Communications
in
Mathematical
Research
29(1)(2013), 14–22

An Extended Multiple Hardy-Hilbert's Integral Inequality*

Hong Yong
(*Department of Mathematics, Guangdong University of Business Studies, Guangzhou, 510320*)

Communicated by Ji You-qing

Abstract: In this paper, by introducing the norm $\|\boldsymbol{x}\|_\alpha$ ($\boldsymbol{x} \in \mathbf{R}^n$), a multiple Hardy-Hilbert's integral inequality with the best constant factor and it's equivalent form are given.

Key words: multiple Hardy-Hilbert's integral inequality, weight function, best constant factor, β-function, Γ-function

2000 MR subject classification: 26D15

Document code: A

Article ID: 1674-5647(2013)01-0014-09

1 Introduction

If $p > 1$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $f \geq 0$, $g \geq 0$, and

$$0 < \int_0^\infty f^p(x)\mathrm{d}x < +\infty, \qquad 0 < \int_0^\infty g^q(x)\mathrm{d}x < +\infty,$$

then the well known Hardy-Hilbert's integral inequality is (see [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin\left(\dfrac{\pi}{p}\right)} \Big(\int_0^\infty f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}} \Big(\int_0^\infty g^q(x)\mathrm{d}x\Big)^{\frac{1}{q}}. \tag{1.1}$$

Its equivalent form is

$$\int_0^\infty \Big(\int_0^\infty \frac{f(x)}{x+y}\mathrm{d}x\Big)^p \mathrm{d}y < \left[\frac{\pi}{\sin\left(\dfrac{\pi}{p}\right)}\right]^p \int_0^\infty f^p(x)\mathrm{d}x, \tag{1.2}$$

where the constant factors in (1.1) and (1.2) are optimal.

Hardy-Hilbert's inequality is important in harmonic analysis, real analysis and operator theory. In recent years, many valuable results (see [2–5]) have been obtained in generalization and improvement of Hardy-Hilbert's inequality. In 1999, Kuang[6] gave a generalization with a parameter λ of (1.1) as follows:

***Received date:** Jan. 26, 2010.

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\mathrm{d}x\mathrm{d}y$$
$$< B^{\frac{1}{p}}\Big(\frac{1}{p},\ \lambda-\frac{1}{q}\Big)B^{\frac{1}{q}}\Big(\frac{1}{q},\ \lambda-\frac{1}{q}\Big)\Big(\int_0^\infty x^{1-\lambda}f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_0^\infty x^{1-\lambda}g^q(x)\mathrm{d}x\Big)^{\frac{1}{q}}, \tag{1.3}$$

where $\max\Big\{\dfrac{1}{p},\ \dfrac{1}{q}\Big\} < \lambda \le 1$, $B(\ \cdot\ ,\ \cdot\ )$ is the β-function. Noticing that the constant factor in (1.3) is not optimal, and the range of values of λ is too narrow, in 2002, Yang[7] gave a new generalization of (1.3) as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\mathrm{d}x\mathrm{d}y$$
$$< B\Big(\frac{p+\lambda-2}{p},\ \frac{q+\lambda-2}{q}\Big)\Big(\int_0^\infty x^{1-\lambda}f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_0^\infty x^{1-\lambda}g^q(x)\mathrm{d}x\Big)^{\frac{1}{q}}, \tag{1.4}$$

where $\lambda > 2-\min\{p,\ q\}$, and the constant factor $B\Big(\dfrac{p+\lambda-2}{p},\ \dfrac{q+\lambda-2}{q}\Big)$ is optimal.

