.e., for $C = A$, B, $A+B$, we have*

$$\lambda_1(C) \le \lambda_2(C) \le \cdots \le \lambda_n(C).$$

Then for each $k = 1,2,\cdots,n$, it holds that

$$\lambda_k(A) + \lambda_1(B) \le \lambda_k(A+B) \le \lambda_k(A) + \lambda_n(B).$$

It is necessary to give some known results for further discussion. Denote

$$S_+^n = \{\eta = (\eta_{ij}) \mid \eta \text{ is a symmetric positive definite matrix of } n \text{ order}\},$$
$$F(\eta) = \det^{\frac{1}{n}} \eta,$$
$$F_{ij}(\eta) = \frac{\partial}{\partial \eta_{ij}} F(\eta).$$

Then we have several lemmas as follows.

Lemma 1.2[2] $\sum\limits_{m=1}^{n} F_{km}(\eta)\eta_{ml} = \dfrac{1}{n}\delta_{kl}F(\eta)$, *where* $\delta_{kl} = \begin{cases} 1, & k = l; \\ 0, & k \neq l. \end{cases}$

Lemma 1.3[2] *For any* $\eta \in S_+^n$, *we have*

$$\operatorname{tr}(F_{ij}(\eta)) = \sum_{i=1}^{n} F_{ii}(\eta) \geq 1.$$

Lemma 1.4[2] *If* $\eta, \zeta \in S_+^n$, *then*

$$\sum_{i,j=1}^{n} F_{ij}(\eta)\zeta_{ij} \geq F(\zeta).$$

The structure of this paper is stated as follows: In Section 2, we show the existence and uniqueness of the admissible solution in Theorem 1.1 by using the method of continuity and comparison theorem. In Section 3, the generalized approach for deriving the structure conditions is presented, and the positive lower bound of $F(D^2u + \sigma(x))$ is obtained. In Section 4, a series of a priori estimates are established.

2 The Method of Continuity and Comparison Theorem

In order to get the existence of admissible solution in Theorem 1.1 by the method of continuity, we consider a family of problems with one parameter $\tau \in [0, 1]$ as follows:

$$-\frac{\partial u^\tau}{\partial t} + \det^{\frac{1}{n}}(D^2 u^\tau + \sigma(x)) = f^\tau(x,t) \qquad \text{in } Q_T, \tag{2.1$_\tau$}$$

$$\alpha(x)\frac{\partial u^\tau}{\partial \nu} + u^\tau = \phi^\tau(x,t) \qquad \text{on } \partial\Omega \times [0, T], \tag{2.2$_\tau$}$$

$$u^\tau = \psi(x, 0) \qquad \text{on } \Omega \times \{t = 0\}, \tag{2.3$_\tau$}$$

where

$$\begin{cases} f^\tau(x,t) = \tau f(x,t) + (1-\tau)f^0(x,t), \\ f^0(x,t) = \det^{\frac{1}{n}}(D^2\psi(x,0) + \sigma(x)), \\ \phi^\tau(x,t) = \tau\phi(x,t) + (1-\tau)\Big[\alpha\dfrac{\partial}{\partial \nu}\psi(x,0) + \psi(x,0)\Big]. \end{cases} \tag{2.4}$$

Obviously, for $\tau = 1$ the problem $(2.1)_\tau$–$(2.3)_\tau$ is just (1.1)–(1.3).

Remark 2.1 If the assumptions of Theorem 1.1 hold, it is easy to find that the admissible solutions to problem $(2.1)_\tau$–$(2.3)_\tau$ satisfy the compatibility condition up to order two uniformly with respect to τ by direct calculations.

Set

$$V = \{v \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_T) \mid \det(D^2 v(x,t) + \sigma(x)) > 0,\ (x,t) \in \bar{Q}_T\},$$
$$S = \{\tau \in [0,1] \mid \text{the problem } (2.1)_\tau\text{–}(2.3)_\tau \text{ has a solution in } V\}.$$

In order to prove the existence of admissible solution in Theorem 1.1 by the method of continuity, we only need to prove that S is nonempty, and also S is a relatively both open and closed set in $[0,1]$.

Let $\tau = 0$. It is obvious that $u^0(x,t) \equiv \psi(x,0)$ is a solution of $(2.1)_\tau$–$(2.3)_\tau$ in V, i.e., S is nonempty.

In order to show that S is a relatively open set in $[0,1]$, we need the following lemma.

Lemma 2.1[6] *Let X_1, X_2 and Σ be Banach spaces, and G be a mapping from an open set U in $X_1 \times \Sigma$ into X_2. If there exists a $(w_0,\ \tau_0) \in U$ satisfying*

(1) $G(w_0,\ \tau_0) = 0$;

(2) *G is differentiable at $(w_0,\ \tau_0)$;*

(3) *the partial Fréchet derivative $G_w(w_0,\ \tau_0)$ is invertible,*

then there exists a neighborhood N of τ_0 in Σ such that the equation

$$G(w,\ \tau) = 0$$

is solvable for each $\tau \in N$ with the solution $w = w_\tau \in X_1$.

