Communications
in
Mathematical
Research
**29**(1)(2013), 41–50

# $\mathcal{F}$-perfect Rings and Modules*

Lu Bo
(*College of Mathematics and Information Science, Northwest Normal University, Lanzhou, 730070*)

Communicated by Du Xian-kun

**Abstract:** Let $R$ be a ring, and let $(\mathcal{F}, \mathcal{C})$ be a cotorsion theory. In this article, the notion of $\mathcal{F}$-perfect rings is introduced as a nontrial generalization of perfect rings and A-perfect rings. A ring $R$ is said to be right $\mathcal{F}$-perfect if $F$ is projective relative to $R$ for any $F \in \mathcal{F}$. We give some characterizations of $\mathcal{F}$-perfect rings. For example, we show that a ring $R$ is right $\mathcal{F}$-perfect if and only if $\mathcal{F}$-covers of finitely generated modules are projective. Moreover, we define $\mathcal{F}$-perfect modules and investigate some properties of them.

**Key words:** $\mathcal{F}$-perfect ring, $\mathcal{F}$-cover, $\mathcal{F}$-perfect module, cotorsion theory, projective module

**2000 MR subject classification:** 16D50, 16D40, 16L30

**Document code:** A

**Article ID:** 1674-5647(2013)01-0041-10

## 1 Introduction

In 1953, Eckmann and Schopf[1] proved the existence of injective envelopes of modules over any associative ring. The dual problem, that is, the existence of projective covers was studied by Bass[2] in 1960. In spite of the existence of injective envelopes over any ring, he proved that over a ring $R$, all right modules have projective covers if and only if $R$ is a right perfect ring. In [3], a ring $R$ is called right almost-perfect if every flat right $R$-module is projective relative to $R$, and proved that a ring is right almost-perfect if and only if flat covers of finitely generated modules are projective. In this article, we introduce the concept of $\mathcal{F}$-perfect rings. We give some characterizations of $\mathcal{F}$-perfect rings. For example, we show that a ring $R$ is right $\mathcal{F}$-perfect if and only if $\mathcal{F}$-covers of finitely generated modules are projective.

***Received date:** Aug. 23, 2010.

Let $\mathcal{X}$ be a class of $R$-modules. We denote

$$\begin{aligned} {}^{\perp}\mathcal{X} &= \ker \operatorname{Ext}^1(\ \cdot\ , X) = \{M \mid \operatorname{Ext}^1(M, X) = 0,\ \forall X \in \mathcal{X}\},\\ \mathcal{X}^{\perp} &= \ker \operatorname{Ext}^1(X,\ \cdot\ ) = \{N \mid \operatorname{Ext}^1(X, N) = 0,\ \forall X \in \mathcal{X}\}. \end{aligned}$$

A pair $(\mathcal{F},\ \mathcal{C})$ of classes of $R$-modules is called a cotorsion theory if $\mathcal{F}^{\perp} = \mathcal{C}$ and $\mathcal{F} = {}^{\perp}\mathcal{C}$ (see [4]). A cotorsion theory $(\mathcal{F},\ \mathcal{C})$ is called complete if every $R$-module has a special $\mathcal{C}$-preenvelope (and a special $\mathcal{F}$-precover) (see [5]). A cotorsion theory $(\mathcal{F},\ \mathcal{C})$ is called perfect if every $R$-module has a $\mathcal{C}$-envelope and an $\mathcal{F}$-cover (see [6, 7]). A cotorsion theory $(\mathcal{F},\ \mathcal{C})$ is said to be hereditary if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathcal{F}$ then $L'$ is also in $\mathcal{F}$, or equivalently, if $0 \to C' \to C \to C'' \to 0$ is exact with $C', C \in \mathcal{C}$ then $C''$ is also in $\mathcal{C}$ (see [8]).

Let $R$ be a ring and $\mathscr{C}$ be a class of $R$-modules which is closed under isomorphic copies. A $\mathscr{C}$-precover of an $R$-module $M$ is a homomorphism $\varphi : F \to M$ with $F \in \mathscr{C}$ such that for any homomorphism $\psi : G \to M$ with $G \in \mathscr{C}$, there exists $\mu : G \to F$ such that $\varphi\mu = \psi$. A $\mathscr{C}$-precover $\varphi : F \to M$ is said to be a $\mathscr{C}$-cover if every endomorphism $\lambda$ of $F$ with $\varphi\lambda = \varphi$ is an automorphism of $F$. Dually, a $\mathscr{C}$-preenvelope and a $\mathscr{C}$-envelope of an $R$-module are defined.

