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Superlinear Fourth-order Elliptic Problem without Ambrosetti and Rabinowitz Growth Condition*

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Abstract: This paper deals with superlinear fourth-order elliptic problem under Navier boundary condition. By using the mountain pass theorem and suitable truncation, a multiplicity result is established for all $\lambda > 0$ and some previous result is extended.

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1 Introduction and Main Results

Fourth-order elliptic problems are usually used to describe some phenomena appeared in different physical, engineering and other sciences. Lazer and McKenna[1] studied the problem of nonlinear oscillation in a suspension bridge and they presented a mathematical model for the bridge that took account of the fact that the coupling provided by the stays connecting the suspension cable to the deck of the road bed is basically nonlinear. Also, Liu and Feng[2] pointed out that this kind of problem furnishes a good model to the static deflection of an elastic plate in a fluid. Ahmed and Harbi[3] indicated that this problem also arises in such as communication satellites, space shuttles, and space stations, which are equipped with large antennas mounted on long flexible masts (beams). Fourth-order elliptic problems have been studied extensively in recent years, and we refer the reader to [4–9] and the references

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therein.

Consider the following fourth-order elliptic problem:

$$\begin{cases} \Delta^2 u + c\Delta u = \lambda f(x, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ^2 is the biharmonic operator, c is a constant, $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain and $f(x, s)$ is a continuous function on $\bar{\Omega} \times \mathbf{R}$.

Denote

$$F(x, s) = \int_0^s f(x, t)\mathrm{d}t,$$
$$H(x, s) = sf(x, s) - 2F(x, s).$$

We assume that $f(x, s)$ satisfies the following hypotheses:

(H1) $\lim\limits_{s\to 0} \dfrac{f(x, s)}{s} = 0$ uniformly for a.e. $x \in \Omega$;

(H2) There exist positive constants C_1 and C_2 such that

$$|f(x, s)| \leqslant C_1 + C_2|s|^p,$$
$$1 \leqslant p < q = \begin{cases} \dfrac{N+2}{N-2}, & N \geqslant 3, \\ +\infty, & N \leqslant 2; \end{cases}$$

(H3) $\lim\limits_{|s|\to+\infty} \dfrac{F(x, s)}{s^2} = +\infty$ uniformly for a.e. $x \in \Omega$;

(H4) There exists a $C_* > 0$ such that

$$H(x, t) \leqslant H(x, s) + C_*$$

for all $0 < t < s$ or $s < t < 0$, $x \in \Omega$.

To obtain nontrivial solutions of the problem (1.1) by applying variational method, one often uses the Ambrosetti-Rabinowitz condition (see [10]), i.e.,

(AR) There are constants $\theta > 0$ and $s_0 > 0$ such that

$$0 < (2 + \theta)F(x, s) \leqslant f(x, s)s, \quad |s| > s_0, \qquad x \in \Omega.$$

This condition ensures the compactness of the corresponding functional, however, it eliminates many nonlinearities. To avoid the condition (AR), many approaches were developed. Costa and Magalhães[11] studied the problem (1.1) via replacing the condition (AR) by

$$\liminf_{s\to\infty} \frac{sf(x, s) - 2F(x, s)}{|s|^\mu} \geqslant k > 0 \qquad \text{uniformly for a.e. } x \in \Omega,$$

where $\mu \geqslant \mu_0 > 0$. Willem and Zou[12] assumed that $H(x, s)$ is increasing in s and

$$sf(x, s) \geqslant 0, \quad s \in \mathbf{R}; \quad sf(x, s) \geqslant C_0|s|^\mu, \quad |s| \geqslant s_0 > 0, \qquad x \in \Omega,$$

where $\mu > 2$ and $C_0 > 0$, in place of the condition (AR). Recently, by using the assumptions (H1)–(H4), Miyagaki and Souto[13] obtained a nontrivial weak solution in the case of second-order elliptic problem.