Actually, we can choose

$$\begin{aligned}
\Sigma &= [0,1],\\
X_1 &= \{w(x,t) \mid w \in C^{2+\beta,1+\beta/2}(\bar{Q}_T),\ w|_{t=0} = 0\},\\
B_1 &= C^{\beta,\beta/2}(\bar{Q}_T),\\
B_2 &= C^{1+\beta,1+\beta/2}(\partial\Omega \times [0,T]),\\
X_2 &= B_1 \times B_2.
\end{aligned}$$

Let

$$U = \{(w,\tau) \mid w(x,t) = v(x,t) - \psi(x,0),\ v(x,t) \in V,\ \tau \in X\}.$$

It is easy to prove that U is an open set in $X_1 \times \Sigma$.

Set

$$\begin{aligned}
G^1(w,\tau) &= -D_t(w+\psi) + \det{}^{\frac{1}{n}}(D^2(w+\psi)+\sigma) - \tau f - (1-\tau)\det{}^{\frac{1}{n}}(D^2\psi+\sigma),\\
G^2(w,\tau) &= \alpha(x)D_\nu(w+\psi) + (w+\psi) - \tau\psi - (1-\tau)(\alpha D_\nu\psi + \psi),\\
G(w,\tau) &= (G^1,G^2)(w,\tau) = (G^1(w,\tau), G^2(w,\tau)).
\end{aligned}$$

Then G^1 and G^2 are mappings from the open set U into B_1 and B_2 respectively, and G is a mapping from U into X_2.

Let $w_0 \in U$, $\tau_0 \in [0,1]$ be such that

$$G^1(w_0,\tau_0) = 0, \qquad G^2(w_0,\tau_0) = 0,$$

i.e.,

$$G(w_0,\tau_0) = 0.$$

It is easy to find that G is differentiable at $(w_0,\ \tau_0)$ if G^1 and G^2 are differentiable at $(w_0,\ \tau_0)$ with the Fréchet derivative

$$G^1_w(w_0,\ \tau_0) = -D_t + F_{ij}(D^2(w_0+\psi)+\sigma)D_{ij},$$
$$G^2_w(w_0,\ \tau_0) = \alpha D_\nu + I,$$

where

$$F_{ij}(r) = \frac{\partial F(r)}{\partial r_{ij}}, \qquad F(r) = \det^{\frac{1}{n}}(r).$$

Since G^1 is a linear parabolic operator and G^2 is an oblique derivative operator, by Theorem 5.3 of Chapter 7 in [7], we know that

$$G_w(w_0,\tau_0) = (G^1_w, G^2_w)(w_0,\tau_0) : X_1 \times \Sigma \to X_2$$

is invertible. Therefore, by Lemma 2.1, there exists a neighborhood $N \subset [0,1]$ of τ_0 such that $N \subset S$. This proves that S is a relatively open set in $[0,1]$.

In order to prove S is a relatively close set in $[0,1]$, we need to establish the following a priori estimate.

Theorem 2.1 *If the assumptions of Theorem* 1.1 *hold, then there exist two positive constants $\alpha \in (0,1)$ and C independent of τ such that*

$$\|u^\tau\|_{C^{2+\alpha,1+\alpha/2}(\bar{Q}_T)} \le C \tag{2.5}$$

holds for all solutions u^τ of the problem $(2.1)_\tau$–$(2.3)_\tau$.

Thus we can prove that S is a relatively close set in $[0,1]$ by Theorem 2.1 and Ascoli-Arzela lemma. It is easy to check that the data of $(2.1)_\tau$ and (1.1) have the same characters. So it suffices to establish the a priori estimate (2.5) for all admissible solutions u of (1.1).

To prove the uniqueness of the admissible solution in Theorem 1.1, we need the following comparison theorem.

Lemma 2.2[3] *Assume that $(a_{ij}(x,t))$ is a non-negative definite matrix, $u \in C^{2,1}(Q_T) \cap C^{1,0}(\bar{Q}_T)$, $\alpha(x) \ge 0$ for any x on $\partial\Omega$, and*

$$\mathcal{L}u \le 0 \qquad \text{in } Q_T, \tag{2.6}$$
$$\alpha(x)\frac{\partial}{\partial\nu}u + u \ge 0 \qquad \text{on } \partial\Omega \times [0,T], \tag{2.7}$$
$$u \ge 0 \qquad \text{on } \Omega \times \{t=0\}, \tag{2.8}$$

where

$$\mathcal{L} = -\frac{\partial}{\partial t} + \sum_{i,j=1}^n a_{ij}D_{ij},$$

and ν is the unit exterior normal at $(x,t) \in \partial\Omega \times [0,T]$. Then

$$u(x,t) \ge 0, \qquad (x,t) \in \bar{Q}_T.$$

Theorem 2.2 *If the assumptions of Theorem* 1.1 *hold, then there exists at most one admissible solution of the problem* (1.1)–(1.3).

Proof. If u_1 and u_2 are two admissible solutions of (1.1)–(1.3), then $\tilde{u} = u_1 - u_2$ satisfies

$$-\frac{\partial \tilde{u}}{\partial t} + F(D^2 u_1 + \sigma(x)) - F(D^2 u_2 + \sigma(x)) = 0, \qquad (x,t) \in \bar{Q}_T.$$

Moreover, we have

$$\begin{cases} -\dfrac{\partial \tilde{u}}{\partial t} + a_{ij}(x,t) D_{ij}\tilde{u} = 0 & \text{in } Q_T, \\ \alpha(x)\dfrac{\partial}{\partial \nu}\tilde{u} + \tilde{u} = 0 & \text{on } \partial\Omega \times [0,T], \\ \tilde{u} = 0 & \text{on } \Omega \times \{t = 0\}, \end{cases}$$

where

$$a_{ij}(x,t) = \int_0^1 F_{ij}(sD^2 u_1 + (1-s)D^2 u_2 + \sigma(x))\mathrm{d}s.$$

By Lemma 2.2, we have

$$\tilde{u} = 0,$$

i.e.,

$$u_1 = u_2.$$

The proof is completed.