At present, for multiple Hardy-Hilbert's integral inequality, many new results have be obtained (see [8–10]). In this paper, by the method of weight function, a higher-dimensional generalization of (1.4) is obtained, and its equivalent form is researched. For the sake of convenience, we introduce the following symbols:

$$\mathbf{R}_+^n = \{\boldsymbol{x} = (x_1,\ \cdots,\ x_n):\ x_1,\ \cdots,\ x_n > 0\},$$
$$\|\boldsymbol{x}\|_\alpha = (x_1^\alpha + \cdots + x_n^\alpha)^{\frac{1}{\alpha}}, \qquad \alpha > 0.$$

Lemma 1.1[11] *If $p_i > 0$, $a_i > 0$, $\alpha_i > 0$, $i = 1, 2, \cdots, n$, and $\Psi(u)$ is a measurable function, then*

$$\int\cdots\int_{x_1,\cdots,x_n>0;(\frac{x_1}{a_1})^{\alpha_1}+\cdots+(\frac{x_n}{a_n})^{\alpha_n}\le 1}\Psi\Big(\Big(\frac{x_1}{a_1}\Big)^{\alpha_1}+\cdots+\Big(\frac{x_n}{a_n}\Big)^{\alpha_n}\Big)x_1^{p_1-1}\cdots x_n^{p_n-1}\mathrm{d}x_1\cdots\mathrm{d}x_n$$
$$= \frac{a_1^{p_1}\cdots a_n^{p_n}\Gamma\Big(\dfrac{p_1}{\alpha_1}\Big)\cdots\Gamma\Big(\dfrac{p_n}{\alpha_n}\Big)}{\alpha_1\cdots\alpha_n\Gamma\Big(\dfrac{p_1}{\alpha_1}+\cdots+\dfrac{p_n}{\alpha_n}\Big)}\int_0^1\Psi(u)u^{\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}-1}\mathrm{d}u. \tag{1.5}$$

Lemma 1.2 *If $p > 1$, $\dfrac{1}{p}+\dfrac{1}{q} = 1$, $n \in \mathbf{Z}_+$, $\alpha > 0$, $\lambda > \max\{n(2-p),\ n(2-q)\}$, and set the weight function*

$$\omega_{\alpha,\lambda}(\boldsymbol{x},\ q) = \int_{\mathbf{R}_+^n}\frac{1}{(\|\boldsymbol{x}\|_\alpha + \|\boldsymbol{x}\|_\alpha)^\lambda}\Big(\frac{\|\boldsymbol{x}\|_\alpha}{\|\boldsymbol{y}\|_\alpha}\Big)^{\frac{2n-\lambda}{q}}\mathrm{d}\boldsymbol{y},$$

then

$$\omega_{\alpha,\lambda}(\boldsymbol{x},\ q) = \|\boldsymbol{x}\|_\alpha^{n-\lambda}\frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{1}{\alpha}\Big)}B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big). \tag{1.6}$$

Proof. By (1.5) one has

$$\omega_{\alpha,\lambda}(\boldsymbol{x},\ q) = \|\boldsymbol{x}\|_\alpha^{\frac{2n-\lambda}{q}}\int_{\mathbf{R}_+^n}\frac{1}{(\|\boldsymbol{x}\|_\alpha + \|\boldsymbol{y}\|_\alpha)^\lambda}\|\boldsymbol{y}\|_\alpha^{\frac{\lambda-2n}{q}}\mathrm{d}\boldsymbol{y}$$

$$= \|\boldsymbol{x}\|_\alpha^{\frac{2n-\lambda}{q}} \lim_{r\to+\infty} \int\cdots\int_{y_1,\cdots,y_n>0; y_1^\alpha+\cdots+y_n^\alpha<r^\alpha} \frac{\left[r\left(\left(\frac{y_1}{r}\right)^\alpha+\cdots+\left(\frac{y_n}{r}\right)^\alpha\right)^{\frac{1}{\alpha}}\right]^{\frac{\lambda-2n}{q}}}{\left[\|\boldsymbol{x}\|_\alpha + r\left(\left(\frac{y_1}{r}\right)^\alpha+\cdots+\left(\frac{y_n}{r}\right)^\alpha\right)^{\frac{1}{\alpha}}\right]^\lambda} \cdot y_1^{1-1}\cdots y_n^{1-1}\mathrm{d}y_1\cdots\mathrm{d}y_n$$

$$= \|\boldsymbol{x}\|_\alpha^{\frac{2n-\lambda}{q}} \lim_{r\to+\infty} \frac{r^n\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \int_0^1 \frac{(ru^{\frac{1}{\alpha}})^{\frac{\lambda-2n}{q}}}{(\|\boldsymbol{x}\|_\alpha + ru^{\frac{1}{\alpha}})^\lambda} u^{\frac{n}{\alpha}-1}\mathrm{d}u$$

