COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
29(1)(2013), 51–60

Dynamics and Long Time Convergence of the Extended Fisher-Kolmogorov Equation under Numerical Discretization*

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Communicated by Ma Fu-ming

Abstract: We present a numerical study of the long time behavior of approximation solution to the Extended Fisher–Kolmogorov equation with periodic boundary conditions. The unique solvability of numerical solution is shown. It is proved that there exists a global attractor of the discrete dynamical system. Furthermore, we obtain the long-time stability and convergence of the difference scheme and the upper semicontinuity $d(\mathcal{A}_{h,\tau}, \mathcal{A}) \to 0$. Our results show that the difference scheme can effectively simulate the infinite dimensional dynamical systems.

Key words: Extended Fisher–Kolmogorov equation, finite difference method, global attractor, long time stability and convergence

2000 MR subject classification: 65M06, 37L99, 35B40, 35Q55, 65M20

Document code: A

Article ID: 1674-5647(2013)01-0051-10

1 Introduction

The Extended Fisher-Kolmogorov (EFK) equation is given by

$$\frac{\partial u}{\partial t} + \beta \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2} + (u^3 - u) = 0, \qquad x \in \Omega,\ t \geq 0 \tag{1.1}$$

with the boundary condition

$$u(0, t) = u(x + L, t), \qquad x \in \Omega,\ t \geq 0 \tag{1.2}$$

and the initial condition

$$u(x, 0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where $\beta > 0$, $0 < L < +\infty$ and $\Omega = (0, L)$ is a bounded domain in $\mathbf{R}$ with boundary $\partial\Omega$, and u_0 is a given L-periodic function.

When $\beta = 0$ in (1.1), the standard Fisher-Kolmogorov equation was obtained (see [1–2]). Adding a stabilizing fourth order derivative term to the standard Fisher-Kolmogorov

***Received date:** Aug. 24, 2010.
Foundation item: The NSF (10871055) of China and the Fundamental Research Funds (HEUCFL20111102) for the Central Universities.

equation, the equation (1.1) is proposed and called as Extended Fisher-Kolmogorov equation (see [3–6]).

The equation (1.1) occurs in a variety of applications such as pattern formation in bistable systems, propagation of domain walls in liquid crystals, travelling waves in reaction diffusion systems and mezoscopic model of a phase transition in a binary system near the Lipschitz point (see [4, 7–9]). In particular, in the phase transitions near critical points (Lipschitz points), the higher order gradient terms in the free energy functional can no longer be neglected and the fourth order derivative becomes important.

There have been a number of papers in the literature dealing with equations similar to the equation (1.1) (see [10–13]). In recent years, attention has been focused on the connection between finite-dimensional dynamical system theory and the long-time behavior of solutions of a priori infinite-dimensional dynamical systems described by partial differential equations. In particular, the techniques have been developed to establish this connection in a rigorous and quantitative way by showing how the dimension of the global attractor may be estimated for some dissipative partial differential equations (see [14–17]). The long-time behavior of the solutions to (1.1) is studied theoretically in [18].

For the long-time computation of partial differential equations, the error estimate are important in both space and time directions. Simo and Armero[19] pointed out that the first order scheme with long time stability and convergence is more effective than the second order scheme. Recently, some useful results about equivalence of equi-attraction and continuous convergence of attractors in different spaces have been given in [20]. To compute a trajectory numerically, long-time computation generally suffers from error accumulation at the unavoidable exponential rates. A numerical trajectory eventually leaves the exact trajectory and no longer shows any information about the original trajectory. On the other hand, for dissipative system such as the EFK equation, if the discretization schemes are appropriately selected, the numerical trajectory is expected to approach a discrete attractor and it eventually enters and stays in a small neighborhood of the attractor.

For this reason, we consider the error estimates for a global attractor. Existence of attractors for the dissipative systems is proved. The remainder of this paper is organized as follows. In Section 2, we describe a new finite difference scheme for the EFK equation and prove that the difference scheme is uniquely solvable. In Section 3, we derive the priori error estimates for numerical solution to obtain the existence of a global attractor. In Section 4, we discuss the long time stability and convergence of the difference scheme and the upper semicontinuity $d(\mathcal{A}_{h,\tau}, \mathcal{A}) \to 0$.