3 A Positive Lower Bound Estimate of $F(D^2 u(x,t) + \sigma(x))$

For convenience of statements, we call a constant depending only on the data of the problem as a controllable constant.

The structure conditions (C_2), (C_2'), (C_2'') are used to estimate a positive lower bound of $F(D^2 u(x,t) + \sigma(x))$. From now on, we show a generalized approach for deriving these structure conditions.

Set

$$\mathcal{L}_u = -\frac{\partial}{\partial t} + F_{ij}(D^2 u + \sigma)D_{ij}, \tag{3.1}$$

where

$$F_{ij}(r) = \frac{\partial F(r)}{\partial r_{ij}}, \qquad F(r) = \det^{\frac{1}{n}}(r).$$

Choose the auxiliary function

$$v(x,t) = u_t(x,t) + at + c,$$

where u is an admissible solution of the problem (1.1)–(1.3), $a \geq 0$ and c are constants to be chosen.

For any $(x,t) \in \bar{Q}_T$,

$$\mathcal{L}_u v = -\frac{\partial u_t}{\partial t} + F_{ij}(D^2 u + \sigma)D_{ij}u_t - a = f_t - a \leq 0$$

provided

$$a = \max\{\max_{\bar{Q}_T} f_t(x,t),\ 0\}.$$

For any $(x,t) \in \partial\Omega \times [0,T]$,

$$\alpha\frac{\partial v}{\partial \nu} + v = \alpha\frac{\partial u_t}{\partial \nu} + u_t + at + c = \phi_t + at + c \geq 0$$

provided

$$c = -\min_{\partial_p Q_T} \phi_t(x,t).$$

For any $(x,t) \in \Omega \times \{t=0\}$,

$$v(x,0) = u_t(x,0) + c = \psi_t(x,0) + c \geq 0$$

provided

$$c = -\min_{\partial_p Q_T} \psi_t(x,0).$$

As the discussion above, setting

$$c = -\min\{\min_{\partial_p Q_T} \phi_t(x,t),\ \min_{\partial_p Q_T} \psi_t(x,0)\},$$
$$a = \max\{0,\ \max_{\bar{Q}_T} f_t(x,t)\},$$

we get $v \geq 0$ by Lemma 2.2. Therefore, in order to obtain

$$F(D^2u(x,t) + \sigma(x)) = u_t + f(x,t) \geq -at - c + f(x,t) \geq \gamma > 0,$$

where γ is a controllable constant, we need only to establish the structure condition

$$\begin{cases} -\min\{\min_{\partial_p Q_T} \phi_t(x,t),\ \min_{\partial_p Q_T} \psi_t(x,0)\} + \min_{\bar{Q}_T} f(x,t) - aT = \gamma > 0, \\ a = \max\{0,\ \max_{\bar{Q}_T} f_t(x,t)\}. \end{cases} \tag{C$_2''$}$$

Thus we have the following theorem.

Theorem 3.1 *If the assumptions of Theorem* 1.1 (*except* (C$_2$) *and* (C$_2'$)) *hold, then there exists a controllable positive constant* γ *such that* $F(D^2u(x,t) + \sigma(x))$ *has a positive lower bound, where* u *is an admissible solution of the problem* (1.1)–(1.3).

Remark 3.1 We can get the structure condition (C$_2$) by choosing the auxiliary function

$$v(x,t) = u_t(x,t) - \frac{1}{2}a|x - x_0|^2 + c,$$

where $a \geq 0$ and c are to be chosen, and x_0 is an arbitrary fixed point in Ω.

Remark 3.2 We can get the structure condition (C$_2'$) by choosing the auxiliary function

$$v(x,t) = u_t(x,t) + f(x,t) - c,$$

where c is to be chosen.

4 A Priori Estimate of $\|u\|_{C^{2+\alpha,1+\alpha/2}(\bar{Q}_T)}$

Theorem 4.1 *If the assumptions of Theorem* 1.1 *hold, then there exists a controllable constant* $C_0 > 0$ *such that*

$$\sup_{\bar{Q}_T} |u| \leq C_0$$

holds for all admissible solutions u *of the problem* (1.1)–(1.3).

Proof. Choose the function

$$w_{\pm}(x,t) = \pm K(t+1) + \psi(x,0),$$

where

$$K = \max\left\{ \max_{\bar{Q}_T} |f(x,t)| + \max_{\bar{\Omega}} F(D^2\psi(x,0)+\sigma), \right.$$
$$\left. \max_{\partial\Omega\times[0,T]} \phi(x,t) + \max_{\partial\Omega} |\psi(x,0)| + \max_{\partial\Omega} \left| \alpha(x)\frac{\partial\psi(x,0)}{\partial\nu} \right| \right\}.$$

Set

$$L = -D_t + a_{ij}(x,t)D_{ij},$$

where

$$a_{ij}(x,t) = \int_0^1 F_{ij}(sD^2w_+(x,t) + (1-s)D^2u(x,t) + \sigma(x))\mathrm{d}s$$

is a positive definite matrix, i.e., L is a linear parabolic operator.