$$= \|\boldsymbol{x}\|_\alpha^{\frac{2n-\lambda}{q}} \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \lim_{r\to+\infty} \int_0^r \frac{1}{(\|\boldsymbol{x}\|_\alpha + u)^\lambda} u^{\frac{\lambda-2n}{q}+n-1}\mathrm{d}u$$

$$= \|\boldsymbol{x}\|_\alpha^{\frac{2n-\lambda}{q}} \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \int_0^\infty \frac{1}{(\|\boldsymbol{x}\|_\alpha + u)^\lambda} u^{\frac{\lambda-2n}{q}+n-1}\mathrm{d}u$$

$$= \|\boldsymbol{x}\|_\alpha^{n-\lambda} \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \int_0^\infty \frac{1}{(1+u)^\lambda} u^{\frac{n(q-2)+\lambda}{q}-1}\mathrm{d}u$$

$$= \|\boldsymbol{x}\|_\alpha^{n-\lambda} \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} B\left(\frac{n(q-2)+\lambda}{q},\ \lambda - \frac{n(q-2)+\lambda}{q}\right)$$

$$= \|\boldsymbol{x}\|_\alpha^{n-\lambda} \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} B\left(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\right),$$

and hence (1.6) is valid.

Lemma 1.3 *If* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ n \in \mathbf{Z}_+,\ \alpha > 0,\ \lambda > \max\{n(2-p),\ n(2-q)\}$, *and* $0 < \varepsilon < n(q-2) + \lambda$, *then*

$$\tilde{\omega}_{\alpha,\lambda}(\boldsymbol{x},\ q) := \int_{\mathbf{R}_+^n} \frac{1}{(\|\boldsymbol{x}\|_\alpha + \|\boldsymbol{y}\|_\alpha)^\lambda} \|\boldsymbol{y}\|_\alpha^{\frac{\lambda-2n-\varepsilon}{q}}\mathrm{d}\boldsymbol{y}$$
$$= \|\boldsymbol{x}\|_\alpha^{n-\lambda+\frac{\lambda-2n-\varepsilon}{q}} \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} B\left(\frac{n(q-2)+\lambda}{q} - \frac{\varepsilon}{q},\ \frac{n(p-2)+\lambda}{p} + \frac{\varepsilon}{q}\right). \qquad (1.7)$$

Proof. By a method similar to the proof of Lemma 1.2, Lemma 1.3 can be proved.

2 Main Result

Theorem 2.1 *If* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ n \in \mathbf{Z}_+,\ \alpha > 0,\ \lambda > \max\{n(2-p),\ n(2-q)\},\ f \geq 0,$ $g \geq 0$, *and*

$$0 < \int_{\mathbf{R}_+^n} \|\boldsymbol{x}\|_\alpha^{n-\lambda} f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x} < \infty, \qquad 0 < \int_{\mathbf{R}_+^n} \|\boldsymbol{x}\|_\alpha^{n-\lambda} g^q(\boldsymbol{x})\mathrm{d}\boldsymbol{x} < \infty, \qquad (2.1)$$

then

$$\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})g(\boldsymbol{y})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$
$$<h_{\alpha,\lambda}(n,p,q)\Big(\int_{\mathbf{R}_+^n}\|\boldsymbol{x}\|_\alpha^{n-\lambda}f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}}\Big(\int_{\mathbf{R}_+^n}\|\boldsymbol{x}\|_\alpha^{n-\lambda}g^q(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{q}}, \tag{2.2}$$

$$\int_{\mathbf{R}_+^n}\|\boldsymbol{y}\|_\alpha^{(\lambda-n)(p-1)}\Big[\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\Big]^p\mathrm{d}\boldsymbol{y}<h_{\alpha,\lambda}^p(n,p,q)\int_{\mathbf{R}_+^n}\|\boldsymbol{x}\|_\alpha^{n-\lambda}f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{2.3}$$

where

$$h_{\alpha,\lambda}(n,p,q)=\frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)}B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big),$$

and the constant factors $h_{\alpha,\lambda}(n,p,q)$ *in* (2.2) *and* $h_{\alpha,\lambda}^p(n,p,q)$ *in* (2.3) *are optimal.*