2 Finite Difference Scheme and Unique Solvability of Difference Approximation

Let $h = L/J$ be the uniform step size in the spatial direction for a positive integer J. Let τ denote the uniform step size in the temporal direction. Denote $V_i^n = V(x_i, t_n)$ for $t_n = n\tau$,

$n = 0, 1, \cdots$ and

$$\mathbf{R}^J_{per} = \{V_i = (V_i)_{0\le i\le J} : V_i \in \mathbf{R} \text{ and } V_{i+J} = V_i,\ 0 \le i \le J\}.$$

We define the difference operator for a function $V_i \in \mathbf{R}^J_{per}$, respectively, as

$$\nabla^+_h V_i = \frac{V_{i+1} - V_i}{h}, \quad \nabla^-_h V_i = \frac{V_i - V_{i-1}}{h}, \quad \Delta_h V_i = \nabla^+_h(\nabla^-_h V_i), \quad \Delta^2_h V_i = \Delta_h(\Delta_h V_i).$$

Furthermore, we define operator $\partial_t V^n$ as

$$\partial_t V^n_i = \frac{V^{n+1}_i - V^n_i}{\tau}.$$

We now introduce the discrete L^2-inner product and the associated norm by

$$(V, W)_h = \sum_{i=1}^{J} V_i W_i h, \quad V,\ W \in \mathbf{R}^J_{per}, \qquad \|V\|_h = (V, V)^{\frac{1}{2}}_h.$$

The discrete H^k-seminorm $|\cdot|_{k,h}$, H^k-norm $\|\cdot\|_{k,h}$ and L^∞-norm $\|\cdot\|_{\infty,h}$ are defined, respectively, as

$$|V|_{k,h} = \|\nabla^{+k}_h V\|_h, \qquad \|V\|_{k,h} = \left(\sum_{l=0}^{k} \|\nabla^{+l}_h V\|^2_h\right)^{\frac{1}{2}}, \qquad \|V\|_{\infty,h} = \max_{1\le i\le J} |V_i|.$$

Let $\Omega_h = \{ih; 0 \le i \le J\}$. It is convenient to let $L^2_{per}(\Omega_h)$ and $H^k_{per}(\Omega_h)$ $(k \ge 1)$ denote the normed vector spaces, respectively, as

$$\{\mathbf{R}^J_{per}, \|\cdot\|_h\}, \qquad \{\mathbf{R}^J_{per}, \|\cdot\|_{k,h}\}.$$

Thanks to the periodicity of the discrete function $V \in H^k_{per}(\Omega_h)$, we have

$$|V|_{k,h} = \|\nabla^{+k}_h V\|_h = \|\nabla^{+l}_h \nabla^{-(k-l)}_h V\|_h, \qquad 0 \le l \le k.$$

Throughout this paper, we denote $c_i > 0$ as a generic constant independent of step sizes h. To obtain some important results, we introduce the following lemmas.

Lemma 2.1[21] *For $V, W \in \mathbf{R}^J_{per}$, there holds*

$$(V, -\nabla^+_h(\nabla^-_h W))_h = (\nabla^+_h V, \nabla^+_h W)_h = (-\nabla^+_h(\nabla^-_h V), W)_h.$$

Lemma 2.2[21] *For $V \in H^n_{per}(\Omega_h)$, we have*

$$\|\nabla^{+k}_h V\|_h \le K_1 \|\nabla^{+n}_h V\|^{\frac{k}{n}}_h \|V\|^{1-\frac{k}{n}}_h \tag{2.1}$$

and

$$\|\nabla^{+k}_h V\|_{\infty,h} \le K_2 \|V\|^{1-\frac{k+\frac{1}{2}}{n}}_h \left(\|\nabla^{+n}_h V\|_h + \frac{\|V\|_h}{L^n}\right)^{\frac{k+\frac{1}{2}}{n}} \tag{2.2}$$

for $0 \le k \le n$, where K_1 and K_2 are constants independent of h and the discrete function V.

Lemma 2.3[21] *For a function $V \in H^1_{per}(\Omega_h)$, the following inequality holds*:

$$c_1 \|V\|^2_h \le |V|^2_{1,h}.$$

According to Lemma 2.3 and (2.1), we have a lemma as follows.