Then, for any $(x,t) \in \bar{Q}_T$, we have

$$\begin{aligned} L(w_+ - u) =& -D_t w_+ + \det{}^{\frac{1}{n}}(D^2w_+ + \sigma) - f \\ =& -K - D_t\psi(x,0) + \det{}^{\frac{1}{n}}(D^2\psi(x,0)+\sigma) - f \\ \le& \ 0. \end{aligned}$$

For any $(x,t) \in \partial\Omega \times [0,T]$, we have

$$\alpha(x)\frac{\partial}{\partial\nu}(w_+ - u) + (w_+ - u) = \alpha(x)\frac{\partial\psi(x,0)}{\partial\nu} + K(t+1) + \psi(x,0) - \phi(x,t) \ge 0.$$

For any $(x,t) \in \bar{\Omega} \times \{t=0\}$, we have

$$w_+(x,0) - u(x,0) = K + \psi(x,0) - \psi(x,0) = K \ge 0.$$

Therefore, by Lemma 2.2, $w_+ \ge u$. Similarly, $w_- \le u$. Then there exists a controllable constant $C_0 > 0$ such that

$$\sup_{\bar{Q}_T} |u| \le C_0.$$

This completes the proof.

Theorem 4.2 *If the assumptions of Theorem* 1.1 *hold, then there exists a controllable constant* $C_1 > 0$ *such that*

$$\sup_{\bar{Q}_T} |Du| \le C_1$$

holds for all admissible solutions u *of the problem* (1.1)–(1.3).

Proof. Step 1. For any $\xi \in S^{n-1}$, set

$$w_{\pm}(x,t,\xi) = \pm D_\xi u + M|x|^2/2,$$

where

$$M = 1 + \Lambda(\sigma) + \sup_{Q_T} |Df|, \qquad \Lambda(\sigma) = \sup_{x\in\Omega,\xi\in S^{n-1}} \lambda_{\max}(\pm D_\xi\sigma(x)).$$

Differentiating (1.1) with respect to ξ, we have

$$-D_t(D_\xi u) + F_{ij}(D^2u+\sigma)D_{ij}(D_\xi u) = D_\xi f - F_{ij}(D^2u+\sigma)D_\xi\sigma_{ij}.$$

Set

$$\mathcal{L}_u = -D_t + F_{ij}(D^2u(x,t) + \sigma(x))D_{ij}.$$

By Lemma 1.4, we have

$$\mathcal{L}_u w_\pm = \pm D_\xi f + F_{ij}(D^2u + \sigma)(M\delta_{ij} \mp D_\xi \sigma_{ij}) \geq \pm D_\xi f + F(MI \mp D_\xi \sigma) \geq 0.$$

By the maximum principle of parabolic equations we have

$$w_\pm \leq \sup_{\partial_p Q_T} w_\pm.$$

There exists a controllable constant $\tilde{C}_1 > 0$ such that

$$|D_\xi u| \leq \sup_{\partial_p Q_T} |D_\xi u| + \tilde{C}_1.$$

Obviously, $D_\xi u$ is known on $\Omega \times \{t = 0\} \times S^{n-1}$, and hence we only need to get a priori estimate of $|D_\xi u|$ on $\partial\Omega \times [0, T] \times S^{n-1}$.

Step 2. For all $(x_0, t_0) \in \partial\Omega \times [0, T]$, by (1.2) and Theorem 4.1, there exists a controllable constant $\hat{C}_1 > 0$ such that

$$|D_\nu u(x_0, t_0)| \leq \hat{C}_1.$$

If we can prove that there exists a controllable constant $\check{C}_1 > 0$ such that

$$|D_\eta u(x_0, t_0)| \leq \check{C}_1, \tag{$*$}$$

where $\nu \cdot \eta = 0$, then for any $\xi \in S^{n-1}$, there exist θ, $\zeta \in [0, 1]$ with $\theta^2 + \zeta^2 = 1$ such that

$$\xi = \theta\nu + \zeta\eta,$$

and

$$D_\xi u = \theta D_\nu u + \zeta D_\eta u, \qquad |D_\xi u| \leq |D_\nu u| + |D_\eta u|.$$

Thus

$$|D_\xi u(x_0, t_0)| \leq \hat{C}_1 + \check{C}_1.$$

Step 3. We now prove that ($*$) holds. Actually, if $u(x, t_0)$ is a convex function of x, following the proof of Theorem 2.2 in [8], we have

$$|D_\eta u(x_0, t_0)| \leq C.$$

If it were not true, we could choose the function

$$\hat{u}(x, t_0) = u(x, t_0) + (\Lambda(\sigma) + 1)\frac{|x|^2}{2}.$$

Since

$$(D^2u(x,t) + \sigma(x)) > 0, \qquad (x, t) \in \bar{Q}_T,$$

by Wyel's theorem we have

$$(D^2u(x,t)) \geq (D^2u(x,t) + \sigma(x)) - \Lambda(\sigma)I, \qquad (D^2u(x,t) + (\Lambda(\sigma) + 1)I) \geq I.$$

Thus $\hat{u}(x, t_0)$ is a convex function of x, and we have

$$|D_\eta \hat{u}(x_0, t_0)| \leq C.$$

At last, noticing that Ω is bounded, we have

$$|D_\eta u(x_0, t_0)| = |D_\eta \hat{u}(x_0, t_0) - (\Lambda(\sigma) + 1)(x_0 \cdot \eta)| \leq C + (\Lambda(\sigma) + 1)|x_0| = \check{C}_1.$$

The proof is completed.