In particular,

(1) *for* $\lambda=n$, *one has*

$$\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})g(\boldsymbol{y})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^n}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$
$$<\frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)}B\Big(\frac{n}{p},\ \frac{n}{q}\Big)\Big(\int_{\mathbf{R}_+^n}f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}}\Big(\int_{\mathbf{R}_+^n}g^q(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{q}},$$

$$\int_{\mathbf{R}_+^n}\Big[\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^n}\mathrm{d}\boldsymbol{x}\Big]^p\mathrm{d}\boldsymbol{y}<\left[\frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)}B\Big(\frac{n}{p},\ \frac{n}{q}\Big)\right]^p\int_{\mathbf{R}_+^n}f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

where the constant factors

$$\frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)}B\Big(\frac{n}{p},\ \frac{n}{q}\Big)$$

and

$$\left[\frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)}B\Big(\frac{n}{p},\ \frac{n}{q}\Big)\right]^p$$

are optimal;

(2) *for* $\alpha=1$, *one has*

$$\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})g(\boldsymbol{y})}{\Big[\sum\limits_{i=1}^n(x_i+y_i)\Big]^\lambda}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$
$$<\frac{1}{(n-1)!}B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big)$$
$$\cdot\Big(\int_{\mathbf{R}_+^n}\Big(\sum_{i=1}^n x_i\Big)^{n-\lambda}f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}}\Big(\int_{\mathbf{R}_+^n}\Big(\sum_{i=1}^n x_i\Big)^{n-\lambda}g^q(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{q}},$$

$$\int_{\mathbf{R}_+^n}\Big(\sum_{i=1}^n y_i\Big)^{(\lambda-n)(p-1)}\Bigg[\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})}{\Big[\sum\limits_{i=1}^n(x_i+y_i)\Big]^\lambda}\mathrm{d}\boldsymbol{x}\Bigg]^p\mathrm{d}\boldsymbol{y}$$
$$<\Big[\frac{1}{(n-1)!}B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big)\Big]^p\int_{\mathbf{R}_+^n}\Big(\sum_{i=1}^n x_i\Big)^{n-\lambda}f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

where the constant factors

$$\frac{1}{(n-1)!}B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big)$$

and

$$\Big[\frac{1}{(n-1)!}B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big)\Big]^p$$

are optimal;

(3) *for* $p=q=2$, *one has*

$$\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})g(\boldsymbol{y})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\ \mathrm{d}\boldsymbol{y}$$
$$<\frac{\Gamma^n\Big(\frac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\frac{n}{\alpha}\Big)}B\Big(\frac{\lambda}{2},\ \frac{\lambda}{2}\Big)\Big(\int_{\mathbf{R}_+^n}\|\boldsymbol{x}\|_\alpha^{n-\lambda}f^2(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac12}\Big(\int_{\mathbf{R}_+^n}\|\boldsymbol{x}\|_\alpha^{n-\lambda}g^2(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac12},$$

$$\int_{\mathbf{R}_+^n}\|\boldsymbol{y}\|_\alpha^{\lambda-n}\Big[\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\Big]^2\mathrm{d}\boldsymbol{y}$$
$$<\left[\frac{\Gamma^n\Big(\frac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\frac{n}{\alpha}\Big)}B\Big(\frac{\lambda}{2},\ \frac{\lambda}{2}\Big)\right]^2\int_{\mathbf{R}_+^n}\|\boldsymbol{x}\|_\alpha^{n-\lambda}f^2(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

where the constant factors

$$\frac{\Gamma^n\Big(\frac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\frac{n}{\alpha}\Big)}B\Big(\frac{\lambda}{2},\ \frac{\lambda}{2}\Big)$$

and

$$\left[\frac{\Gamma^n\Big(\frac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\frac{n}{\alpha}\Big)}B\Big(\frac{\lambda}{2},\ \frac{\lambda}{2}\Big)\right]^2$$

are optimal.