In [4] a ring $R$ is called right almost-perfect if every flat right $R$-module is projective relative to $R$; equivalently, flat covers of finitely generated right $R$-modules are projective. It was shown that right perfect rings are right almost-perfect, and right almost-perfect rings are semiperfect, but not conversely. In Section 2, we introduce the notion of $\mathcal{F}$-perfect rings as a generalization of the notion of almost-perfect rings, that is, we call a ring $R$ $\mathcal{F}$-perfect in case $F$ is projective relative to $R$ for any $F \in \mathcal{F}$. We give some characterizations of $\mathcal{F}$-perfect rings. For example, in Theorem 2.1 we show that a ring $R$ is right $\mathcal{F}$-perfect if and only if $\mathcal{F}$-covers of finitely generated modules are projective. And in Theorem 2.3 we prove that a ring $R$ is right $\mathcal{F}$-perfect if and only if for every right $R$-modules $F$ with $F \in \mathcal{F}$, if

$$F = P + U,$$

where $P$ is a finitely generated projective summand of $F$ and $U \leq F$, then

$$F = P \oplus V \qquad \text{for some } V \leq U.$$

In Section 3, we introduce the notion of $\mathscr{F}$-perfect modules, that is, let $(\mathcal{F},\ \mathcal{C})$ be a perfect cotorsion theory. We call an $R$-module $M$ $\mathscr{F}$-perfect in case the $\mathscr{F}$-cover of every factor module of $M$ is projective. We show that $\mathscr{F}$-perfectness is closed under factor modules, extensions, and finite direct sums. Also some characterizations of $\mathscr{F}$-perfect modules are given.

Throughout this article, all rings are associative with identity, and all modules are unitary right modules unless stated otherwise. For a ring $R$, let $J = J(R)$ be the Jacobson radical of $R$. $(\mathcal{F},\ \mathcal{C})$ denotes a cotorsion theory. $\mathcal{F}$ (resp., $\mathcal{C}$) denotes the $\mathcal{F}$ (resp., $\mathcal{C}$) of the cotorsion theory $(\mathcal{F},\ \mathcal{C})$ unless stated otherwise.

General background materials can be found in [4, 9–10].

## 2 $\mathcal{F}$-perfect Rings

Let $R$ be a ring, and $(\mathcal{F},\ \mathcal{C})$ be a cotorsion theory.

**Lemma 2.1**[11] *Let $U$ be an $R$-module.*

(1) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $R$-modules and $U$ is $M$-projective, then $U$ is projective relative to both $M'$ and $M''$.*

(2) *If $U$ is projective relative to each $R$-module $M_i$ $(1 \le i \le n)$, then $U$ is $\bigoplus_{i=1}^{n} M_i$-projective.*

*Moreover, if $U$ is finitely generated and $M_\alpha$ $(\alpha \in A)$, then $U$ is projective relative to $\oplus_A M_\alpha$.*

**Definition 2.1** *Let $(\mathcal{F},\ \mathcal{C})$ be a cotorsion theory. A ring $R$ is called right $\mathcal{F}$-perfect if every right $R$-module $F$ with $F \in \mathcal{F}$ is projective relative to $R$. Left $\mathcal{F}$-perfect rings are defined similarly. If $R$ is both left and right $\mathcal{F}$-perfect, then $R$ is called an $\mathcal{F}$-perfect ring.*

**Remark 2.1** Let $R$ be a ring.

(1) Let $\mathcal{F}$ be the class of flat right $R$-modules. Then $R$ is $\mathcal{F}$-perfect if and only if $R$ is $A$-perfect.

(2) Let $\mathcal{F}$ be the class of right $R$-modules of flat dimension at most $n$. Then $\mathcal{F}$-perfect rings are $A$-perfect, but $A$-perfect rings are not necessarily $\mathcal{F}$-perfect.