Theorem 4.3 *If the assumptions of Theorem* 1.1 *hold, then there exists a controllable constant* $C_2 > 0$ *such that*

$$\sup_{\bar{Q}_T} |D_t u| \le C_2$$

holds for all admissible solutions u *of the problem* (1.1)–(1.3).

Proof. A priori estimate

$$D_t u + f = \det^{\frac{1}{n}}(D^2 u + \sigma) \ge \gamma$$

in Theorem 3.1 yields

$$D_t u \ge \gamma - f \ge \tilde{C_2},$$

where $\tilde{C_2}$ is a controllable constant.

In order to get the upper bound of $D_t u$, denoting $v = D_t u$, differentiating (1.1)–(1.2) with respect to t, we have

$$\begin{cases} -D_t v + F_{ij}(D^2 u + \sigma) D_{ij} v = f_t & \text{in } Q_T, \\ \alpha D_\nu v + v = \phi_t & \text{on } \partial\Omega \times [0, T], \\ v(x, 0) = \det^{\frac{1}{n}}(D^2 u(x, 0) + \sigma) - f(x, 0) & \text{on } \Omega \times \{t = 0\}. \end{cases}$$

Choose

$$w(x) = \frac{M}{2}|x|^2 - K,$$

where

$$\begin{aligned} M &= \sup_{Q_T} |D_t f|, \\ K &= \max\Big\{ M \sup_{\partial\Omega\times[0,T]} \Big(|\alpha| \Big| D_\nu \Big(\frac{|x|^2}{2} \Big) \Big| + \frac{|x|^2}{2} + |\phi_t| \Big), \\ &\qquad \sup_{\Omega} \Big(\frac{M}{2}|x|^2 + |\det^{\frac{1}{n}}(D^2 u(x, 0) + \sigma)| + |f(x, 0)| \Big) \Big\}. \end{aligned}$$

Following the proof of Theorem 4.1 we get the upper bound of $D_t u$. Thus the proof of Theorem 4.3 is completed.

By Theorems 3.1 and 4.3, we have the following proposition.

Proposition 4.1 *If the assumptions of Theorem* 1.1 *hold, then there exists a controllable constant* $\Gamma > 0$ *such that*

$$\gamma \le \det^{\frac{1}{n}}(D^2 u(x, t) + \sigma(x)) = D_t u + f \le \Gamma, \qquad (x, t) \in \bar{Q}_T \tag{H$_1$}$$

for all admissible solutions u *of the problem* (1.1)–(1.3).

In order to get the estimate of $\sup_{\bar{Q}_T} |D^2 u|$, we need the following lemma.

Lemma 4.1[3] *Assume that* $h \in C^{2,1}(\bar{Q}_T)$, $h|_{\partial\Omega\times[0,T]} = 0$, *and*

$$|h(x, 0)| \le \hat{C}_3, \quad |Dh(x, 0)| \le \check{C}_3, \quad |-D_t h + F_{ij} D_{ij} h| \le \kappa \sum_{i=1}^n F_{ii}, \qquad x \in \Omega,$$

where $\hat{C}_3$, $\check{C}_3$, κ *are positive constants,* (F_{ij}) *is a positive definite matrix and* $\mathrm{tr}(F_{ij}) \geq 1$. *Then there exists a controllable constant* $C_3 > 0$ *such that*

$$\sup_{\partial\Omega\times[0,T]} |Dh| \leq C_3.$$

Theorem 4.4 *If the assumptions of Theorem* 1.1 *hold, and* $u \in C^{4,2}(\bar{Q}_T)$ *is an admissible solution of the problem* (1.1)–(1.3), *then there exists a controllable constant* $C_4 > 0$ *such that*

$$\sup_{\bar{Q}_T} |D^2u| \leq C_4. \tag{H$_2$}$$

Proof. Step 1. Denote $\bar{\Lambda}(\sigma) = \sup\limits_{x\in\Omega} \lambda_{\max}(\sigma(x))$. Notice that $(D^2u + \sigma(x)) > 0$, Wyel's theorem yields

$$\lambda(D^2u) \geq -\bar{\Lambda}(\sigma).$$

Step 2. In order to get the upper bound of $D_{\xi\xi}u$, $\xi \in S^{n-1}$, we choose the function $v(x,t,\xi) : \bar{Q}_T \times S^{n-1} \to R$,

$$v(x,t,\xi) = D_{\xi\xi}u - \tilde{v}(x,t,\xi) + K|x|^2,$$

where $\tilde{v}$ is given by

$$\tilde{v}(x,t,\xi) = 2(\xi\cdot\nu)\tilde{\xi}_i(D_i\Phi - D_i\nu_k D_k u),$$

in which, ν is a $C^3(\bar{\Omega})$ extension of the unit exterior normal vector on $\partial\Omega$,

$$\tilde{\xi}_i = (\xi - (\xi\cdot\nu)\nu)_i, \qquad \Phi = (\phi - u)/\alpha(x),$$

and K is a positive constant to be chosen.