Proof. By the Hölder's inequality, one has

$$A:=\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})g(\boldsymbol{y})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$
$$=\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^{\frac{\lambda}{p}}}\Big(\frac{\|\boldsymbol{x}\|_\alpha}{\|\boldsymbol{y}\|_\alpha}\Big)^{\frac{2n-\lambda}{pq}}\frac{g(\boldsymbol{y})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^{\frac{\lambda}{q}}}\Big(\frac{\|\boldsymbol{y}\|_\alpha}{\|\boldsymbol{x}\|_\alpha}\Big)^{\frac{2n-\lambda}{pq}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$
$$\le\Big[\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^n}\frac{f^p(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\Big(\frac{\|\boldsymbol{x}\|_\alpha}{\|\boldsymbol{y}\|_\alpha}\Big)^{\frac{2n-\lambda}{q}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\Big]^{\frac1p}$$

$$\cdot\Big[\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^n}\frac{g^q(\boldsymbol{y})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\Big(\frac{\|\boldsymbol{y}\|_\alpha}{\|\boldsymbol{x}\|_\alpha}\Big)^{\frac{2n-\lambda}{p}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\Big]^{\frac{1}{q}}$$

$$=\Big[\int_{\mathbf{R}_+^n}f^p(\boldsymbol{x})\Big(\int_{\mathbf{R}_+^n}\frac{1}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\Big(\frac{\|\boldsymbol{x}\|_\alpha}{\|\boldsymbol{y}\|_\alpha}\Big)^{\frac{2n-\lambda}{q}}\mathrm{d}\boldsymbol{y}\Big)\mathrm{d}\boldsymbol{x}\Big]^{\frac{1}{p}}$$

$$\cdot\Big[\int_{\mathbf{R}_+^n}g^q(\boldsymbol{y})\Big(\int_{\mathbf{R}_+^n}\frac{1}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\Big(\frac{\|\boldsymbol{y}\|_\alpha}{\|\boldsymbol{x}\|_\alpha}\Big)^{\frac{2n-\lambda}{p}}\mathrm{d}\boldsymbol{x}\Big)\mathrm{d}\boldsymbol{y}\Big]^{\frac{1}{q}}$$

$$=\Big(\int_{\mathbf{R}_+^n}f^p(\boldsymbol{x})\omega_{\alpha,\lambda}(\boldsymbol{x},\ q)\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}}\Big(\int_{\mathbf{R}_+^n}g^q(\boldsymbol{y})\omega_{\alpha,\lambda}(\boldsymbol{y},\ p)\mathrm{d}\boldsymbol{y}\Big)^{\frac{1}{q}}.$$

According to the condition of taking equality in the Hölder's inequality, if this inequality takes the form of an equality, then there exist constants C_1 and C_2 with $C_1^2+C_2^2\neq 0$ such that

$$\frac{C_1f^p(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\Big(\frac{\|\boldsymbol{x}\|_\alpha}{\|\boldsymbol{y}\|_\alpha}\Big)^{\frac{2n-\lambda}{q}}=\frac{C_2g^q(\boldsymbol{y})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\Big(\frac{\|\boldsymbol{y}\|_\alpha}{\|\boldsymbol{x}\|_\alpha}\Big)^{\frac{2n-\lambda}{p}}\qquad\text{a.e. in }\mathbf{R}_+^n\times\mathbf{R}_+^n.$$

It follows that

$$C_1\|\boldsymbol{x}\|_\alpha^{2n-\lambda}f^p(\boldsymbol{x})=C_2\|\boldsymbol{y}\|_\alpha^{2n-\lambda}g^q(\boldsymbol{y})=C\ \text{(constant)}\qquad\text{a.e. in }\mathbf{R}_+^n\times\mathbf{R}_+^n.$$

