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Two Generator Subsystems of Lie Triple System*

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Abstract: For a Lie triple system T over a field of characteristic zero, some sufficient conditions for T to be two-generated are proved. We also discuss to what extent the two-generated subsystems determine the structure of the system T. One of the main results is that T is solvable if and only if every two elements generates a solvable subsystem. In fact, we give an explicit two-generated law for the two-generated subsystems.

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1 Introduction

Lie triple system (L.t.s.) is a generalization of the concept of Lie algebra, since every Lie algebra L is also an L.t.s. with the multiplication

$$[x,\ y,\ z] := [x,\ [y,\ z]].$$

And also every L.t.s. is a subsystem of an L.t.s. coming from a Lie algebra, due to the concept of standard imbedding. Hence L.t.s. is strongly linked to Lie algebra, and many results of Lie algebra can be generalized in an appropriate form, to the L.t.s. (see [1–5]).

In this section we recall some definitions and facts about L.t.s. We start with the definition of an L.t.s. (see [1–2]).

Definition 1.1 *A Lie triple system* (*L.t.s.*) *is a vector space T over a field $\mathbb{K}$, which is closed with respect to a trilinear multiplication* $[\ \cdot,\ \cdot,\ \cdot\ ]$ *and satisfies*

$$[xxy] = 0, \tag{1.1}$$

$$[xyz] + [yzx] + [zxy] = 0, \tag{1.2}$$

$$[uv[xyz]] = [[uvx]yz] + [x[uvy]z] + [xy[uvz]], \tag{1.3}$$

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where $u, v, x, y, z \in T$.

A derivation of an L.t.s. T is a linear transformation D of T such that

$$D[xyz] = [(Dx)yz] + [x(Dy)z] + [xy(Dz)], \qquad x, y, z \in T.$$

For $x, y, z \in T$, define linear transformations $L(\ \cdot, \ \cdot\ )$, $R(\ \cdot, \ \cdot\ )$ on the vector space T by

$$L(x, y)(z) = R(y, z)(x) = [xyz].$$

We can see by the definition of T that all $L(x, y)$, $x, y \in T$, are derivations. A derivation D of the form

$$D = \sum L(x_i,\ y_i), \qquad x_i, y_i \in T$$

is called an inner derivation.

A subspace U of T is called an ideal of T if for all $x, y \in T$, $u \in U$ we have $[uxy] \in U$. For any submodule V in T, the centralizer $Z_T(V)$ of V in T is defined by

$$Z_T(V) = \{x \in T \mid [xvt] = [tvx] = 0,\ t \in T,\ v \in V\}.$$

In particular, $Z_T(T)$ is called the center of T and denoted simply by $Z(T)$. An L.t.s. T is called abelian if it satisfies

$$[xyz] = 0, \qquad x, y, z \in T.$$

For an ideal V of an L.t.s. T, define the lower central series (see [6]) for V by

$$V^0 := V$$

and

$$V^{n+1} := [V^n TV] + [VTV^n], \qquad n \geq 0.$$

Then V is called T-nilpotent if $V^m = 0$ for some m. It is called nilpotent if it is T-nilpotent. Put

$$V^{(0)} := V, \qquad V^{(n+1)} := [V^{(n)}TV^{(n)}].$$

Then V is called solvable if there is a positive integer k for which $V^{(k)} = 0$. T is solvable if it is a solvable ideal.

The Frattini subsystem $F(T)$ of an L.t.s. T is the intersection of all maximal subsystems of T. The Frattini ideal $\phi(T)$ is the largest ideal of T contained in $F(T)$.

For a subset U of an L.t.s. T, $\langle U \rangle$ denotes the subsystem of T generated by U.

2 Two Generated Subsystems of L.t.s.

In this section we give some results about two generated subsystems of L.t.s., which are generalisations of corresponding results of Lie algebras (see [7]).

Definition 2.1 *Let T be an L.t.s. For arbitrary $x, y \in T$, the subsystem $\langle x,\ y \rangle$ generated by x, y is called a two-generated subsystem. T is called two-generated if it is a two-generated subsystem of itself.*

Lemma 2.1 *An L.t.s. T is two-generated if and only if $T/\phi(T)$ is two-generated.*

Proof. Let π be the natural homomorphism from T to $T/\phi(T)$, that is,

$$\pi : T \to T/\phi(T), \qquad x \mapsto \bar{x}.$$

Evidently, $T = \langle x,\ y\rangle$ implies that

$$T/\phi(T) = \langle \bar{x},\ \bar{y}\rangle.$$

Now suppose that

$$T/\phi(T) = \langle \bar{x},\ \bar{y}\rangle.$$

Then

$$T = \langle x,\ y\rangle + \phi(T).$$

If $T \neq \langle x,\ y\rangle$, then there exists a maximal subsystem M of T such that $\langle x,\ y\rangle \subseteq M$. Since by definition $\phi(T)$ is contained in every maximal subsystem of T, we have

$$\langle x,\ y\rangle + \phi(T) \subseteq M \subsetneq T,$$

which is a contradiction and the proof is completed.