Rewrite

$$\tilde{v}(x,t,\xi) = a_k D_k u + b,$$

where

$$a_k = 2(\xi\cdot\nu)(-\tilde{\xi}_k/\alpha(x) - \tilde{\xi}_i D_i\nu_k), \qquad b = 2(\xi\cdot\nu)\tilde{\xi}_i D_i\Phi.$$

Then

$$\begin{aligned}
&- D_t v + F_{ij}(D^2u+\sigma(x))D_{ij}v \\
= &- D_{\xi\xi t}u + a_k D_{kt}u + D_t b + D_t a_k D_k u + F_{ij}(D^2u+\sigma(x))D_{ij\xi\xi}u \\
&- a_k F_{ij}(D^2u+\sigma(x))D_{ijk}u - D_k u F_{ij}(D^2u+\sigma(x))D_{ij}a_k \\
&- F_{ij}(D^2u+\sigma(x))D_{ij}b + 2KF_{ii}(D^2u+\sigma(x)) \\
&- 2D_i a_k F_{ij}(D^2u+\sigma(x))[D_{jk}u+\sigma_{jk}] + 2F_{ij}(D^2u+\sigma(x))D_i a_k\sigma_{jk}.
\end{aligned} \tag{4.1}$$

Step 3. From now on, we prove that we can choose $K > 0$ large enough so that the right hand term of (4.1) is positive.

By Lemma 1.2, we have

$$F_{ik}(D^2u+\sigma(x))[D_{kj}u+\sigma_{kj}] = \frac{1}{n}\delta_{ij}\det{}^{\frac{1}{n}}(D^2u+\sigma(x)) > 0. \tag{4.2}$$

Differentiating (1.1) twice with respect to $\xi \in S^{n-1}$, we get

$$\begin{aligned}
&- D_{t\xi\xi}u + F_{ij}(D^2u+\sigma(x))D_{ij\xi\xi}u \\
= &- F_{ij,kl}(D^2u+\sigma(x))(D_{ij\xi}u + D_\xi\sigma_{ij}(x))(D_{kl\xi}u + D_\xi\sigma_{ij}(x)) \\
&+ D_{\xi\xi}f(x,t) - F_{ij}(D^2u+\sigma(x))D_{\xi\xi}\sigma_{ij}(x).
\end{aligned}$$

Since F is concave, it holds that

$$-D_{t\xi\xi}u + F_{ij}(D^2u + \sigma(x))D_{ij\xi\xi}u \geq D_{\xi\xi}f(x,t) - F_{ij}(D^2u + \sigma(x))D_{\xi\xi}\sigma_{ij}(x). \tag{4.3}$$

Differentiating (1.1) with respect to x_k, and multiplying by a_k, we get

$$a_kD_{kt}u - a_kF_{ij}(D^2u + \sigma(x))D_{ijk}u = -a_kD_kf(x,t) + a_kF_{ij}(D^2u + \sigma(x))D_k\sigma_{ij}(x). \tag{4.4}$$

Since $(F_{ij}(D^2u + \sigma(x)))$ is a positive definite matrix, we have

$$|F_{ij}(D^2u + \sigma(x))| \leq \frac{1}{2}[F_{ii}(D^2u + \sigma(x)) + F_{jj}(D^2u + \sigma(x))]. \tag{4.5}$$

At last, by Lemma 1.3, we have

$$\sum_{k=1}^{n} F_{kk}(D^2u + \sigma(x)) \geq 1.$$

Substituting (4.2)–(4.5) into (4.1), and noticing that the other terms in the right hand side of (4.1) are all controllable, for all $(x,t) \in \bar{Q}_T$ we have

$$-D_tv + F_{ij}(D^2u + \sigma(x))D_{ij}v \geq (2K - C)\sum_{k=1}^{n} F_{kk}(D^2u + \sigma(x)) - C > 0,$$

where C is a controllable constant, and K is a large enough controllable positive constant. By means of the maximum principle of parabolic equations, we know that the maximum of v is attained on ∂_pQ_T.

Step 4. Since v is known on $\Omega \times \{t = 0\} \times S^{n-1}$, we only need to estimate v on $\partial\Omega \times (0,T] \times S^{n-1}$. Assume that the maximum of v is attained at (x^0, t^0, ξ). Then we need only to estimate $v(x^0, t^0, \xi)$.

Now, we complete the estimate in the following four cases.

Case 1. Estimate of $|D_{\eta\nu}u(x^0, t^0)|$, where $\nu \cdot \eta = 0$.

Set

$$\delta_i = (\delta_{ij} - \nu_i\nu_j)D_j.$$

Applying δ_i to (1.2) ($D_\nu u = \Phi$), we have

$$\delta_i\nu_kD_ku + \nu_k\delta_iD_ku = \delta_i\Phi, \tag{4.6}$$

and multiplying (4.6) with η_i, we get

$$\eta_i\delta_i\nu_kD_ku + \eta_i\nu_k\delta_{ij}D_{jk}u - \nu_k\eta_i\nu_i\nu_jD_{jk}u = \eta_i\delta_i\Phi.$$

Since $\eta_i\nu_i = 0$, we have

$$\eta_i\nu_k\delta_{ij}D_{jk}u = D_{\eta\nu}u.$$

It holds that

$$\eta_i\delta_i\nu_kD_ku + D_{\eta\nu}u = \eta_i\delta_i\Phi,$$

which implies that there exists a controllable positive constant C such that

$$|D_{\eta\nu}u(x^0, t^0)| \leq C.$$

Case 2. Estimate of $|D_{\eta\eta}u(x^0, t^0)|$, where $\nu \cdot \eta = 0$.