Without loss of generality, suppose that $C_1\neq 0$. Then

$$\|\boldsymbol{x}\|_\alpha^{n-\lambda}f^p(\boldsymbol{x})=\frac{C}{C_1}\|\boldsymbol{x}\|_\alpha^{-n}\qquad\text{a.e. in }\mathbf{R}_+^n,$$

which contradicts (2.1). Hence

$$A<\Big(\int_{\mathbf{R}_+^n}f^p(\boldsymbol{x})\omega_{\alpha,\lambda}(\boldsymbol{x},\ q)\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}}\Big(\int_{\mathbf{R}_+^n}g^q(\boldsymbol{y})\omega_{\alpha,\lambda}(\boldsymbol{y},\ p)\mathrm{d}\boldsymbol{y}\Big)^{\frac{1}{q}}.$$

Further, by (1.6), one has that (2.2) is valid.

For $0<a<b<\infty$, set

$$g_{a,b}(\boldsymbol{y})=\begin{cases}\|\boldsymbol{y}\|_\alpha^{(\lambda-n)(p-1)}\Big(\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\Big)^{p-1}, & a<\|\boldsymbol{y}\|_\alpha<b;\\ 0, & 0<\|\boldsymbol{y}\|_\alpha\le a\text{ or }\|\boldsymbol{y}\|_\alpha\ge b,\end{cases}$$

$$\tilde g(\boldsymbol{y})=\|\boldsymbol{y}\|_\alpha^{(\lambda-n)(p-1)}\Big(\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\Big)^{p-1},\qquad \boldsymbol{y}\in\mathbf{R}_+^n.$$

By (2.1), for sufficiently small $a>0$ and sufficiently large $b>0$, one has

$$0<\int_{a<\|\boldsymbol{y}\|_\alpha<b}\|\boldsymbol{y}\|_\alpha^{n-\lambda}g_{a,b}^q(\boldsymbol{y})\mathrm{d}\boldsymbol{y}<\infty.$$

Hence, by (2.2), one has

$$\int_{a<\|\boldsymbol{y}\|_\alpha<b}\|\boldsymbol{y}\|_\alpha^{n-\lambda}\tilde g^q(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

$$=\int_{a<\|\boldsymbol{y}\|_\alpha<b}\|\boldsymbol{y}\|_\alpha^{(\lambda-n)(p-1)}\Big(\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\Big)^p\mathrm{d}\boldsymbol{y}$$

$$=\int_{a<\|\boldsymbol{y}\|_\alpha<b}\|\boldsymbol{y}\|_\alpha^{(\lambda-n)(p-1)}\Big(\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\Big)^{p-1}\Big(\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\Big)\mathrm{d}\boldsymbol{y}$$

$$=\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})g_{a,b}(\boldsymbol{y})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$

$$
\begin{aligned}
&< \frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)} B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big) \\
&\quad \cdot \Big(\int_{\mathbf{R}_+^n} \|\boldsymbol{x}\|_\alpha^{n-\lambda} f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}} \Big(\int_{\mathbf{R}_+^n} \|\boldsymbol{y}\|_\alpha^{n-\lambda} g_{a,b}^q(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\Big)^{\frac{1}{q}} \\
&= \frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)} B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big) \\
&\quad \cdot \Big(\int_{\mathbf{R}_+^n} \|\boldsymbol{x}\|_\alpha^{n-\lambda} f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}} \Big(\int_{a<\|\boldsymbol{y}\|_\alpha<b} \|\boldsymbol{y}\|_\alpha^{n-\lambda} \tilde{g}^q(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\Big)^{\frac{1}{q}},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\int_{a<\|\boldsymbol{y}\|_\alpha<b} \|\boldsymbol{y}\|_\alpha^{n-\lambda} \tilde{g}^q(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&< \left[\frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)} B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big)\right]^p \int_{\mathbf{R}_+^n} \|\boldsymbol{x}\|_\alpha^{n-\lambda} f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

For $a \to 0^+$, $b \to +\infty$, we get

$$
\begin{aligned}
&\int_{\mathbf{R}_+^n} \|\boldsymbol{y}\|_\alpha^{n-\lambda} \tilde{g}^q(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&\le \left[\frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)} B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big)\right]^p \int_{\mathbf{R}_+^n} \|\boldsymbol{x}\|_\alpha^{n-\lambda} f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Hence, by (2.1), one has

$$
0 < \int_{\mathbf{R}_+^n} \|\boldsymbol{y}\|_\alpha^{n-\lambda} \tilde{g}^q(\boldsymbol{y})\mathrm{d}\boldsymbol{y} < \infty.
$$

