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Existence and Blow-up of Solutions for a Nonlinear Parabolic System with Variable Exponents*

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Abstract: In this paper, we study a nonlinear parabolic system with variable exponents. The existence of classical solutions to an initial and boundary value problem is obtained by a fixed point theorem of the contraction mapping, and the blow-up property of solutions in finite time is obtained with the help of the eigenfunction of the Laplace equation and a delicated estimate.

Key words: blow-up, existence, nonlinear parabolic system, variable exponent

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1 Introduction

In this paper, we study the existence and blow-up of solutions to the nonlinear parabolic system

$$\begin{cases} u_t = \Delta u + \alpha f(v), & (x,t) \in Q_T, \\ v_t = \Delta v + \beta g(u), & (x,t) \in Q_T, \\ u(x,t) = 0, \quad v(x,t) = 0, & (x,t) \in S_T, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\alpha > 0$, $\beta > 0$ are constants, $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $Q_T = \Omega \times [0, T)$ with $0 < T < \infty$, S_T denotes the lateral boundary of the cylinder Q_T, and

$$f(v) = v^{p_1(x)}, \qquad g(u) = u^{p_2(x)}$$

are source terms. We also assume that the exponents

$$p_1(x),\ p_2(x):\ \Omega \to (1, +\infty)$$

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satisfy the following conditions:

$$1 < p_1^- = \inf_{x\in\Omega} p_1(x) \leqslant p_1(x) \leqslant p_1^+ = \sup_{x\in\Omega} p_1(x) < +\infty, \tag{1.2}$$

$$1 < p_2^- = \inf_{x\in\Omega} p_2(x) \leqslant p_2(x) \leqslant p_2^+ = \sup_{x\in\Omega} p_2(x) < +\infty. \tag{1.3}$$

When p_1, p_2 are constants, Escobedo and Herrero[1] investigated the boundedness and blow up of solutions to the problem (1.1). Furthermore, the authors also studied the uniqueness and global existence for some solutions (see [2]), and there have been also many results about the existence, boundedness and blow up property of the solutions (see [3–6]).

The motivation of our work is mainly due to [7], where the authors studied the following parabolic problem involving a variable exponent:

$$\begin{cases} u_t = \Delta u + f(u), & (x,t) \in \Omega \times [0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T), \end{cases} \tag{1.4}$$

where $\Omega \in \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and the source term is of the form

$$f(u) = a(x)u^{p(x)}$$

or

$$f(u) = a(x)\int_\Omega u^{q(x)}(y,t)\mathrm{d}y.$$

The parabolic problems with sources as the one in (1.4) can be used to model chemical reactions, heat transfer or population dynamic, etc. The readers can refer to [8–14] and the references therein.

However, to the best of our knowledge, there are no results about the existence, blow-up properties of solutions to the systems of parabolic equations with variable exponents. Our main aim in this paper is to study the problem (1.1) and to obtain the existence and blow up results of the solutions.

Our main results are stated in the next section, including some preliminary results and local existence of solutions to the problem (1.1). The blow-up of solutions is obtained and proved in Section 3.

2 Existence of Solutions

This section is devoted to the proof of existence of solutions to the problem (1.1). We give the following definition.

Definition 2.1 *We say that the solution* $(u(x,t), v(x,t))$ *for the problem* (1.1) *blows up in finite time if there exists an instant* $T^* < \infty$ *such that*

$$|||(u,v)||| \to \infty \qquad \textit{as } t \to T^*,$$

where

$$|||(u,v)||| = \sup_{t\in[0,T)} \{\|u(t)\|_\infty + \|v(t)\|_\infty\},$$

and

$$\|u(t)\|_\infty = \|u(\cdot,t)\|_{L^\infty(\Omega)}, \qquad \|v(t)\|_\infty = \|v(\cdot,t)\|_{L^\infty(\Omega)}.$$

Our first result is the following theorem.

Theorem 2.1 *Let $\Omega \subset \mathbf{R}^N$ be a bounded smooth domain. Assume that $p_1(x)$ and $p_2(x)$ satisfy the conditions (1.2)-(1.3), and $u_0(x)$ and $v_0(x)$ are nonnegative, continuous and bounded. Then there exists a T^0 with $0 < T^0 \le \infty$ such that the problem (1.1) has a nonnegative and bounded solution (u,v) in Q_{T^0}.*

Proof. Let us consider the equivalence systems of (1.1)

$$\begin{cases} u(x,t) = \displaystyle\int_\Omega g_1(x,y,t)u_0(y)\mathrm{d}y + \alpha\int_0^t\int_\Omega g_1(x,y,t-s)v^{p_1(y)}(y,s)\mathrm{d}y\mathrm{d}s, \\ v(x,t) = \displaystyle\int_\Omega g_2(x,y,t)v_0(y)\mathrm{d}y + \beta\int_0^t\int_\Omega g_2(x,y,t-s)u^{p_2(y)}(y,s)\mathrm{d}y\mathrm{d}s, \end{cases}$$

where $g_i(x,y,t)$ $(i=1,2)$ are the corresponding Green functions. Then the existence and uniqueness of solutions for a given $(u_0(x), v_0(x))$ could be obtained by a fixed point theorem.