(3) Let $\mathcal{F}$ be the class of $n$-flat right $R$-modules. If $R$ is $A$-perfect, then $R$ is $\mathcal{F}$-perfect (since $(\mathcal{F}_n,\ \mathcal{C}_n)$ is a complete hereditary cotorsion, where $\mathcal{F}_n$ (resp., $\mathcal{C}_n$) denotes the class of modules all $n$-flat (resp., $n$-cotorsion) right $R$-modules. And $n$-flat right $R$-modules is flat (see [12])). But if $R$ is $\mathcal{F}$-perfect, then $R$ is not necessarily $A$-perfect.

(4) Let $R$ be a right coherent ring, and

$$\mathcal{F} = \mathcal{F}\mathscr{P}_n,$$

where $\mathcal{F}\mathscr{P}_n$ is the class of all right $R$-modules of $FP$-injective dimension at most $n$. Then $(\mathcal{F}\mathscr{P}_n,\ \mathcal{F}\mathscr{P}_n^{\perp})$ is a perfect cotorsion theory (see [13]). But $A$-perfect rings are not necessarily $\mathcal{F}$-perfect and $\mathcal{F}$-perfect rings are not necessarily $A$-perfect.

**Lemma 2.2** *Let $(\mathcal{F},\ \mathcal{C})$ be a cotorsion theory, and $\phi : F \to M$ be an $\mathcal{F}$-cover of the $R$-module $M$. If $F$ is projective, then $\phi : F \to M$ is a projective cover of $M$.*

*Proof.* Since $\phi : F \to M$ is an $\mathcal{F}$-cover of the $R$-module $M$, $\phi$ is an epimorphism. Now let $L + \ker\phi = F$ with $L \le F$. So $\phi|_L : L \to M$ is an epimorphism. By the projectivity of $F$, there is $\lambda : F \to L \subseteq F$ such that

$$\phi\lambda = \phi.$$

Since $\phi : F \to M$ is an $\mathcal{F}$-cover of the $R$-module $M$, $\lambda$ is an automorphism of $F$, and hence

$$L = F.$$

Therefore,

$$\ker\phi \ll F,$$

and so $\phi : F \to M$ is a projective cover of $M$.

**Lemma 2.3**[10] *Let $f : F \to M$ be an $\mathcal{F}$-cover of the $R$-module $M$, and $K = \ker f$. Then $\mathrm{Ext}^1_R(G, K) = 0$ for any $G \in \mathcal{F}$.*

**Theorem 2.1** *Let $R$ be a ring. For the following statements:*

(1) *$R$ is right $\mathcal{F}$-perfect;*

(2) *$R$ is semiperfect and $\mathcal{F}$-covers of finitely generated $R$-modules are finitely generated;*

(3) *Finitely generated $\mathcal{F}$ right $R$-modules are projective and $\mathcal{F}$-covers of finitely generated right $R$-modules are finitely generated;*

(4) *$\mathcal{F}$-covers of finitely generated right $R$-modules are projective;*

(5) *$\mathcal{F}$-covers of cyclic right $R$-modules are projective,*

*we have* (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) *and* (1) $\Rightarrow$ (2).

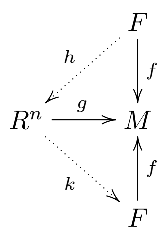
*Proof.* (1)$\Rightarrow$(2). Let $M$ be a finitely generated right $R$-module and $f : F \to M$ be an $\mathcal{F}$-covers of $M$. Suppose that $g : R^n \to M$ is an epimorphism. Since $F$ is $R$-projective, by Lemma 2.1, $F$ is $R^n$-projective. So there exists $h : F \to R^n$ such that

$$gh = f.$$

As $f : F \to M$ is a flat cover of $M$, there exists $k : R^n \to F$ such that

$$fk = g.$$

Thus we have the following commutative diagram:




Therefore,

$$fkh = f.$$

By the definition of an $\mathcal{F}$-cover, $kh$ must be an automorphism of $F$. Thus $k : R^n \to F$ is a split epimorphism. That is, $F$ is a finitely generated projective $R$-module. By Lemma 2.4, $f : F \to M$ is a projective of $M$, and hence $R$ is semiperfect.