Applying δ_i twice to (1.2) ($D_\nu u = \Phi$), we have

$$D_ku\delta_i\delta_j\nu_k + \delta_i\nu_k\delta_jD_ku + \delta_j\nu_k\delta_iD_ku + \nu_k\delta_i\delta_jD_ku = \delta_i\delta_j\Phi, \tag{4.7}$$

and multiplying (4.7) with $\xi_i\xi_j$, we get

$$\begin{aligned} D_{\xi\xi\nu}u &= \nu_k\xi_i\xi_j D_{ijk}u \\ &= \xi_i\xi_j\delta_i\delta_j\Phi - \xi_i\xi_j D_k u\delta_i\delta_j\nu_k \\ &\quad - \delta_i\nu_k\delta_j D_k u\xi_i\xi_j - \delta_j\nu_k\delta_i D_k u\xi_i\xi_j + (\delta_i\nu_j)\xi_i\xi_j D_{\nu\nu}u. \end{aligned} \tag{4.8}$$

Then

$$\begin{aligned} D_{\xi\xi\nu}u &= \nu_k\xi_i\xi_j D_{ijk}u \\ &= \xi_i\xi_j\delta_i\delta_j\Phi - \xi_i\xi_j D_k u\delta_i\delta_j\nu_k - \delta_i\nu_k\delta_j D_k u\xi_i\xi_j - \delta_j\nu_k\delta_i D_k u\xi_i\xi_j + (\delta_i\nu_j)\xi_i\xi_j D_{\nu\nu}u \\ &\le (-1/\alpha)D_{ij}u\xi_i\xi_j - 2(\delta_i\nu_k)D_{jk}u\xi_i\xi_j + (\delta_i\nu_j)\xi_i\xi_j D_{\nu\nu}u + C, \end{aligned}$$

i.e.,

$$D_{\xi\xi}u \le -\alpha(x)D_{\xi\xi\nu}u - 2\alpha(x)\delta_i\nu_k D_{jk}u\xi_i\xi_j + \alpha(x)\delta_i\nu_j\xi_i\xi_j D_{\nu\nu}u + C. \tag{4.9}$$

Since the maximum of v is attained at $(x^0, t^0, \xi) \in \partial\Omega \times [0, T] \times S^{n-1}$ and $a_k = 0$ (by $\xi \cdot \nu = 0$), we have

$$0 \le D_\nu v = D_{\xi\xi\nu}u - a_k D_{\nu k}u - (D_\nu a_k)D_k u - D_\nu b + 2K(x\cdot\nu),$$

which implies that

$$-\alpha(x)D_{\xi\xi\nu}u \le C\alpha, \tag{4.10}$$

where C is a controllable positive constant. Moreover, by the positive definite property of $(\delta_i\nu_k)$ and $(D_{jk}u + \sigma_{jk}(x))$, we have

$$\begin{aligned} -2\alpha(x)\delta_i\nu_k D_{jk}u\xi_i\xi_j &= -2\alpha(x)\delta_i\nu_k(D_{jk}u + \sigma_{jk})\xi_i\xi_j + 2\alpha(x)\delta_i\nu_k\sigma_{jk}\xi_i\xi_j \\ &\le 2\alpha(x)\delta_i\nu_k\sigma_{jk}\xi_i\xi_j. \end{aligned} \tag{4.11}$$

Substituting (4.10)–(4.11) into (4.9), we have

$$D_{\xi\xi}u(x^0, t^0) = D_{\eta\eta}u(x^0, t^0) \le C(1 + D_{\nu\nu}u(x^0, t^0)).$$

Case 3. Estimate of $|D_{\xi\xi}u(x^0, t^0)|$, where $\xi \ne \eta$ and $\xi \ne \nu$.

For all $\xi \in S^{n-1}$, rewrite

$$\xi = p\eta + q\nu,$$

where

$$p = (\xi\cdot\eta), \qquad q = (\xi\cdot\nu), \qquad p^2 + q^2 = 1.$$

At the point $(x^0, t^0, \xi) \in \partial\Omega \times [0, T] \times S^{n-1}$, it holds that

$$\tilde{v} = 2(\xi\cdot\nu)\tilde{\xi}_i(D_i\Phi - D_i\nu_k D_k u) = 2qpD_{\eta\nu}u,$$
$$D_{\xi\xi}u = p^2 D_{\eta\eta}u + q^2 D_{\nu\nu}u + \tilde{v}.$$

Then

$$D_{\xi\xi}u - \tilde{v} + K|x|^2 = p^2 D_{\eta\eta}u + q^2 D_{\nu\nu}u + (p^2 + q^2)K|x|^2,$$

i.e.,

$$v(x^0, t^0, \xi) = p^2 v(x^0, t^0, \eta) + q^2 v(x^0, t^0, \nu).$$

Since the maximum of v is attained at $(x^0, t^0, \xi) \in \partial\Omega \times [0, T] \times S^{n-1}$, we have

$$v(x^0, t^0, \nu) \ge v(x^0, t^0, \xi) \ge v(x^0, t^0, \eta),$$

which implies that

$$D_{\xi\xi}u(x^0,t^0) \le C(1 + D_{\nu\nu}u(x^0,t^0)).$$

Case 4. Estimate of $|D_{\nu\nu}u(x^0,t^0)|$.