We introduce the following iteration scheme:

$$\begin{aligned} &u_1(x,t) = 0, \\ &v_1(x,t) = 0, \\ &u_{n+1}(x,t) = \int_\Omega g_1(x,y,t)u_0(y)\mathrm{d}y + \alpha\int_0^t\int_\Omega g_1(x,y,t-s)v_n^{p_1(x)}(y,s)\mathrm{d}y\mathrm{d}s, \\ &v_{n+1}(x,t) = \int_\Omega g_2(x,y,t)v_0(y)\mathrm{d}y + \beta\int_0^t\int_\Omega g_2(x,y,t-s)u_n^{p_2(x)}(y,s)\mathrm{d}y\mathrm{d}s. \end{aligned}$$

The convergence of the sequence $\{(u_n,v_n)\}$ follows by showing that

$$\begin{cases} \Phi_1(v) = \alpha\displaystyle\int_0^t\int_\Omega g_1(x,y,t-s)v^{p_1(y)}(y,s)\mathrm{d}y\mathrm{d}s, \\ \Phi_2(u) = \beta\displaystyle\int_0^t\int_\Omega g_2(x,y,t-s)u^{p_2(y)}(y,s)\mathrm{d}y\mathrm{d}s, \end{cases}$$

is a contraction mapping in the set E_T which will be defined later. Now, we define

$$\Psi(u,v) = (\Phi_1(v), \Phi_2(u)),$$

where

$$\Phi_1(v) = \alpha\int_0^t\int_\Omega g_1(x,y,t-s)v^{p_1(y)}(y,s)\mathrm{d}y\mathrm{d}s,$$

$$\Phi_2(u) = \beta\int_0^t\int_\Omega g_2(x,y,t-s)u^{p_2(y)}(y,s)\mathrm{d}y\mathrm{d}s.$$

We also denote

$$\Psi(u,v) - \Psi(w,z) = (\Phi_1(v) - \Phi_1(z),\ \Phi_2(u) - \Phi_2(w)).$$

For an arbitrary $T > 0$, define

$$E_T = \{C^{1,2}(\Omega_T) \cap C(\overline{\Omega_T}) \mid |||(u,v)||| \le M\},$$

where $\Omega_T = \Omega \times [0,T],\ M > |||(u_0(x), v_0(x))|||$ is a fixed positive constant.

Then we claim that Ψ is a strict contraction on E_T. In fact, for any fixed $x \in \Omega$, we have

$$\xi_1^{p_1(x)} - \eta_1^{p_1(x)} = p_1(x) w_1^{p_1(x)-1}(\xi_1 - \eta_1)$$

and

$$\xi_2^{p_2(x)} - \eta_2^{p_2(x)} = p_2(x) w_2^{p_2(x)-1}(\xi_2 - \eta_2),$$

where

$$w_1 = s_1\xi_1 + (1-s_1)\eta_1, \qquad s_1 \in (0,1)$$

and

$$w_2 = s_2\xi_2 + (1-s_2)\eta_2, \qquad s_2 \in (0,1).$$

And we always have

$$\||p_i(x) w_i^{p_i(x)-1}(\xi_i - \eta_i)|\| \le p_i^+ (2M)^{p_i^+ - 1} \|\xi_i - \eta_i\|_\infty, \qquad i = 1, 2. \tag{2.1}$$

Now, we define

$$\mu_i(t) = \sup_{x\in\overline{\Omega},\, 0\le\tau<t} \int_0^\tau \int_\Omega g_i(x,\, y,\, t-s)\mathrm{d}y\mathrm{d}s, \qquad i = 1, 2.$$