(1)$\Rightarrow$(3). By the proof of (1)$\Rightarrow$(2), $\mathcal{F}$-covers of finitely generated right $R$-modules are finitely generated. Now we show that finitely generated $\mathcal{F}$ right $R$-modules are projective. Let $M$ be a finitely generated right $R$-module with $M \in \mathcal{F}$. Then there exists a projective cover $p : P \to M$ with $P$ finitely generated. Since $R$ is right $\mathcal{F}$-perfect, any $F \in \mathcal{F}$ is $P$-projective by Lemma 2.1. That is, for any homomorphism $f : F \to M$, there exists $g : F \to P$ such that

$$pg = f.$$

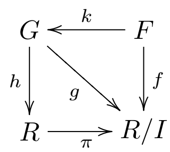
So $p : P \to M$ is an $\mathcal{F}$-cover of $M$, and hence $K \in \mathcal{C}$ by Lemma 2.3. That is,

$$\mathrm{Ext}^1_R(M, K) = 0,$$

the sequence $0 \to K \to P \to M \to 0$ is split, and therefore $M$ is projective.

(3)⇒(4) and (4)⇒(5) are clear.

(5)⇒(1). Let $F \in \mathcal{F}$, $I$ be an ideal of $R$, $\pi : R \to R/I$ be the natural epimorphism and $f : F \to R/I$ be a homomorphism, $g : G \to R/I$ be a $\mathcal{F}$-cover of $R/I$. By hypothesis, $G$ is projective, and hence there is $h : G \to R$ such that $g = \pi h$. There exists $k : F \to G$ such that $f = gk$ by the definition of $\mathcal{F}$-cover.




Put $\bar{f} = hk$. Then $\pi \bar{f} h = \pi hk = f$. Hence $R$ is a right $\mathcal{F}$-perfect ring.

**Corollary 2.1** ([3], Theorem 3.7) *For a ring $R$, the following statements are equivalent*:

(a) *$R$ is right $A$-perfect*;

(b) *$R$ is semiperfect, and flat covers of finitely generated right $R$-modules are finitely generated*;

(c) *Finitely generated flat right $R$-modules are projective, and flat covers of finitely generated right $R$-modules are finitely generated*;

(d) *Flat covers of finitely generated right $R$-modules are projective*;

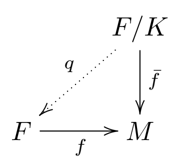
(e) *Flat covers of cyclic right $R$-modules are projective.*

**Lemma 2.4** *Let $f : F \to M$ be an $\mathcal{F}$-cover of the $R$-module $M$. If $K \subseteq \ker f$ and $F/K \in \mathcal{F}$, then $K = 0$.*

*Proof.* Suppose that $K \leq \ker f$ and $F/K \in \mathcal{F}$. Let $p : F \to F/K$ be the natural epimorphism. So $f$ induces $\bar{f} : F/K \to M$ such that

$$f = \bar{f} p.$$

Since $F/K \in \mathcal{F}$ and $f : F \to M$ be an $\mathcal{F}$-cover of the $R$-module $M$, there exists $q : F/K \to F$ with $fq = \bar{f}$. That is, we get the following commutative diagram:




Therefore,

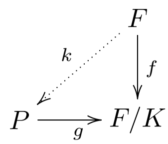
$$f = \bar{f} p = fqp.$$

Thus $qp$ is an automorphism of $F$ and so

$$K = \ker p \subseteq \ker qp = 0.$$

**Theorem 2.2** *Let $R$ be a ring. Then $R$ is right $\mathcal{F}$-perfect if and only if for any $F \in \mathcal{F}$, and $K \leq F$ if $F/K$ is cyclic (finitely generated), then $F = P \oplus Q$ with $Q \subseteq K$ and $P$ is a projective $R$-module.*

*Proof.* Suppose that $R$ is right $\mathcal{F}$-perfect. Let $F$ be a right $R$-module with $F \in \mathcal{F}$ and $K \le F$ with $F/K$ being cyclic (finitely generated). Suppose that $g : P \to F/K$ is an $\mathcal{F}$-cover of $F/K$ and $f : F \to F/K$ is the natural epimorphism. Since $R$ is right $\mathcal{F}$-perfect, $P$ is projective, and so there is $h : P \to F$ with $fh = g$. By the definition of the $\mathcal{F}$-cover, there exists $k : F \to P$ with $f = gk$, i.e., we have the commutative diagram:




Thus $g = gkh$. Therefore, $kh$ is an automorphism of $P$, and so

$$F = \mathrm{im} h \oplus \ker k.$$

Hence $\mathrm{im} h \cong P$ is projective, and

$$\ker k \subseteq \ker f = K.$$

Conversely, let $M$ be a cyclic (finitely generated) $R$-module and $f : F \to M$ be an $\mathcal{F}$-cover of $M$. Since $F/\ker f \cong M$ is cyclic (finitely generated), by hypothesis,

$$F = P \oplus Q,$$

where $Q \subseteq K$, and $P$ is a projective $R$-module. So $F/Q \cong P$ is projective. By Lemma 2.4, $Q = 0$. Therefore, $F = P$ is projective, and so $R$ is right $\mathcal{F}$-perfect by Theorem 2.1.