For the fourth-order problem (1.1), Zhang and Li[14] obtained at least two nontrivial solutions by means of Morse theory and local linking when f is sublinear at infinity. By using the linking theorem, Qian and Li[15] obtained one nontrivial solution if f is superlinear and satisfies the Ambrosetti-Rabinowitz condition, and two nontrivial solutions if f is asymptotically linear as s is large enough. An and Liu[2] also established the existence of at least one nontrivial solution if f is asymptotically linear at infinity. In this paper, we consider the fourth-order problem (1.1) when f is superlinear but the Ambrosetti-Rabinowitz condition is not required. Applying the mountain pass theorem, we obtain at least two nontrivial solutions for all $\lambda > 0$.

Our main result is as follows.

Theorem 1.1 *Assume that* (H1)–(H4) *hold and* $c < \lambda_1$, *where* λ_1 *denotes the first eigenvalue of* $-\Delta$ *in* $H_0^1(\Omega)$. *Then, for all* $\lambda > 0$, *the problem* (1.1) *has at least two nontrivial solutions, one of which is positive and the other is negative.*

Remark 1.1 Note that the condition (H3) is weaker than (AR) (see [13]). Let

$$F(x, s) = s^2 \ln(|s| + 1).$$

It is easy to see that F satisfies assumptions (H1)–(H4) but not (AR) condition.

Remark 1.2 Note that (H4) is weaker than the following condition:

(i) There exists an $s_0 > 0$ such that $\dfrac{f(x,s)}{s}$ is increasing in $s > s_0$ and decreasing in $s < -s_0$ for all $x \in \Omega$.

In previous works, many authors (see [16–17]) used the condition (i) to assure that the corresponding energy functional satisfies the Cerami condition. In this paper, our arguments show that the condition (i) implies that the energy functional satisfies Palais-Smale condition.

2 Preliminary Results

Let $H = H^2(\Omega) \cap H_0^1(\Omega)$ be a Hilbert space equipped with the inner product

$$(u, v)_H = \int_\Omega (\Delta u \Delta v + \nabla u \nabla u)\mathrm{d}x$$

and the deduced norm

$$\|u\|_H^2 = \int_\Omega |\Delta u|^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x.$$

Let λ_k ($k \in \mathbf{N}$) be the eigenvalues and φ_k ($k \in \mathbf{N}$) be the corresponding eigenfunctions of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where each eigenvalue λ_k is repeated according to the multiplicity. Recall that $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_k \to +\infty$ and $\varphi_1 > 0$ for $x \in \Omega$. It is easily seen that

$$\Lambda_k = \lambda_k(\lambda_k - c)$$

are eigenvalues of the problem

$$\begin{cases} \Delta^2 u + c\Delta u = \Lambda u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

and the corresponding eigenfunctions are still φ_k.

Assume that $c < \lambda_1$. We denote by $\|\cdot\|$ the norm in H which is given by

$$\|u\|^2 = \int_\Omega |\Delta u|^2 \mathrm{d}x - c\int_\Omega |\nabla u|^2 \mathrm{d}x.$$

It is easy to show that the norm $\|\cdot\|$ is an equivalent norm on H and the following Poincaré inequality holds:

$$\|u\|^2 \geqslant \Lambda_1 \|u\|_{L^2}^2, \qquad u \in H. \tag{2.3}$$

We say that $u \in H$ is a weak solution to problem (1.1), if u satisfies

$$\int_\Omega (\Delta u \Delta v - c\nabla u \nabla v - \lambda f(x,u)v)\mathrm{d}x = 0, \qquad v \in H^*,$$

where H^* is the dual space of H.

It is well known that the weak solution of problem (1.1) is equivalent to the critical point of the Euler-Lagrange functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2)\mathrm{d}x - \lambda\int_\Omega F(x,u)\mathrm{d}x, \qquad u \in H.$$

Obviously, $I_\lambda \in C^1(H, \mathbf{R})$ and

$$I'_\lambda(u)\cdot v = \int_\Omega (\Delta u \Delta v - c\nabla u \nabla v - \lambda f(x,u)v)\mathrm{d}x, \qquad u, v \in H.$$