Lemma 2.4 *For a function $V \in H^k_{per}(\Omega_h)$, we have*

$$c_k |V|^2_{k-1,h} \le |V|^2_{k,h}, \qquad 2 \le k \le 4.$$

Lemma 2.5 *For $s \in \mathbf{R}$, there hold*

$$\frac{1}{2}s^4 - c_5 \le (s^3 - s)s \le \frac{3}{2}s^4 + c_5, \qquad c_5 > 0, \tag{2.3}$$

$$\frac{3}{2}s^2 - c_6 \le (s^3 - s)' \le \frac{9}{2}s^2 + c_6, \qquad c_6 > 0. \tag{2.4}$$

We propose a new difference scheme for the solution of the problem (1.1)–(1.3) as follows:

$$\partial_t U_i^n + \beta \Delta_h^2 U_i^{n+1} - \Delta_h U_i^{n+1} + [(U_i^{n+1})^3 - U_i^{n+1}] = 0, \qquad 0 \le i \le J,\ n \ge 0, \tag{2.5}$$

$$U_i^n = U_{i+J}^n, \qquad 0 \le i \le J,\ n \ge 0, \tag{2.6}$$

$$U_i^0 = u_0(ih), \qquad i = 0, 1, \cdots, J. \tag{2.7}$$

Below, we prove the solvability of the discrete system (2.5)–(2.7).

Theorem 2.1 *The difference scheme* (2.5)–(2.7) *is uniquely solvable.*

Proof. For $\phi \in H_{per}^1(\Omega_h)$, we define a discrete function ϕ as follows:

$$\frac{\phi_i - \lambda U_i^n}{\tau} + \lambda\beta\Delta_h^2\phi_i - \lambda\Delta_h\phi_i + \lambda(\phi_i^3 - \phi_i) = 0, \qquad 0 \le i \le J,\ n \ge 0, \tag{2.8}$$

where $0 \le \lambda \le 1$. It defines a mapping $\phi = T_\lambda(\phi)$ of $H_{per}^1(\Omega_h)$ into itself. Obviously, the mapping $T_\lambda(\phi)$ is continuous for any $\phi \in H_{per}^1(\Omega_h)$. Since a difference solution is a fixed point of T_1, it only needs to prove the existence of the fixed point of T_1, i.e., it is sufficient to prove the uniform boundedness for the mapping T_λ with respect to the parameter $0 \le \lambda \le 1$ by the Leray-Schauder fixed point theorem. Taking an inner product of (2.8) with ϕ and using Lemma 2.1, we obtain

$$\frac{\|\phi\|_h^2 - \|U^n\|_h^2}{2\tau} + \lambda\beta|\phi|_{2,h}^2 + \lambda|\phi|_{1,h}^2 + \lambda(\phi^3 - \phi, \phi)_h \le 0, \qquad n \ge 0. \tag{2.9}$$

By (2.3) and (2.9), we get

$$\|\phi\|_h^2 \le \|U^n\|_h^2 + 2c_5 L\tau, \qquad n \ge 0.$$

This means that $\|\phi\|_h^2$ is uniformly bounded with respect to the parameter $0 \le \lambda \le 1$. Thus the solution of the difference scheme (2.5) with boundary conditions (2.6) exists.

Let U^{n+1} and V^{n+1} be the solutions of the discrete system (2.5)-(2.6) with initial conditions U^0 and V^0, respectively. Then $\varepsilon^{n+1} = U^{n+1} - V^{n+1}$ satisfies that

$$\partial_t \varepsilon_i^n + \beta\Delta_h^2\varepsilon_i^{n+1} - \Delta_h\varepsilon_i^{n+1} + \left\{[(U_i^{n+1})^3 - U_i^{n+1}] - [(V_i^{n+1})^3 - V_i^{n+1}]\right\} = 0, \tag{2.10}$$

$$\varepsilon_i^n = \varepsilon_{i+J}^n, \qquad 0 \le i \le J,\ n \ge 0, \tag{2.11}$$

$$\varepsilon_i^0 = U_i^0 - V_i^0, \qquad 0 \le i \le J. \tag{2.12}$$

Computing the inner product of (2.10) with ε^{n+1}, and using Lemma 2.1, we obtain

$$\frac{\|\varepsilon^{n+1}\|_h^2 - \|\varepsilon^n\|_h^2}{2\tau} + (((U^{n+1})^3 - U^{n+1}) - ((V^{n+1})^3 - V^{n+1}), \varepsilon^{n+1})_h \le 0. \tag{2.13}$$

It follows from (2.4), (2.13) and the mean value theorem that

$$\|\varepsilon^n\|_h^2 \le \frac{1}{1 - 2c_6\tau}\|\varepsilon^{n-1}\|_h^2 \le \cdots \le \mathrm{e}^{\frac{2c_6 n\tau}{1-2c_6\tau}}\|\varepsilon^0\|_h^2, \qquad n = 0, 1, 2, \cdots \tag{2.14}$$

(2.14) determines U_i^{n+1} uniquely. This completes the proof.