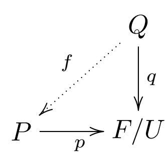
**Theorem 2.3** *A ring $R$ is right $\mathcal{F}$-perfect if and only if for every right $R$-module $F$ with $F \in \mathcal{F}$; if $F = P + U$, where $P$ is a finitely generated projective summand of $F$ and $U \le F$, then*

$$F = P \oplus V \qquad for\ some\ V \le U.$$

*Proof.* Suppose that $R$ is right $\mathcal{F}$-perfect. Let $F$ be a right $R$-module with $F \in \mathcal{F}$ and $F = P + U$, where $P$ is a finitely generated projective summand of $F$ and $U \le F$. Assume that $F = P \oplus Q$, and $p : P \to F/U$ and $q : Q \to F/U$ be the canonical mappings. Since $Q \in \mathcal{F}$ and $R$ is right $\mathcal{F}$-perfect, by Lemma 2.1, $Q$ is $P$-projective. So there exists $f : Q \to P$ such that

$$pf = q,$$

that is, we have the following commutative diagram:




This means that for any $x \in Q$,

$$x + U = f(x) + U,$$

and hence $(1 - f)(Q) \subseteq U$.

Now we show that

$$F = P \oplus (1 - f)Q.$$

We have

$$F = P + Q \subseteq P + f(Q) + (1 - f)(Q) = P + (1 - f)(Q).$$

Now let $x \in P \cap (1 - f)(Q)$. So

$$x = (1 - f)(y) \qquad \text{for some } y \in Q.$$

Thus

$$y = x + f(y) \in P \cap Q = 0,$$

and so $x$=0.

Conversely, let $G \in \mathcal{F}$. We show that $G$ is $R$-projective, and so $R$ is right $\mathcal{F}$-perfect.

Suppose that $p : R \to W$ is an epimorphism and $g : G \to W$ is an homomorphism. Let

$$F = G \oplus R \in \mathcal{F}$$

and

$$U = (x, y) \in F : g(x) + p(y) = 0.$$

Since $p$ is epimorphism,

$$U + R = F.$$

By hypothesis,

$$F = V \oplus R \qquad \text{for some } V \subseteq U.$$

Let $f : F \to R$ be the projection with respect to the decomposition

$$F = V \oplus R.$$

Let $h = f|_G : G \to R$. Since

$$(1 - f)(G) \subseteq (1 - f)(F) = V \subseteq U,$$

for any $x \in G$, we have

$$(1 - h)(x) = (x, -h(x)) \in U,$$

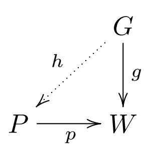
and so

$$g(x) - ph(x) = 0,$$

that is,

$$g = ph,$$

i.e., we have the following commutative diagram:




Consequently, $R$ is right $\mathcal{F}$-perfect.

**Proposition 2.1** *Let $F$ be an $R$-module with $F \in \mathcal{F}$ and $f : F \to M$ be an epimorphism. If $\ker f \in \mathcal{C}$, then $f : F \to M$ is an $\mathcal{F}$-precover of $M$.*

*Proof.* Since $f : F \to M$ is an epimorphism, the sequence

$$0 \to \ker f \to F \to M \to 0$$

is exact. This induces the exact sequence

$$\mathrm{Hom}_R(X, F) \to \mathrm{Hom}_R(X, M) \to \mathrm{Ext}^1_R(X, \ker f)$$

for any $X \in \mathcal{F}$. By hypothesis,

$$\mathrm{Ext}^1_R(X,\ \ker f) = 0,$$

and so

$$\mathrm{Hom}_R(X, F) \to \mathrm{Hom}_R(X, M) \to 0$$

is exact. Hence $f : F \to M$ is an $\mathcal{F}$-precover of $M$.