It is obvious that

$$\mu_i(t) \to 0 \qquad \text{as } t \to 0^+.$$

Then by (2.1), we get

$$\begin{aligned}
&\|\Phi_1(v) - \Phi_1(z)\|_\infty + \|\Phi_2(u) - \Phi_2(w)\|_\infty \\
\le{}& \alpha\Big\| \int_0^t \int_\Omega g_1(x, y, \tau - s)(v_n^{p_1(x)} - z_n^{p_1(x)})\mathrm{d}y\mathrm{d}s \Big\|_\infty \\
&+ \beta\Big\| \int_0^t \int_\Omega g_2(x, y, \tau - s)(u_n^{p_2(x)} - w_n^{p_2(x)})\mathrm{d}y\mathrm{d}s \Big\|_\infty \\
\le{}& (\mu_1(t) + \mu_2(t))\Gamma(2M)^{\max\{p_1^+, p_2^+\}-1}(\|p_1\|_\infty + \|p_2\|_\infty)(\|v - z\|_\infty + \|u - w\|_\infty) \\
\le{}& (\mu_1(t) + \mu_2(t))\Gamma(2M)^{\max\{p_1^+, p_2^+\}-1}(\|p_1\|_\infty + \|p_2\|_\infty) \sup_{t\in[0,T)} (\|v - z\|_\infty + \|u - w\|_\infty) \\
={}& (\mu_1(t) + \mu_2(t))\Gamma(2M)^{\max\{p_1^+, p_2^+\}-1}(\|p_1\|_\infty + \|p_2\|_\infty)\|\|(v - z, u - w)\|\|,
\end{aligned}$$

where $\Gamma = \max\{\alpha,\, \beta\}$. Since $\mu_i(t)$ $(i = 1, 2)$ are sufficiently small as $t \to 0$, we get

$$\begin{aligned}
&\|\|\Psi(u, v) - \Psi(w, z)\|\| \\
={}& \|(\Phi_1(v) - \Phi_1(z), \Phi_2(u) - \Phi_2(w))\| \\
\le{}& \sup_{t\in[0,T)} \{\|\Phi_1(v) - \Phi_1(z)\|_\infty + \|\Phi_2(u) - \Phi_2(w)\|_\infty\} \\
\le{}& (\mu_1(t) + \mu_2(t))\Gamma(2M)^{\max\{p_1^+ p_2^+\}-1}(\|p_1\|_\infty + \|p_2\|_\infty)\|\|(v - z, u - w)\|\|.
\end{aligned}$$

Hence, Ψ is a strict contraction mapping. This completes the proof.

3 Blow up of Solutions

This section is devoted to the blow up of solutions to the problem (1.1). We first give the following lemma.

Lemma 3.1 *Let $y(t)$ be a solution of*

$$y'(t) \geq a y^r(t), \qquad y(0) > 0,$$

where $r > 1$ and $a > 0$. Then $y(t)$ cannot be globally defined, and

$$y(t) \geq \Big(y(0)^{1-r} - \frac{r-1}{a} t\Big)^{-1/(r-1)}.$$

This lemma follows by a direct integration, and gives an upper bound for the blow up time. The following theorem gives the main result of this section.

Theorem 3.1 *Let $\Omega \subset \mathbf{R}^N$ be a bounded smooth domain and (u, v) a positive solution of the problem (1.1) with $p_1(x)$ and $p_2(x)$ satisfying the conditions (1.2)-(1.3). Then for a sufficiently large initial datum $(u_0(x),\ v_0(x))$, there exists a finite time $T^* > 0$ such that*

$$\sup_{0 \leq t \leq T^*} |||(u, v)||| = +\infty.$$

Proof. Let λ_1 be the first eigenvalue of

$$-\Delta \varphi = \lambda \varphi, \qquad x \in \Omega$$

with homogeneous Dirichlet boundary condition and φ a positive eigenfunction satisfying

$$\int_\Omega \varphi \mathrm{d}x = 1.$$

Set

$$\eta(t) = \int_\Omega (u + v) \varphi \mathrm{d}x.$$

We get

$$\begin{aligned} \eta'(t) = & \int_\Omega (u_t + v_t) \varphi \mathrm{d}x \\ = & \int_\Omega (\Delta u + \Delta v) \varphi \mathrm{d}x + \int_\Omega (\alpha v^{p_1(x)} + \beta u^{p_2(x)}) \varphi \mathrm{d}x \\ = & -\lambda \eta + \int_\Omega (\alpha v^{p_1(x)} + \beta u^{p_2(x)}) \varphi \mathrm{d}x. \end{aligned}$$

We now deal with the term

$$\int_\Omega (\alpha v^{p_1(x)} + \beta u^{p_2(x)}) \varphi \mathrm{d}x.$$

For each $t > 0$, we divide Ω into the following four sets:

$$\begin{aligned} \Omega_{11} &= \{x \in \Omega : v(x,t) < 1,\ u(x,t) < 1\}, \\ \Omega_{12} &= \{x \in \Omega : v(x,t) < 1,\ u(x,t) \geq 1\}, \\ \Omega_{21} &= \{x \in \Omega : v(x,t) \geq 1,\ u(x,t) < 1\}, \\ \Omega_{22} &= \{x \in \Omega : v(x,t) \geq 1,\ u(x,t) \geq 1\}. \end{aligned}$$