By (2.2), one has

$$
\begin{aligned}
&\int_{\mathbf{R}_+^n} \|\boldsymbol{y}\|_\alpha^{(\lambda-n)(p-1)} \Big(\int_{\mathbf{R}_+^n} \frac{f(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha + \|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\Big)^p \mathrm{d}\boldsymbol{y} \\
&= \int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^n} \frac{f(\boldsymbol{x})\tilde{g}(\boldsymbol{y})}{(\|\boldsymbol{x}\|_\alpha + \|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\
&< \frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)} B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big) \\
&\quad \cdot \Big(\int_{\mathbf{R}_+^n} \|\boldsymbol{x}\|_\alpha^{n-\lambda} f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}} \Big(\int_{\mathbf{R}_+^n} \|\boldsymbol{y}\|_\alpha^{n-\lambda} \tilde{g}^q(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\Big)^{\frac{1}{q}} \\
&< \frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)} B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big) \\
&\quad \cdot \Big(\int_{\mathbf{R}_+^n} \|\boldsymbol{x}\|_\alpha^{n-\lambda} f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}} \Big[\int_{\mathbf{R}_+^n} \|\boldsymbol{y}\|_\alpha^{(\lambda-n)(p-1)} \Big(\int_{\mathbf{R}_+^n} \frac{f(\boldsymbol{x})}{(\|\boldsymbol{x}\|_\alpha + \|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\Big)^p \mathrm{d}\boldsymbol{y}\Big]^{\frac{1}{q}}.
\end{aligned}
$$

Hence (2.3) can be obtained.

By (2.3), one can also obtain (2.2). Hence (2.2) and (2.3) are equivalent.

If the constant factor $h_{\alpha,\lambda}(n,p,q)$ in (2.2) is not optimal, then there exists a positive constant $K < h_{\alpha,\lambda}(n,p,q)$ such that

$$\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^n}\frac{f(\boldsymbol{x})g(\boldsymbol{y})}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} < K\Big(\int_{\mathbf{R}_+^n}\|\boldsymbol{x}\|_\alpha^{n-\lambda}f^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}}\Big(\int_{\mathbf{R}_+^n}\|\boldsymbol{y}\|_\alpha^{n-\lambda}g^q(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\Big)^{\frac{1}{q}}. \tag{2.4}$$

In particular, setting

$$f(\boldsymbol{x})=\|\boldsymbol{x}\|_\alpha^{\frac{\lambda-2n-\varepsilon}{p}},\qquad g(\boldsymbol{y})=\|\boldsymbol{y}\|_\alpha^{\frac{\lambda-2n-\varepsilon}{q}},$$

by (2.4) and the properties of limit, we see that there exists a sufficiently small $a>0$ such that

$$\begin{aligned}&\int_{\|\boldsymbol{x}\|_\alpha>a}\int_{\mathbf{R}_+^n}\frac{1}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\|\boldsymbol{x}\|_\alpha^{\frac{\lambda-2n-\varepsilon}{p}}\|\boldsymbol{y}\|_\alpha^{\frac{\lambda-2n-\varepsilon}{q}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\ &< K\Big(\int_{\|\boldsymbol{x}\|_\alpha>a}\|\boldsymbol{x}\|_\alpha^{n-\lambda}\|\boldsymbol{x}\|_\alpha^{\lambda-2n-\varepsilon}\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}}\Big(\int_{\|\boldsymbol{y}\|_\alpha>a}\|\boldsymbol{y}\|_\alpha^{n-\lambda}\|\boldsymbol{y}\|_\alpha^{\lambda-2n-\varepsilon}\mathrm{d}\boldsymbol{y}\Big)^{\frac{1}{q}}\\ &= K\int_{\|\boldsymbol{x}\|_\alpha>a}\|\boldsymbol{x}\|_\alpha^{-n-\varepsilon}\mathrm{d}\boldsymbol{x}.\end{aligned}$$