3 Global Attractor of Discrete Dynamical System

In this section, we consider the existence of a global attractor for the semigroup $\{(S_{h,\tau})^n\}_{n\geq 0}$ associated with the discrete system (2.5)–(2.7). Then the semigroup $\{(S_{h,\tau})^n\}_{n\geq 0}$ acting on $H^1_{per}(\Omega_h)$, $(S_{h,\tau})^n : H^1_{per}(\Omega_h) \to H^1_{per}(\Omega_h)$ for every $n \geq 0$ is defined by

$$U^n = (S_{h,\tau})^n U^0.$$

To obtain the existence of a global attractor, we introduce the following lemmas.

Lemma 3.1 *Suppose that u_0 is smooth enough. Then the solution of the difference scheme* (2.5)*,* (2.6) *and* (2.7) *is estimated as follows*:

$$\|U^n\|_h^2 \leq (\mathrm{e}^{-\frac{\alpha\tau}{1+\alpha\tau}})^n \|U^0\|_h^2 + \frac{2c_5 L}{\alpha}, \qquad n = 0, 1, 2, \cdots,$$

where $\alpha = 2(\beta c_1 + c_1 \cdot c_2)$. Furthermore, there exists a constant $\rho_0 > \sqrt{\dfrac{2c_5 L}{\alpha}}$ such that the ball

$$B_0^h = \{U \in L^2_{per}(\Omega_h); \|U\|_h \leq \rho_0\}$$

is a bounded absorbing set in $L^2_{per}(\Omega_h)$ under the semigroup $\{(S_{h,\tau})^n\}_{n\geq 0}$.

Proof. Taking an inner product of (2.5) with U^{n+1}, we have

$$\frac{\|U^{n+1}\|_h^2 - \|U^n\|_h^2}{2\tau} + \beta|U^{n+1}|_{2,h}^2 + |U^{n+1}|_{1,h}^2 + ((U^{n+1})^3 - U^{n+1}, U^{n+1})_h \leq 0.$$

An application of Lemma 2.3, Lemma 2.4 and (2.3) yields

$$\frac{\|U^{n+1}\|_h^2 - \|U^n\|_h^2}{2\tau} + (\beta c_1 \cdot c_2 + c_1)\|U^{n+1}\|_h^2 \leq c_5 L, \tag{3.1}$$

Let $\alpha = 2(\beta c_1 \cdot c_2 + c_1)$. Then, it follows from (3.1) that

$$\|U^{n+1}\|_h^2 \leq \frac{1}{1+\alpha\tau}\|U^n\|_h^2 + \frac{2c_5 L\tau}{1+\alpha\tau} \leq \cdots \leq (\mathrm{e}^{-\frac{\alpha\tau}{1+\alpha\tau}})^{n+1}\|U^0\|_h^2 + \frac{2c_5 L}{\alpha}, \qquad n \geq 0.$$

This completes the proof.

Lemma 3.2 *Suppose that u_0 is smooth enough. Then the solution of the difference scheme* (2.5)*,* (2.6) *and* (2.7) *is estimated as follows*:

$$|U^n|_{1,h}^2 \leq \frac{1}{c_2}(\mathrm{e}^{-\frac{\gamma\tau}{1+\gamma\tau}})^n |U^0|_{2,h}^2 + \frac{\kappa}{c_2\gamma}, \qquad n = 0, 1, 2, \cdots,$$

where

$$\gamma = \beta c_3 c_4 + 2c_3, \qquad \kappa = \frac{1}{\beta}\rho_0^2(\rho_0^2 + 1)^2.$$

Furthermore, there exists a constant $\rho_1 > \rho_0 + \sqrt{\dfrac{\kappa}{c_2\gamma}}$ such that the ball

$$B_1^h = \{U \in H^1_{per}(\Omega_h); \|U\|_{1,h} \leq \rho_1\}$$

is a bounded absorbing set in $H^1_{per}(\Omega_h)$ under the semigroup $\{(S_{h,\tau})^n\}_{n\geq 0}$.