Let

$$u^+ = \max\{u,\, 0\},$$
$$u^- = \min\{u,\, 0\}.$$

Consider the problem

$$\begin{cases} \Delta^2 u + c\Delta u = \lambda f^+(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where

$$f^+(x,t) = \begin{cases} f(x,t), & t \geqslant 0; \\ 0, & t < 0. \end{cases}$$

Define the corresponding functional $I_\lambda^+ : H \to \mathbf{R}$ as follows:

$$I_\lambda^+(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2)\mathrm{d}x - \lambda\int_\Omega F^+(x,u)\mathrm{d}x, \qquad u \in H,$$

where

$$F^+(x,u) = \int_0^u f^+(x,s)\mathrm{d}s.$$

Obviously, $I_\lambda^+ \in C^1(H, \mathbf{R})$. Let u be a critical point of I_λ^+, which implies that u is a weak solution of (2.4). Furthermore, by the weak maximum principle, it follows that $u \geqslant 0$ in Ω. Thus u is also a solution of the problem (1.1) and

$$I_\lambda(u) = I_\lambda^+(u).$$

Similarly, we can define

$$f^-(x,t) = \begin{cases} f(x,t), & t \leqslant 0; \\ 0, & t > 0, \end{cases} \tag{2.5}$$

and

$$I_\lambda^-(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2)\mathrm{d}x - \lambda\int_\Omega F^-(x,u)\mathrm{d}x, \qquad u \in H,$$

where

$$F^-(x,u)=\int_0^u f^-(x,s)\mathrm{d}s.$$

It is easy to see that $I_\lambda^- \in C^1(H,\mathbf{R})$ and if v is a critical point of I_λ^-, then it is a solution of the problem (1.1) with

$$I_\lambda(v)=I_\lambda^-(v).$$

Now we prove that the functionals I_λ^+ and I_λ^- have the mountain pass geometry.

Lemma 2.1 *Under the assumption* (H3), I_λ^+ *and* I_λ^- *are unbounded from below.*

Proof. (H3) implies that for all $M>0$ there exists $C_M>0$ such that

$$F^+(x,s)\geqslant Ms^2-C_M, \qquad x\in\Omega,\ s>0. \tag{2.6}$$

Taking $\phi\in H$ with $\phi>0$, from (2.6) we obtain

$$\begin{aligned} I_\lambda^+(t\phi) &\leqslant \frac{t^2}{2}\|\phi\|^2-\lambda\int_\Omega Mt^2\phi^2\mathrm{d}x+\lambda\int_\Omega C_M\mathrm{d}x \\ &= t^2\Big(\frac{1}{2}\|\phi\|^2-\lambda M\int_\Omega \phi^2\mathrm{d}x\Big)+\lambda C_M|\Omega|, \end{aligned} \tag{2.7}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Let

$$M=\frac{\|\phi\|^2}{2\lambda\displaystyle\int_\Omega \phi^2\mathrm{d}x}+1.$$

Then

$$\lim_{t\to+\infty} I_\lambda^+(t\phi)=-\infty. \tag{2.8}$$

For I_λ^-, by using an analogous argument we can find some $\phi_*\in H$ with $\phi_*<0$ such that

$$\lim_{t\to+\infty} I_\lambda^-(t\phi_*)=-\infty. \tag{2.9}$$

The proof is completed.

Lemma 2.2 *Assume that* (H1) *and* (H2) *hold. Then there exist* ρ, $R>0$ *such that*

$$I_\lambda^\pm(u)\geqslant R,$$

if

$$\|u\|=\rho.$$

Proof. We just consider the case of I_λ^+. The case of I_λ^- can be dealt with similarly.