Then, we have

$$\begin{aligned}
&\int_\Omega (\alpha v^{p_1(x)} + \beta u^{p_2(x)})\varphi \mathrm{d}x \\
= &\int_{\Omega_{11}} (\alpha v^{p_1(x)} + \beta u^{p_2(x)})\varphi \mathrm{d}x + \int_{\Omega_{12}} (\alpha v^{p_1(x)} + \beta u^{p_2(x)})\varphi \mathrm{d}x \\
&+ \int_{\Omega_{21}} (v^{\alpha p_1(x)} + \beta u^{p_2(x)})\varphi \mathrm{d}x + \int_{\Omega_{22}} (\alpha v^{p_1(x)} + \beta u^{p_2(x)})\varphi \mathrm{d}x \\
\ge &\int_{\Omega_{12}} \beta u^p \varphi \mathrm{d}x + \int_{\Omega_{21}} \alpha v^p \varphi \mathrm{d}x + \int_{\Omega_{22}} (\alpha v^p + \beta u^{p)})\varphi \mathrm{d}x \\
\ge &\int_{\Omega_{12}} \beta u^p \varphi \mathrm{d}x + \int_{\Omega_{21}} \alpha v^p \varphi \mathrm{d}x + \int_{\Omega} (\alpha v^p + \beta u^{p)})\varphi \mathrm{d}x \\
&- \int_{\Omega_{12}} (\alpha v^p + \beta u^p)\varphi \mathrm{d}x - \int_{\Omega_{21}} (\alpha v^p + \beta u^p)\varphi \mathrm{d}x - \int_{\Omega_{11}} (\alpha v^p + \beta u^{p)})\varphi \mathrm{d}x \\
\ge &\, \Gamma \int_\Omega (v^p + u^p)\varphi \mathrm{d}x - \Gamma \int_{\Omega_{12}} v^p \varphi \mathrm{d}x - \Gamma \int_{\Omega_{21}} u^p \varphi \mathrm{d}x - \Gamma \int_{\Omega_{11}} (v^p + u^{p)})\varphi \mathrm{d}x \\
\ge &\, \Gamma \int_\Omega (v^p + u^p)\varphi \mathrm{d}x - 4\Gamma \int_\Omega \varphi \mathrm{d}x,
\end{aligned}$$

where

$$p = \min\{p_1^-,\, p_2^-\}, \qquad \Gamma = \min\{\alpha,\, \beta\}.$$

In view of the convex property of

$$f(w) = w^r, \qquad r > 1$$

and by using Jensen's inequality again, we obtain

$$\begin{aligned}
&\int_\Omega (\alpha v^{p_1(x)} + \beta u^{p_2(x)})\varphi \mathrm{d}x \\
\ge &\, \Gamma \int_\Omega (v^p + u^p)\varphi \mathrm{d}x - 4\Gamma \int_\Omega \varphi \mathrm{d}x \\
\ge &\, \Gamma \int_\Omega 2\Big(\frac{v+u}{2}\Big)^p \varphi \mathrm{d}x - 4\Gamma \int_\Omega \varphi \mathrm{d}x \\
\ge &\, \frac{\Gamma}{2^{p-1}} \eta^p(t) - 4\Gamma.
\end{aligned}$$

Then, we get

$$\eta'(t) \ge -\lambda_1 \eta(t) + \frac{\Gamma}{2^{p-1}} \eta^p(t) - 4\Gamma.$$

Since

$$\begin{aligned}
\eta(t) &= \int_\Omega (u+v)\varphi \mathrm{d}x \\
&\le (\|u(\,\cdot\,,t)\|_{L^\infty(\Omega)} + \|v(\,\cdot\,,t)\|_{L^\infty(\Omega)}) \int_\Omega \varphi \mathrm{d}x \\
&\le |||(u,v)|||,
\end{aligned}$$

we know that the result follows from Lemma 3.1 for $\eta(0)$ large enough. The proof is completed.

Remark 3.1 Assume that the source terms in (1.1) are of the form

$$f(v) = \int_\Omega v^{p_1(x)} \mathrm{d}x, \qquad g(u) = \int_\Omega u^{p_2(x)} \mathrm{d}x.$$

Then Theorem 3.1 also holds.

In fact, defining

$$\eta(t) = \int_{\Omega} (u + v)\varphi \mathrm{d}x$$

and repeating the previous argument, we can obtain the same result.

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