Proof. Taking an inner product of (2.5) with $\Delta_h^2 U^{n+1}$, by Lemma 2.1, we have

$$\frac{|U^{n+1}|_{2,h}^2 - |U^n|_{2,h}^2}{2\tau} + \beta|U^{n+1}|_{4,h}^2 + |U^{n+1}|_{3,h}^2 + ((U^{n+1})^3 - U^{n+1}, \Delta_h^2 U^{n+1})_h \leq 0. \tag{3.2}$$

Using Young's inequality, we obtain

$$\frac{|U^{n+1}|_{2,h}^2-|U^n|_{2,h}^2}{2\tau}+\beta|U^{n+1}|_{4,h}^2+|U^{n+1}|_{3,h}^2\le\frac{1}{2\beta}\|(U^{n+1})^3-U^{n+1}\|_h^2+\frac{\beta}{2}|U^{n+1}|_{4,h}^2,$$

It follows from Lemmas 2.4 and 3.1 that

$$\frac{|U^{n+1}|_{2,h}^2-|U^n|_{2,h}^2}{2\tau}+\Big(\frac{\beta}{2}c_3c_4+c_3\Big)|U^{n+1}|_{2,h}^2\le\frac{1}{2\beta}\rho_0^2(\rho_0^2+1)^2,\qquad n\ge 0. \tag{3.3}$$

Let $\gamma=\beta c_3c_4+2c_3$, $\kappa=\frac{1}{\beta}\rho_0^2(\rho_0^2+1)^2$. Thus, (3.3) yields

$$|U^{n+1}|_{2,h}^2\le\frac{1}{1+\gamma\tau}|U^n|_{2,h}^2+\frac{\kappa\tau}{1+\gamma\tau}\le\cdots\le(\mathrm{e}^{-\frac{\gamma\tau}{1+\gamma\tau}})^{n+1}|U^0|_{2,h}^2+\frac{\kappa}{\gamma},\qquad n\ge 0. \tag{3.4}$$

Hence, we obtain from Lemma 2.4 that

$$|U^n|_{1,h}^2\le\frac{1}{c_2}(\mathrm{e}^{-\frac{\gamma\tau}{1+\gamma\tau}})^{n}|U^0|_{2,h}^2+\frac{\kappa}{c_2\gamma},\qquad n=0,1,2,\cdots$$

This completes the proof.

According to the Lemma 3.1, Lemma 3.2 and (2.2), we have

Theorem 3.1 *Assume that u_0 is sufficiently regular. Let U^n be the solution of the difference scheme* (2.5)–(2.7)*. Then, there exists a positive constant C independent of h and of τ such that*

$$\|U^n\|_{\infty,h}\le C,\qquad n\ge 0.$$

Obviously, the family of operators $\{(S_{h.\tau})^n\}_{n\ge0}$ satisfy the semigroup properties

$$(S_{h,\tau})^m(S_{h,\tau})^n=(S_{h,\tau})^{m+n},\quad\forall\, m,n\ge 0,\qquad (S_{h,\tau})^0=I.$$

By the above estimates, there exists a bounded set B_1^h which is absorbing in $H^1_{per}(\Omega_h)$ under $\{(S_{h.\tau})\}_{n\ge0}$. Using Theorem 1.1 in [22], we obtain the following theorem.

Theorem 3.2 *Suppose that the conditions of Lemma* 3.2 *are satisfied. Then the discrete dynamical system associated with the finite difference scheme* (2.5)–(2.7) *possesses a global attractor $\mathcal{A}_{h,\tau}$ in $H^1_{per}(\Omega_h)$, and*

$$\mathcal{A}_{h,\tau}=\bigcap_{n\ge0}\overline{\bigcup_{m\ge n}(S_{h,\tau})^mB_1^h}.$$

4 Long Time Convergence and Stability

Define the net function $u_i^n=u(x_i,t^n)$. Let $E_i^n=u_i^n-U_i^n$ and $f(u)=u-u^3$.

To prove long-time convergence, we make discussions for different cases.

(i) f is a smooth function satisfying

$$f'(s)\le 0,\qquad s\in R. \tag{4.1}$$

Then, we can obtain Proposition 4.1 as follows.