Definition 2.2 *Suppose that V is a subspace of an L.t.s. T. The normalizer $N_T(V)$ of V in T is defined by*

$$N_T(V) = \langle V,\ N(T,\ V)\rangle,$$

where $N(T, V)$ is given by

$$N(T, V) = \{x \in T \mid [xvt], [xtv] \in V,\ t \in T,\ v \in V\}.$$

Lemma 2.2 (1) *If V is a subspace of T, then $N(T,\ V)$ is a subsystem of T;*

(2) *If V is a subsystem of T, then $N_T(V) = V + N(T,\ V)$;*

(3) *If V is a subsystem of T, then V is an ideal of $N_T(V)$;*

(4) *If V is an ideal of T, then $N_T(V)$ is an ideal of T.*

Proof. (1) Notice that by definition $[xyz] \in V$ if two of x, y, z belong to V and $N(T,\ V)$ respectively. Now for all $x, y, z \in N(T, V)$, $v \in V$, $t \in T$, by the identity (3) we have

$$[[xyz]vt] = [xy[zvt]] - [z[xyv]t] - [zv[xyt]] \in V.$$

Similarly, $[[xyz]tv] \in V$.

(2) Clearly, we only need to prove that $V + N(T, V)$ is a subsystem. If all x, y, z belong to V or $N(T,\ V)$, then by assumption and (1), $[xyz] \in V + N(T, V)$. Otherwise, by the definition of $N(T,\ V)$, we also have $[xyz] \in V + N(T, V)$.

(3) Clearly, $V \subseteq N_T(V)$. Then (3) follows from the definition of $N(T,\ V)$ and (2).

(4) We only need to prove that $N(T, V)$ is an ideal of T. By using the identity (3), the proof is similar to that of (1).

Lemma 2.3 *If S is a subsystem of a nilpotent L.t.s. T and $S \neq T$, then $N_T(S) \neq S$.*

Proof. Since T is nilpotent, there is a p such that

$$T^0 = T \supseteq T^1 \supseteq \cdots \supseteq T^p = 0.$$

By assumption, $S \neq T$, so there is a k such that

$$T^k \subsetneq S, \qquad T^{k+1} \subsetneq S.$$

Now, for all $x \in T^k - S$, $s \in S$, $t \in T$, we have

$$[xst], [xts] \in T^{k+1} \subseteq S,$$

that is, $x \in N(T, S) \subseteq N_T(S)$. So $N_T(S) \neq S$. The proof is completed.

Theorem 2.1 *Let T be a simple L.t.s. over a field $\mathbb{K}$. Assume that every proper subsystem of T is nilpotent. Then the following hold*:

(1) *$M_1 \cap M_2 = 0$ for every pair of different maximal subsystem M_1 and M_2 of T*;

(2) *there is no $x(\neq 0) \in T$ such that $R(x, x)$ is a nilpotent linear transformation on T*;

(3) *T is two-generated.*

Proof. (1) Let M be a maximal subsystem of T. Assume that there exists a proper subsystem S of T such that $\dim M \cap S$ is maximal. By Lemma 2.3, the nilpotency of M and S implies that

$$N_M(M \cap S) \neq M \cap S, \qquad N_S(M \cap S) \neq M \cap S.$$

Let U be the subsystem of T generated by $N_M(M \cap S)$ and $N_S(M \cap S)$. Since $M \cap S$ is a non-zero ideal of U, it follows from the simplicity of T that $U \neq T$. Moreover, $S \cap M$ is properly contained in $U \cap M$, which contradicts our choice of S.

(2) Suppose that $0 \neq x \in T$ and $R(x, x)$ is nilpotent. Let H be a maximal subsystem of T containing x. Let $T = T_0 + T_1$ be the fitting decomposition of T relative to H. Since H is a maximal subsystem of T, we have $H = T_0$. Since $R(x, x)$ acts nilpotently on T_1, there exists $0 \neq y \in T_1$ such that $[yxx] = 0$. Hence $y \in Z_T(x)$ and $Z_T(x)$ is not contained in H. Take a maximal subsystem M of T containing $Z_T(x)$. We get $H \neq M$ and $H \cap M \supseteq x \neq 0$, which contradicts (1).

(3) Assume $T \neq \langle x,\ y\rangle$ for every $x, y \in T$. Let M be a maximal subsystem of T. Take $0 \neq x \in M$ and $y \in T - M$. There exists a maximal subsystem S of T containing $\langle x,\ y\rangle$. We have $S \cap M \neq 0$, which contradicts (1).

It is known that in group theory, a finite group is solvable if and only if every two elements generate a solvable subgroup. The same is true for Lie algebra g. In fact, Grunewald *et al.* (cf. [7]) have given an explicit two-variable law for a Lie algebra L. They defined a reasonable sequence $\{e_n(x, y)\}_{n=1}^{\infty}$ for $x, y \in L$ by

$$\begin{aligned} &e_1 = [x, y], \\ &e_1' = [e_1, x], \quad e_1'' = [e_1, y], \quad e_2 = [e_1',\ e_1''], \quad \cdots, \\ &e_n' = [e_n,\ x], \quad e_n'' = [e_n,\ y], \quad e_{n+1} = [e_n',\ e_n''], \quad \cdots, \end{aligned} \tag{2.1}$$

and proved the following:

Theorem 2.2 *Let L be a finite dimensional Lie algebra over an infinite field $\mathbb{K}$ of characteristic $p > 5$. Then L is solvable if and only if for some n the identity $e_n \equiv 0$ holds in L.*

We can generalize this to L.t.s.