Take $\alpha\in\Big(2,\dfrac{2N}{N-2}\Big)$. (H1) and (H2) imply that for all given $\epsilon>0$, there exists a $C_\epsilon>0$ such that

$$F^+(x,s)\leqslant\frac{\epsilon}{2}s^2+C_\epsilon s^\alpha, \qquad x\in\Omega,\ s>0. \tag{2.10}$$

Combining (2.10) and the Poincaré inequality as well as the Sobolve embedding, we have

$$\begin{aligned} I_\lambda^+(u) &\geqslant \frac{1}{2}\|u\|^2-\frac{\lambda\epsilon}{2}\int_\Omega |u|^2\mathrm{d}x-\lambda C_\epsilon\int_\Omega |u|^\alpha\mathrm{d}x \\ &\geqslant \Big(\frac{1}{2}-\frac{\lambda\epsilon}{2\Lambda_1}\Big)\|u\|^2-C_s\|u\|^\alpha, \end{aligned} \tag{2.11}$$

where C_s is a positive constant. In (2.11), taking $\epsilon > 0$ such that

$$\frac{1}{2} - \frac{\lambda\epsilon}{2\Lambda_1} \geqslant \frac{1}{4}$$

and choosing

$$\|u\| = \rho > 0$$

small enough, we can find an $R > 0$ such that

$$I_\lambda^+(u) \geqslant R,$$

if

$$\|u\| = \rho.$$

This completes the proof.

Now, we prove that every Palais-Smale sequence of $I_\lambda^\pm$ is relatively compact.

We recall that a sequence $\{u_n\} \subset H$ is said to be a Palais-Smale sequence of the functional Φ provided that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ in H^*.

Lemma 2.3 *Suppose that* (H2)–(H4) *hold. Then for all* $\lambda > 0$, *every Palais-Smale sequence of* $I_\lambda^\pm$ *has a convergent subsequence.*

Proof. We just prove the case of I_λ^+. The arguments for the case of I_λ^- are similar.

Since Ω is bounded and (H2) holds, if $\{u_n\}$ is bounded in H, by using the Sobolve embedding and the standard procedures, we can get a subsequence converges strongly. So we need only to show that $\{u_n\}$ is bounded in H.

Assume that $\{u_n\} \subset H$ is a Palais-Smale sequence of I_λ^+, i.e.,

$$I_\lambda^+(u_n) \to c_\lambda, \quad (I_\lambda^+)'(u_n) \to 0 \qquad \text{as } n \to \infty. \tag{2.12}$$

We suppose, by contradiction, that passing to a subsequence, if necessary,

$$\|u_n\| \to +\infty \qquad \text{as } n \to +\infty.$$

Set

$$\omega_n := \frac{u_n}{\|u_n\|}.$$

Then

$$\|\omega_n\| = 1. \tag{2.13}$$

Passing to a subsequence, if necessary, we may assume that there exists an $\omega \in H$ such that

$$\begin{aligned} &\omega_n \rightharpoonup \omega \quad \text{weakly in } H,\ n \to +\infty, \\ &\omega_n \to \omega \quad \text{strongly in } L^2(\Omega),\ n \to +\infty, \\ &\omega_n(x) \to \omega(x) \quad \text{a.e. in } \Omega,\ n \to +\infty. \end{aligned}$$

We claim that

$$\omega(x) \equiv 0 \qquad \text{a.e. in } \Omega.$$

In fact, we denote

$$\Omega^* := \{x \in \Omega,\ \omega(x) \neq 0\}.$$

If

$$\Omega^* \neq \emptyset,$$

then for $x \in \Omega^*$,

$$|u_n(x)| \to +\infty.$$

By (H3) we have

$$\lim_n \frac{F^+(x, u_n(x))}{(u_n(x))^2}(\omega_n(x))^2 = +\infty. \tag{2.14}$$

The Fatou Lemma and (2.12) imply

$$\begin{aligned}
&\int_\Omega \lim_n \frac{F^+(x, u_n(x))}{(u_n(x))^2}(\omega_n(x))^2 \mathrm{d}x \qquad (2.15)\\
&= \int_\Omega \lim_n \frac{F^+(x, u_n(x))}{(u_n(x))^2} \cdot \frac{(u_n(x))^2}{\|u_n(x)\|^2} \mathrm{d}x \\
&\leqslant \lim_n \frac{1}{\|u_n(x)\|^2} \int_\Omega F^+(x, u_n(x)) \mathrm{d}x \\
&= \lim_n \frac{1}{\lambda \|u_n(x)\|^2} \Big(\frac{1}{2}\|u_n(x)\|^2 - I_\lambda^+(u_n) \Big) \\
&= \frac{1}{2\lambda}. \qquad (2.16)
\end{aligned}$$