Proposition 4.1 *Suppose that f is a smooth function satisfying* (4.1) *and the solution $u(x,t)$ of* (1.1)–(1.3) *is sufficiently regular. Then, the solution of the difference scheme* (2.5), (2.6) *and* (2.7) *converges to the solution of the problem* (1.1)–(1.3) *in the discrete $L^2_{per}(\Omega_h)$-norm and the rate of convergence is $O(\tau+h^2)$.*

Proof. Making use of Taylor's expansion, we find

$$\partial_t u_i^n + \beta\Delta_h^2 u_i^{n+1} - \Delta_h u_i^{n+1} = f(u_i^{n+1}) + R_i^{n+1}, \qquad 0 \le i \le J,\ n \ge 0, \tag{4.2}$$

$$u_i^n = u_{i+J}^n, \qquad 0 \le i \le J,\ n \ge 0, \tag{4.3}$$

$$u_i^0 = u(ih, 0), \qquad 0 \le i \le J, \tag{4.4}$$

where R_i^{n+1} is the truncation errors of the difference scheme (4.2). It can be easily obtained that

$$\max_{1\le i\le J} |R_i^{n+1}| \le M(\tau + h^2), \qquad n \ge 0, \tag{4.5}$$

and the constant M is independent of τ and h.

Subtracting (4.2) from (2.5), we find

$$\partial_t E_i^n + \beta\Delta_h^2 E_i^{n+1} - \Delta_h E_i^{n+1} = [f(u_i^{n+1}) - f(U_i^{n+1})] + R_i^{n+1}, \quad 0 \le i \le J,\ n \ge 0, \tag{4.6}$$

$$E_i^n = E_{i+J}^n, \qquad 0 \le i \le J,\ n \ge 0, \tag{4.7}$$

$$E_i^0 = 0, \qquad 0 \le i \le J. \tag{4.8}$$

Taking in (4.6) the inner product with E^{n+1} and using Lemma 2.1, we obtain

$$\begin{aligned} &\frac{\|E^{n+1}\|_h^2 - \|E^n\|_h^2}{2\tau} + \beta|E^{n+1}|_{2,h}^2 + |E^{n+1}|_{1,h}^2 \\ \le& (f(u^{n+1}) - f(U^{n+1}), E^{n+1})_h + |(R^n, E^{n+1})_h|. \end{aligned} \tag{4.9}$$

An application of the mean value theorem and (4.1) yield

$$\frac{\|E^{n+1}\|_h^2 - \|E^n\|_h^2}{2\tau} + \beta|E^{n+1}|_{2,h}^2 + |E^{n+1}|_{1,h}^2 \le |(R^n, E^{n+1})_h| \le \|R^n\|_h \cdot \|E^{n+1}\|_h.$$

Using Young's inequality, Lemmas 2.3 and 2.4, we obtain

$$\frac{\|E^{n+1}\|_h^2 - \|E^n\|_h^2}{2\tau} + (\beta c_1 c_2 + c_1)\|E^{n+1}\|_h^2 \le \frac{1}{4(\beta c_1 c_2 + c_1)}\|R^n\|_h^2 + (\beta c_1 c_2 + c_1)\|E^{n+1}\|_h^2.$$

Then, we get

$$\|E^n\|_h^2 \le \|E^0\|_h^2 + \frac{\tau}{2(\beta c_1 c_2 + c_1)} \sum_{m=1}^{n} \|R^m\|_h^2. \tag{4.10}$$

Combining (4.5), (4.8) and (4.10), we find

$$\|u^n - U^n\|_h \le O(\tau + h^2), \qquad n = 0, 1, 2, \cdots, \tag{4.11}$$

which completes the proof of Proposition 4.1.

(ii) f is a smooth function satisfying

$$f'(s) > 0, \qquad s \in R. \tag{4.12}$$

In the following Proposition 4.2, our argument is based on some hypothesis as follows.

First, we consider the corresponding stationary problem

$$-\Delta_h u = f(u), \qquad x \in \Omega,\ t \ge 0, \tag{4.13}$$

$$u(x,t) = u(x+L, t), \qquad x \in \Omega,\ t \ge 0. \tag{4.14}$$

For any positive function $\omega \in C(\bar{\Omega})$, we consider the eigenvalue problem with weight

$$-\Delta_h \varphi = \mu\omega\varphi \qquad \text{in } \Omega,$$

$$\varphi(x,t) = \varphi(x+L, t), \qquad x \in \Omega,\ t \ge 0.$$

Let $\mu_1[\omega]$ denote its smallest eigenvalue. Recall that $\mu_1[\omega]$ can be characterized as

$$\mu_1[\omega] = \inf_{\varphi \in H_{per}^1} \frac{|\varphi|_{1,h}^2}{(\omega\varphi, \varphi)_h} > 0.$$