On the other hand, by (1.7), one has

$$\begin{aligned}&\int_{\|\boldsymbol{x}\|_\alpha>a}\int_{\mathbf{R}_+^n}\frac{1}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\|\boldsymbol{x}\|_\alpha^{\frac{\lambda-2n-\varepsilon}{p}}\|\boldsymbol{y}\|_\alpha^{\frac{\lambda-2n-\varepsilon}{q}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\ &=\int_{\|\boldsymbol{x}\|_\alpha>a}\|\boldsymbol{x}\|_\alpha^{\frac{\lambda-2n-\varepsilon}{p}}\int_{\mathbf{R}_+^n}\frac{1}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\|\boldsymbol{y}\|_\alpha^{\frac{\lambda-2n-\varepsilon}{q}}\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\ &=\int_{\|\boldsymbol{x}\|_\alpha>a}\|\boldsymbol{x}\|_\alpha^{\frac{\lambda-2n-\varepsilon}{p}}\tilde{\omega}_{\alpha,\lambda}(\boldsymbol{x},\ q)\mathrm{d}\boldsymbol{x}\\ &=\frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)}B\Big(\frac{n(q-2)+\lambda}{q}-\frac{\varepsilon}{q},\ \frac{n(p-2)+\lambda}{p}+\frac{\varepsilon}{q}\Big)\int_{\|\boldsymbol{x}\|_\alpha>a}\|\boldsymbol{x}\|_\alpha^{-n-\varepsilon}\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Hence,

$$\frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)}B\Big(\frac{n(q-2)+\lambda}{q}-\frac{\varepsilon}{q},\ \frac{n(p-2)+\lambda}{p}+\frac{\varepsilon}{q}\Big)<K.$$

For $\varepsilon\to 0^+$, one has

$$h_{\alpha,\lambda}(n,p,q)=\frac{\Gamma^n\Big(\dfrac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\dfrac{n}{\alpha}\Big)}B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big)\le K,$$

which contradicts the fact that $K<h_{\alpha,\lambda}(n,p,q)$. Hence the constant factor $h_{\alpha,\lambda}(n,p,q)$ in (2.2) is optimal.

Since (2.2) and (2.3) are equivalent, the constant factor $h^p_{\alpha,\lambda}(n,p,q)$ in (2.3) is also optimal.

The proof of Theorem 2.1 is completed.

References

[1] Hardy G H, Littewood J E, Polya G. Inequalities. London: Gambridge Univ. Press, 1952.

[2] Zhao C J, Debnath L. Some new type Hilbert integral inequalities. *J. Math. Anal. Appl.*, 2001, **262**: 411–418.

[3] Hu K. On Hilbert's inequality and it's applications. *Adv. in Math.* (*China*), 1993, **22**: 160–163.

[4] Pachpatte B G. On some new inequalities similar to Hilbert's inequality. *J. Math. Anal. Appl.*, 1998, **226**: 166–179.

[5] Gao M Z, Tan L. Some improvements on Hilbert's integral inequality. *J. Math. Anal. Appl.*, 1999, **229**: 682–689.

[6] Kuang J C. On new extensions of Hilbert's integtal inequality. *J. Math. Anal. Appl.*, 1999, **235**: 608–614.

[7] Yang B C. On a generalization of Hardy-Hilbert's inequality. *Chinese Ann. Math. Ser. A*, 2002, **23**(5): 247–254.

[8] Hong Y. All-sided generalization about Hardy-Hilbert integral inequalities. *Acta Math. Sinica*, 2001, **44**(4): 619–626.

[9] Yang B C. On a multiple Hardy-Hilbert's integral inequality. *Chinese Ann. Math. Ser. A*, 2003, **24**(6): 743–750.

[10] Yang B C, Themistocles M. On the way of weight coefficient and research for the Hilbert-type inequalities. *Math. Inequal. Appl.*, 2003, **6**: 625–658.

[11] Fichtingoloz G M. A Course in Differential and Integral Calculus. Beijing: Renmin Jiaoyu Publisgers, 1957.