Differentiating (1.1) with respect to x_k, we get

$$-D_{kt}u + F_{ij}(D^2u + \sigma(x))D_{ijk}u = D_k f(x,t) - F_{ij}(D^2u + \sigma(x))D_k\sigma_{ij}(x).$$

Set

$$h(x,t) = \nu_k D_k u - \phi.$$

Then $h \in C^{2,1}(\bar{Q}_T)$, $h|_{\partial\Omega\times[0,T]} = 0$, and $|h|$, $|Dh|$ are all bounded on $\bar{\Omega} \times \{t = 0\}$. By simple calculations, we have

$$\begin{aligned}
&- D_t h + F_{ij}(D^2u + \sigma(x))D_{ij}h \\
={}& \nu_k D_k f(x,t) - D_t u/\alpha(x) + 2F_{ij}(D^2u + \sigma)D_i u D_j(1/\alpha(x)) \\
&+ uF_{ij}(D^2u + \sigma)D_{ij}(1/\alpha(x)) + F_{ij}(D^2u + \sigma)D_{ij}u/\alpha(x) \\
&+ 2F_{ij}(D^2u + \sigma)D_i(\nu_k)D_{jk}u + F_{ij}(D^2u + \sigma)D_{ij}\nu_k D_k u \\
&- F_{ij}(D^2u + \sigma)D_{ij}(\phi/\alpha) + \phi_t/\alpha(x) - \nu_k F_{ij}(D^2u + \sigma)D_k\sigma_{ij}(x) \\
&+ 2D_i(\nu_k)F_{ij}(D^2u + \sigma)\sigma_{ik}(x) - 2D_i(\nu_k)F_{ij}(D^2u + \sigma)\sigma_{ik}(x) \\
&+ (1/\alpha(x))F_{ij}(D^2u + \sigma)\sigma_{ij}(x) - (1/\alpha(x))F_{ij}(D^2u + \sigma)\sigma_{ij}(x).
\end{aligned}$$

Following the discussion of (4.1), we can find a controllable constant $\kappa > 0$ such that

$$|-D_t h + F_{ij}(D^2u + \sigma(x))D_{ij}h| \le \kappa \sum_{k=i}^{n} F_{ii}(D^2u + \sigma(x)).$$

By Lemma 4.1, there exists a controllable constant $C_3 > 0$ such that

$$\sup_{\partial\Omega\times[0,T]} |Dh| \le C_3,$$

which implies that

$$\sup_{\partial\Omega\times[0,T]} |D_{\nu\nu}u| \le C.$$

The proof is completed.

By Theorems 4.1–4.4, we get the estimate of $\|u\|_{C^{2,1}(\bar{Q}_T)}$. Similarly to the Chapter 14 in [9], it is easy to get the estimate of $\|u\|_{C^{2+\beta,1+\beta/2}(\bar{Q}_T)}$ provided that (1.1) is a uniformly parabolic equation. The following lemma implies that (1.1) is a uniformly parabolic equation.

Lemma 4.2 *Assume that $u \in C^{4,2}(\bar{Q}_T)$ is an admissible solution of the problem* (1.1)–(1.3)*. If* (H$_1$) *and* (H$_2$) *hold, then there exist two positive constants λ and Λ such that*

$$\lambda|\xi|^2 \le F_{ij}(D^2u + \sigma(x))\xi_i\xi_j \le \Lambda|\xi|^2, \qquad \xi \in \mathbf{R}^n. \tag{4.12}$$

Proof. Let $0 < \lambda_1 \le \cdots \le \lambda_n$ be the eigenvalues of $(D^2u + \sigma(x))$. Noticing that $D_t u$ and f are bounded, we can find that

$$\det{}^{\frac{1}{n}}(D^2u + \sigma(x)) = f + D_t u$$

is bounded. Then

$$0 < \lambda_k \le \bar{C}, \qquad k = 1, \cdots, n,$$

where $\bar{C} > 1$ is controllable. Diagonalizing $(D^2u + \sigma(x))$, by (H_1) we have

$$0 < \gamma^n \le \lambda_k \cdot (\bar{C})^{n-1}, \qquad \bar{C} \ge \lambda_k \ge \frac{\gamma^n}{(\bar{C})^{n-1}} > 0.$$

Let $\mu_1 \ge \cdots \ge \mu_n > 0$ be the eigenvalues of $(D^2u + \sigma(x))^{-1}$. Then

$$\mu_k = (\lambda_k)^{-1}$$

and

$$0 < \frac{1}{\bar{C}} \le \mu_k \le \frac{(\bar{C})^{n-1}}{\gamma^n}, \qquad k = 1, \cdots, n.$$

Diagonalizing $F_{ij}(D^2u+\sigma(x))$, by Lemma 1.2, we see that the eigenvalues of $(F_{ij}(D^2u+\sigma(x))$ are

$$\frac{1}{n}\det{}^{\frac{1}{n}}(D^2u + \sigma(x))\mu_k, \qquad k = 1, \cdots, n.$$

Thus we can choose $\lambda = \dfrac{\gamma}{n\bar{c}}$, $\Lambda = \dfrac{(\bar{C})^{n-1}\Gamma}{n\gamma^n}$ such that (4.12) holds. Then the proof is completed.

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