We assume that (4.13)-(4.14) has a classical solution $\underline{u}$, which is linearized stable in the sense that, for some real number δ,

$$\frac{1}{\mu_1[f^{'}(\underline{u})]} \leq \delta < 1.$$

Note that $\mu_1[f^{'}(\underline{u})]$ is well defined because of (4.12). We also assume that $H^1_{per}(\Omega)$ is such that (1.1)–(1.3) has a global classical solution u and $u(t) \to \underline{u}$ in $L^2_{per}(\Omega)$ as $t \to \infty$. Our results depend on upper bounds B_1 and B_2 for the $\|\cdot\|_{\infty,h}$ of u and the $\|\cdot\|_{2,h}$ of u_0

$$\|u(\cdot,t)\|_{\infty,h} \leq B_1, \quad t \geq 0, \qquad \|u_0\|_{2,h} \leq B_2.$$

In order to show the convergence of the finite difference approximate solutions, the following lemma is needed.

Lemma 4.1 *Suppose that f is a smooth function satisfying* (4.12) *and the solution $u(x,t)$ of* (1.1)–(1.3) *is sufficiently regular. For each $B_1, B_2 > 0$ and $0 < \delta < 1$, assume that*

(1) $\underline{u}$ *is a solution of* (4.13)-(4.14) *with*

$$\frac{1}{\mu_1[f^{'}(\underline{u})]} \leq \delta; \tag{4.15}$$

(2) $u_0 \in H^1_{per}(\Omega)$ *is such that* $u(t) \to \underline{u}$ *in* $L^2_{per}(\Omega)$ *as* $t \to \infty$;

(3) u_0 *and the corresponding solution u of* (1.1)–(1.3) *satisfy the bounds*

$$\|u(\cdot,t)\|_{\infty,h} \leq B_1, \quad t \geq 0, \qquad \|u_0\|_{2,h} \leq B_2. \tag{4.16}$$

Then, there is a number δ_1 such that

$$\frac{1}{\mu_1[\omega_i^n]} \leq \delta_1 < 1, \qquad n \geq N > 0, \quad N \in \mathbf{Z}^+ \tag{4.17}$$

where

$$\omega_i^n = \int_0^1 f^{'}(\xi U_i^n - (1-\xi)u_i^n)\mathrm{d}\xi, \qquad i = 0, 1, \cdots, J.$$

We are now ready for the proof of Proposition 4.2.

Proposition 4.2 *Let $N \in \mathbf{Z}^+$. Under the assumptions of Lemma* 4.2, *for τ sufficiently small, the solution of the difference scheme* (2.5), (2.6) *and* (2.7) *converges to the solution of the problem* (1.1)–(1.3) *in the discrete $L^2_{per}(\Omega_h)$-norm and the rate of convergence is $O(\tau + h^2)$.*

Proof. An application of the mean value theorem and Theorem 3.3, (4.9) yields

$$\frac{\|E^{n+1}\|_h^2 - \|E^n\|_h^2}{2\tau} \leq \Big(3C^2 + \frac{3}{2}\Big)\|E^{n+1}\|_h^2 + \frac{1}{2}\|R^{n+1}\|_h^2. \tag{4.18}$$

Let $\kappa = 6C^2 + 3$. Then (4.5), (4.8) and (4.18) yield

$$\|E^n\|_h^2 \leq \frac{1}{1-\kappa\tau}\|E^{n-1}\|_h^2 + \frac{\tau}{1-\kappa\tau}\|R^n\|_h^2 \leq \cdots \leq \frac{1}{\kappa}(\mathrm{e}^{\frac{\kappa n\tau}{1-\kappa\tau}})M^2L(\tau + h^2)^2. \tag{4.19}$$

For $0 < n \leq N$, it follows from (4.19) that

$$\|E^n\|_h = O(\tau + h^2). \tag{4.20}$$

We now turn to the long-time estimate. Using (4.9) and Lemma 4.2, we obtain

$$\begin{aligned}\frac{\|E^{n+1}\|_h^2 - \|E^n\|_h^2}{2\tau} + \beta|E^{n+1}|_{2,h}^2 &\leq |(R^{n+1}, E^{n+1})_h| \\ &\leq \|R^{n+1}\|_h \cdot \|E^{n+1}\|_h, \qquad n > N. \end{aligned} \tag{4.21}$$