Hence Ω^* has zero measure. Consequently,

$$\omega(x) \equiv 0 \qquad \text{a.e. in } \Omega.$$

As in [18], we take $t_n \in [0, 1]$ such that

$$I_\lambda^+(t_n u_n) = \max_{t \in [0,1]} I_\lambda^+(t u_n),$$

which implies that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I_\lambda^+(t u_n)\Big|_{t=t_n} &= t_n \|u_n\|^2 - \lambda \int_\Omega f^+(x,\, t_n u_n) u_n \mathrm{d}x \\
&= 0. \qquad (2.17)
\end{aligned}$$

Since

$$(I_\lambda^+)'(t_n u_n) \cdot (t_n u_n) = t_n^2 \|u_n\|^2 - \lambda \int_\Omega f^+(x,\, t_n u_n) t_n u_n \mathrm{d}x,$$

together with (2.16) it follows that

$$\begin{aligned}
(I_\lambda^+)'(t_n u_n) \cdot (t_n u_n) &= t_n \frac{\mathrm{d}}{\mathrm{d}t} I_\lambda^+(t u_n)\Big|_{t=t_n} \\
&= 0.
\end{aligned}$$

Hence, by (H4) we obtain

$$\begin{aligned}
2I_\lambda^+(t u_n) &\leqslant 2I_\lambda^+(t_n u_n) - (I_\lambda^+)'(t_n u_n) \cdot (t_n u_n) \\
&= \lambda \int_\Omega (t_n u_n f^+(x,\, t_n u_n) - 2F^+(x,\, t_n u_n)) \mathrm{d}x \\
&\leqslant \lambda \int_\Omega (u_n f^+(x,\, u_n) - 2F^+(x,\, u_n) + C_*) \mathrm{d}x \\
&= 2I_\lambda^+(u_n) - (I_\lambda^+)'(u_n) \cdot u_n + \lambda C_* |\Omega| \\
&= 2c_\lambda + \lambda C_* |\Omega|. \qquad (2.18)
\end{aligned}$$

On the other hand, for all $R_0 > 0$,

$$2I_\lambda^+(R_0\omega_n) = R_0^2 - 2\lambda\int_\Omega F^+(x, R_0\omega_n)\mathrm{d}x$$
$$= R_0^2 + o(1),$$

which contradicts (2.17) for R_0 and n large. This completes the proof.

3 Proof of the Main Result

Proof of Theorem 1.1 By (H1), it is easily seen that

$$I_\lambda^\pm(0) = 0.$$

From Lemma 2.1 we know that there exists an

$$e \in H, \qquad \|e\| > \rho,$$

such that

$$I_\lambda^\pm(e) < 0.$$

In addition, Lemma 2.2 implies that there exist ρ, $R > 0$ such that

$$I_\lambda^\pm(u)\big|_{\partial B_\rho} \geqslant R.$$

Define

$$\Gamma = \{\gamma : [0, 1] \to H \,|\, \gamma \text{ is continuous and } \gamma(0) = 0,\ \gamma(1) = e\},$$

and

$$c_\lambda^\pm = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda^\pm(\gamma(t)).$$

By Lemma 2.3 we can see that $I_\lambda^\pm$ satisfies the Palais-Smale condition. By the mountain pass theorem, we know that c_λ^+ is a critical value of I_λ^+ and there is at least one nontrivial critical point $u_{\lambda,+} \in H$ such that

$$I_\lambda^+(u_{\lambda,+}) = c_\lambda^+.$$

Clearly,

$$u_{\lambda,+} \geqslant 0.$$

Then the strong maximum principle implies

$$u_{\lambda,+}(x) > 0, \qquad x \in \Omega.$$

Thus $u_{\lambda,+}$ is a positive solution of the problem (1.1). By an analogous argument we know that there exists at least one negative solution $u_{\lambda,-} \in H$ of the problem (1.1), which is a nontrivial critical point of I_λ^-. Hence, the problem (1.1) admits at least one positive solution and one negative solution.

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