**Theorem 2.4** *Let $R$ be a ring and $I$ any right ideal of $R$. Let $(\mathcal{F},\ \mathcal{C})$ be a cotorsion theory such that if $C \in \mathcal{C}$ as an $R/I$-module, then $C \in \mathcal{C}$ as $R$-module. Then $R$ is right $\mathcal{F}$-perfect if and only if $I \in \mathcal{C}$.*

*Proof.* Let $F$ be a right $R$-module with $F \in \mathcal{F}$, and $I$ be a right ideal of $R$. The exact sequence

$$0 \to I \to R \to R/I \to 0$$

induces the exact sequence

$$0 \to \mathrm{Hom}_R(F, I) \to \mathrm{Hom}_R(F, R) \to \mathrm{Hom}_R(F, R/I) \to \mathrm{Ext}^1_R(F, I).$$

Since $I \in \mathcal{C}$,

$$\mathrm{Ext}^1_R(F, I) = 0,$$

and so $\mathrm{Hom}_R(F, R) \to \mathrm{Hom}_R(F, R/I)$ is an epimorphism. Therefore, $F$ is projective relative to $R$, and hence $R$ is right $\mathcal{F}$-perfect.

Conversely, suppose that $R$ is right $\mathcal{F}$-perfect. Let $J = J(R)$. By Theorem 2.1, $\mathcal{F}$-covers of cyclic right $R$-modules are projective, and hence $\mathcal{F}$-covers and projective covers of cyclic right $R$-modules are the same. Since the natural map $p : R \to R/J$ is the projective cover of the cyclic right $R$-module $R/J$, it is also its $\mathcal{F}$-cover. Thus, by Lemma 2.3,

$$J = \ker p \in \mathcal{C}.$$

Furthermore, $R/J$ is a semisimple ring, and so $R/J$ is injective as an $R/J$-module. By hypothesis, $R/J$ is injective as an $R$-module, and so $R/J \in \mathcal{C}$ as a right $R$-module. Now consider the exact sequence

$$0 \to J \to R \to R/J \to 0.$$

Since $\mathcal{C}$ is closed under extensions, $R \in \mathcal{C}$. Let $I$ be a proper right ideal of $R$, and let $F \in \mathcal{F}$. The exact sequence

$$0 \to I \to R \to R/I \to 0$$

induces the exact sequence

$$\mathrm{Hom}_R(F, R) \to \mathrm{Hom}_R(F, R/I) \to \mathrm{Ext}^1_R(F, I) \to \mathrm{Ext}^1_R(F, R) = 0.$$

Since $R$ is right $\mathcal{F}$-perfect, $\mathrm{Hom}_R(F, R) \to \mathrm{Hom}_R(F, R/I)$ is an epimorphism. Therefore,

$$\mathrm{Ext}^1_R(F, I) = 0,$$

and hence $I \in \mathcal{C}$.

# 3 $\mathcal{F}$-perfect Modules

In this section, we assume that $(\mathcal{F},\ \mathcal{C})$ is a perfect cotorsion theory.

**Definition 3.1** *Let $M$ and $N$ be $R$-modules. Then $N$ is said to be $M$-cyclic (respectively, finitely $M$-generated) if there is an epimorphism $M \to N$ (respectively, $M^n \to N$ for some $n \geq 1$).*

**Definition 3.2** *We call an $R$-module $M$ $\mathcal{F}$-perfect if $\mathcal{F}$-cover of every $M$-cyclic $R$-module is projective.*

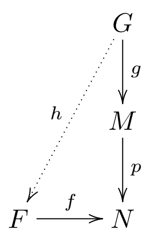
**Proposition 3.1** *Let $M$ be an $R$-module. Then $M$ is $\mathcal{F}$-perfect if and only if every $R$-module $F \in \mathcal{F}$ is $M$-projective and the $\mathcal{F}$-cover of $M$ is projective.*

*Proof.* Suppose that $M$ is $\mathcal{F}$-perfect. Let $F$ be an $R$-module with $F \in \mathcal{F}$. We show that $F$ is $M$-projective. Let $p : M \to N$ be an epimorphism, and $f : F \to N$ be a homomorphism. Suppose that $g : G \to N$ is an $\mathcal{F}$-cover of $N$. So there is $h : F \to G$ with $gh = f$. Since $M$ is $\mathcal{F}$-perfect, $G$ is projective. Thus there is $q : G \to M$ with $pq = g$. Therefore,

$$pqh = gh = f.$$

So $F$ is $M$-projective. It is easy to prove that the $\mathcal{F}$-cover of $M$ is projective.