Using Young's inequality, from Lemmas 2.3 and 2.4 we obtain

$$\frac{\|E^{n+1}\|_h^2 - \|E^n\|_h^2}{2\tau} + \beta c_1 c_2 \|E^{n+1}\|_h^2 \le \frac{1}{4\beta c_1 c_2}\|R^{n+1}\|_h^2 + \beta c_1 c_2 \|E^{n+1}\|_h^2, \qquad n > N.$$

It implies that

$$\begin{aligned}\|E^{n+1}\|_h^2 &\le \|E^n\|_h^2 + \frac{\tau}{2\beta c_1 c_2}\|R^{n+1}\|_h^2 \le \cdots \\ &\le \|E^0\|_h^2 + \frac{\tau}{2\beta c_1 c_2}\sum_{m=1}^{n+1}\|R^m\|_h^2, \qquad n > N. \qquad (4.22)\end{aligned}$$

Combining (4.8) and (4.22), we find

$$\|E^{n+1}\|_h^2 \le \frac{\tau}{2\beta c_1 c_2}\sum_{m=1}^{n+1}\|R^m\|_h^2 \equiv O(\tau + h^2)^2, \qquad n > N. \qquad (4.23)$$

Thus from (4.23), we get

$$\|u^n - U^n\|_h = O(\tau + h^2), \qquad n > N.$$

Combining (4.20) and (4.23), we find

$$\|u^n - U^n\|_h = O(\tau + h^2), \qquad n = 1, 2, \cdots.$$

This completes the proof.

Theorem 4.1 *Under the conditions of the Proposition* 4.1 *or Proposition* 4.2*, the solution of the difference scheme* (2.5)*,* (2.6) *and* (2.7) *converges to the solution of the problem* (1.1)–(1.3) *in the discrete* $L^2_{per}(\Omega_h)$*-norm and the rate of convergence is* $O(\tau + h^2)$.

Below, we can similarly prove stability of the difference solution.

Theorem 4.2 *Under the conditions of the Theorem* 4.1*, the solution of the problem* (2.5)*,* (2.6) *and* (2.7) *is long-time stable for initial data in the discrete* $L^2_{per}(\Omega_h)$*-norm.*

Now, we discuss the upper semiconsciousness of approximate attractor $\mathcal{A}_{h,\tau}$.

Theorem 4.3 *Suppose that the following conditions are satisfied:*

(1) $\{H_\eta\}_{0\le\eta\le\eta_0}$ *is a family of closed subspaces of a Banach space* H*, satisfying that* $\bigcup_{0\le\eta\le\eta_0} H_\eta$ *is dense in* H;

(2) $\{S_\eta(t) : H_\eta \to H_\eta\}_{t\ge 0}$ *are linear semi-group of operator,* $\mathcal{A}_\eta \subset H_\eta$ *and* $\mathcal{A} \subset H$ *are the global attractors of* $S_\eta(t)$ *and* $S(t)$*, respectively*;

(3) *For every compact interval* $I \subset (0, +\infty)$,

$$\delta_\eta(I) = \sup_{u_0\in H_\eta}\sup_{t\in I}\operatorname{dist}(S_\eta(t)u_0, S(t)u_0) \to 0 \qquad \text{as } \eta \to 0.$$

Then $\mathcal{A}_\eta$ *converges to* $\mathcal{A}$ *in the sense of semi-distance:*

$$\operatorname{dist}(\mathcal{A}_\eta, \mathcal{A}) \to 0 \qquad \text{as } \eta \to 0,$$

where

$$\operatorname{dist}(\mathcal{A}_\eta, \mathcal{A}) = \sup_{u\in\mathcal{A}_\eta}\inf_{v\in\mathcal{A}}\|u - v\|_H.$$

Let $H = H^1_{per}(\Omega)$, $H_\eta = H^1_{per}(\Omega_h)$. Obviously, $H^1_{per}(\Omega_h)$ is dense in $H = H^1_{per}(\Omega)$, and the operator $\{(S_{h,\tau})^n\}_{n\ge 0} : H^1_{per}(\Omega_h) \to H^1_{per}(\Omega_h)$ is continuous. By Theorems 3.2, 4.1 and 4.3, we have

Theorem 4.4 *Suppose that the conditions of Theorem* 4.1 *are satisfied. Then we have*

$$\mathrm{dist}(\mathcal{A}_{h,\tau}, \mathcal{A}) \to 0, \qquad as\ \tau \to 0,\ h \to 0.$$

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