Definition 2.3 *The reasonable sequences* $\{f_n(x,y)\}_{n=1}^{\infty}$ *and* $\{g_n(x,y)\}_{n=1}^{\infty}$ *of an L.t.s.* T *are defined by*

$$\begin{aligned} &f_1 = [xyx], \quad g_1 = [xyy], \\ &f_2 = [f_1 g_1 x], \quad g_2 = [f_1 g_1 y], \quad \cdots, \\ &f_{n+1} = [f_n g_n x], \quad g_{n+1} = [f_n g_n y], \quad \cdots \end{aligned} \tag{2.2}$$

Theorem 2.3 *Let* T *be a finite dimensional L.t.s. over a field of characteristic zero. Then* T *is solvable if and only if for some* n *the identity* $f_n \equiv 0$ *(or* $g_n \equiv 0$*) holds in* T*. So* T *is solvable if and only if every two elements generate a solvable subsystem.*

Proof. We only need to prove the first assertion. If T is solvable, then it satisfies an identity of the form $f_n \equiv 0$ (or $g_n \equiv 0$) since for any $x, y \in T$ the value $f_n \equiv 0$ (or $g_n \equiv 0$) belongs to the corresponding term of the derived series $T^{(n)}$. To prove sufficiency, we use the standard imbedding. If T is not solvable, then $L(T) = T \oplus L(T,T)$ is not solvable. So Theorem 2.2 tells us that for any $n \in \mathbb{N}$, $e_n(x,y) \equiv 0$ can not hold. Notice that by definition $f_n(x,y)$ (resp. $g_n(x,y)$) in T is just $e'_n(x,y)$ (resp. e''_n) in $L(T)$, so, both $f_n \equiv 0$ and $g_n \equiv 0$ cannot hold. The theorem is proved.

Corollary 2.1 *Let* T *be an L.t.s. over a field of characteristic zero and* $L(T)$ *be its standard imbedding. Then every two elements of* T *generates a solvable subsystem of* T *if and only if every two elements of* $L(T)$ *generates a solvable subalgebra of* $L(T)$*.*

Recall that the Engel sequence $\{v_i\}$ on a Lie algebra L is defined by

$$v_i(x,y) = [\cdots[[x, \underbrace{y], y]\cdots, y]}_{i} \qquad x, y \in L.$$

Similarly we give the following:

Definition 2.4 *Let* T *be a Lie triple system. The Engel sequence* $\{w_i\}$ *on* T *is given by*

$$w_i(x,y) = [\cdots[[x\, \underbrace{yy]yy]\cdots yy]}_{i}, \qquad x, y \in T.$$

For the Engel sequence $\{w_i\}$ of a Lie triple system T, we have the following theorem.

Theorem 2.4 *Let* T *be a Lie triple system, and* $\{w_i\}$ *the Engel sequence of* T*. If* $w_k \equiv w_l$ *holds in* T *for* $k \neq l$*, then* T *is solvable.*

Proof. Let $L(T) = T \dotplus L(T,T)$ be the standard imbedding of T and $\{v_i\}$ the Engel sequence of $L(T)$. Then it is easy to see that for all $x, y \in T$ we have

$$w_i(x,y) = v_{2i}(x,y).$$

Hence that $w_k \equiv w_l$ holds in T implies that $v_{2k} \equiv v_{2l}$ holds in $L(T)$, which means that $L(T)$ is solvable (see Proposition 3.3 of [7]). Therefore T is solvable. The proof is completed.

We conclude this paper by the following collection of some classes of L.t.s. which behave like solvable Lie triple systems. For short, we will say that the property $\mathcal{P}$ satisfies the condition

(∗) if for every L.t.s. T all of its two-generated proper subsystems possess the property $\mathcal{P}$, then either T itself possesses the property $\mathcal{P}$ or T is two-generated.
Then we have

Theorem 2.5 (1) *The class of abelian L.t.s. satisfies the condition* (∗);

(2) *The class of nilpotent L.t.s. satisfies the condition* (∗);

(3) *The class of solvable L.t.s. satisfies the condition* (∗).

Proof. (1) is clear.

(2) By assumption, for all $x, y \in T$, $\langle x,\ y\rangle$ is nilpotent. Thus

$$R^n(x,x)(y) = 0$$

for some n (cf. [6]). Hence

$$R^d(x,x)(y) = 0,$$

where d is the dimension of the standard imbedding $L(T)$ of T. Since the above assertion holds for any $y \in T$, $R(x,x)$ is nilpotent for all $x \in T$. It follows from Engel's theorem for Lie triple systems that T is nilpotent.

(3) It follows from Theorem 2.4.

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