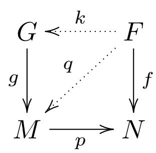
Conversely, let $N$ be an $M$-cyclic $R$-module and $f : F \to N$ be an $\mathcal{F}$-cover of $N$. We want to show that $F$ is projective. Let $p : M \to N$ be an epimorphism and $g : G \to M$ be an $\mathcal{F}$-cover of $M$. There is $h : G \to F$ such that $fh = pg$ (by the definition of $\mathcal{F}$-cover), that is, we have the commutative diagram:




Since every $F \in \mathcal{F}$ is $M$-projective, there exists $q : F \to M$ with $pq = f$. Again by the definition of flat cover, there exists $k : F \to G$ with $gk = q$. Thus

$$f = pq = pgk = fhk,$$

i.e., we have the commutative diagram:




Therefore, $hk$ is an automorphism of $F$, and hence $F$ is isomorphic to a summand of $G$. Since $G$ is projective, $F$ is also projective. Consequently, $M$ is $\mathcal{F}$-perfect.

**Corollary 3.1** *The class of $\mathcal{F}$-perfect modules is closed under factor modules and extensions. In particular, for modules $M_1,\ M_2,\ \cdots,\ M_n$, the sum $\bigoplus_{i=1}^{n} M_i$ is $\mathcal{F}$-perfect if and only if each $M_i$ is $\mathcal{F}$-perfect.*

*Proof.* By the definition of $\mathcal{F}$-perfect modules and Proposition 3.1 the proof is clear.

**Proposition 3.2** *An $R$-module $M$ is $\mathcal{F}$-perfect if and only if for any $R$-module $F \in \mathcal{F}$ and any submodule $K$ of $F$, if $F/K$ is finitely $M$-generated (or $M$-cyclic), then $F = P \oplus Q$ with $P$ projective and $Q \subseteq K$.*

*Proof.* The proof is similar to that of Theorem 2.2.

# References

[1] Eckmann B, Schopf A. Über injektive moduln. *Archiv. Math.*, 1953, **4**: 75–78.

[2] Bass H. Finitistic dimension and a homological generalization of semi-primary rings. *Trans. Amer. Math. Soc.*, 1960, **95**: 466–488.

[3] Amini A, Ershad M, Sharif H. Rings over which flat covers of finitely generated modules are projective. *Comm. Algebra*, 2008, **36**: 2862–2871.

[4] Enochs E E, Jenda O M G. Relative Homological Algebra. Berlin: Walter de Gruyter, 2000.

[5] Trlifaj J. Covers, Envelopes, and Cotorsion Theories: Lecture Notes for the Workshop, Homological Methods in Module Theory. Cortona, September 10–16, 2000.

[6] Enochs E E, Jenda O M G, Torrecillas B, Xu J. Torsion Theory with Respect to Ext. Research Report 98–11. Department of Mathematics, University of Kentucky, May, 1998.

[7] García Rozas J R. Covers and Envelopes in the Category of Complexes of Modules. Boca Raton, FL: Chapman & Hall/CRC, 1999.

[8] Enochs E E, Jenda O M G, Lopez-Ramos J A. The existence of Gorenstein flat covers. *Math. Scand.*, 2004, **94**: 46–62.

[9] Rotman J J. An Introduction to Homological Algebra. New York: Academic Press, 1979.

[10] Xu J Z. Flat Covers of Modules: Lecture Notes in Math. **1634**. New York: Spring-Verlag, 1996.

[11] Anderson F W, Fuller K R. Rings and Categories of Modules. New York: Spring-Verlag, 1992.

[12] Mao L X, Ding N Q. Relative cotorsion modules and relative flat modules. *Comm. Algebra*, 2006, **34**: 2303–2317.

[13] Mao L X, Ding N Q. Envelopes and covers by modules of finite FP-injective and flat dimensions. *Comm. Algebra*, 2007, **35**: 833–849.