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Stochastic Nonlinear Beam Equations with Lévy Jump

Chen Feng
(*College of Mathematics, Jilin University, Changchun, 130012*)

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Abstract: In this paper, we study stochastic nonlinear beam equations with Lévy jump, and use Lyapunov functions to prove existence of global mild solutions and asymptotic stability of the zero solution.

Key words: stochastic extensible beam equation, Lévy jump, Lyapunov function, stability

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1 Introduction

Consider a wide class of abstract stochastic beam equations with Lévy jump in a separable Hilbert space H:

$$\begin{cases} u_{tt}+A^2u+g(u,u_t)+m(\|B^{\frac{1}{2}}u\|^2)Bu=\sigma(u,u_t)\dot{W}+\displaystyle\int_Z f(u,u_t,z)\tilde{N}(\mathrm{d}z,t), \\ u(0)=u_0, \qquad u_t(0)=u_1, \end{cases} \tag{1.1}$$

where A and B are positive self-adjoint operators, m is a nonnegative function in $C^1([0,+\infty))$, W is a Wiener process, and $\tilde{N}$ is the compensated Poisson measure.

In [1], a model for the transversal deflection of an extensible beam of a natural length l was proposed as follows:

$$\frac{\partial^2u}{\partial t^2}+\gamma\frac{\partial^4u}{\partial x^4}=\left[a+b\int_0^l\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x\right]\frac{\partial^2u}{\partial x^2}. \tag{1.2}$$

Chow and Menaldi[2] considered a stochastic extensible beam equation which describes large amplitude vibrations of an elastic panel excited by aerodynamic forces. They studied

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E-mail address: chenfengmath@163.com (Chen F).

a stochastic beam equation as follows:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}-\Big[a+b\int_0^l\Big(\frac{\partial u}{\partial x}\Big)^2\mathrm{d}x\Big]\frac{\partial^2 u}{\partial x^2}+\gamma\frac{\partial^4 u}{\partial x^4}\\
&=g\Big(t,\ x,\ \frac{\partial u}{\partial t},\ \frac{\partial u}{\partial x}\Big)+\sum_{k=1}^{\infty}\sigma_k\Big(t,\ x,\ \frac{\partial u}{\partial t},\ \frac{\partial u}{\partial x}\Big)\dot{\omega}_k, \qquad x\in[0,\ l],\ t\geq 0, \qquad (1.3)\\
&u(t,\ 0)=u(t,\ l)=\frac{\partial u}{\partial x}(t,\ 0)=\frac{\partial u}{\partial x}(t,\ l)=0, \qquad t\geq 0.
\end{aligned}$$

Brzeźniak *et al.*[3] studied (1.3), gave a wide class of abstract stochastic beam equations, proved nonexplosion of mild solutions to (1.3), and established the asymptotic stability of the zero solution when the damping term g is of the form

$$g(u,\ u_t)=\beta u_t, \qquad \beta\geq 0.$$

Recently, the stochastic partial differential equations driven by jump processes have been studied by some scholars. Albeverio *et al.*[4] studied the stochastic reaction diffusion equations driven by Poisson random measures, and established the existence and uniqueness of the solution under growth and Lipschitz conditions. Fournier[5] used Malliavin calculus to study the continuity of the law of the weak solution of the stochastic reaction diffusion equations driven by Poisson random measures. Mueller[6] constructed a minimal solution for the stochastic heat equation driven by non-negative Lévy noise with coefficients of polynomial growth. Mytnik[7] established a weak solution for the stochastic partial equation driven by a one sided, α-stable noise without negative jumps. Röckner and Zhang[8] studied the stochastic evolution equations driven by both Brownian motion and Poisson point processes, and obtained the existence and uniqueness results of the equations.

The purpose of the present paper is to deal with the stochastic beam equations driven by white noise and Poisson noise. The model describes large amplitude vibrations and rare events with low frequency and sudden occurrence vibrations of an elastic panel excited by aerodynamic forces. We prove the existence of global solutions of the system (1.1), by using the technique of constructing a proper Lyapunov function (see [9]), i.e., we assume that there exists a Lyapunov function $V(u):\mathcal{H}\to\mathbf{R}$ of the system (1.1) such that

$$V_R=\inf_{\|u\|\geq R}V(u)\to\infty \quad \text{as } R\to\infty, \qquad \frac{\mathrm{d}V}{\mathrm{d}t}<c_1V,$$

and the coefficient of system (1.1) satisfies some conditions, then there exists a global solution of the system (1.1) for all $t\geq t_0$.

The stability result in this paper depends on the form of $g(u,u_t)$. We can establish the asymptotic stability of zero solution when the damping term g is of the form

$$g(u,u_t)=\beta u_t+h(|u|^2)u, \qquad \beta\geq 0,$$

where $h\in C^1([0,\infty))$ is a nonnegative function.

2 Notations and Preliminaries

Let H be a separable Hilbert space with the norm and the inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Suppose that

(H1) $A : \mathrm{Dom}(A) \to H$ and $B : \mathrm{Dom}(B) \to H$ are self-adjoint operators in H, $A \geq \mu I$ for some $\mu > 0$, $B > 0$, $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(B)$ and $B \in \mathscr{L}(\mathrm{Dom}(A), H)$, where $\mathrm{Dom}(A)$ is endowed with the graph norm $\|x\|_{\mathrm{Dom}(A)} \equiv \|Ax\|$.

(H2) W is a Wiener process in another real separable Hilbert space U with a covariance operator Q, and is defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ such that $\mathscr{F}_0$ contains all P-null sets.

(H3) Let $p = (p(t))$, $t \in D_p$ be a stationary $\mathscr{F}_t$-Poisson point process on a measure space $(Z, \mathcal{Z}, \nu)$, where ν is a σ-finite measure, and $N(\mathrm{d}t, \mathrm{d}x)$ be the Poisson counting measure associated with p, i.e.,

$$N(t, A) = \sum_{s \in D_p, s \leq t} I_A(p(s)).$$

Let

$$\tilde{N}(\mathrm{d}t, \mathrm{d}x) := N(\mathrm{d}t, \mathrm{d}x) - \mathrm{d}t\nu(\mathrm{d}x)$$

be the compensated Poisson measure on $[0, T] \times \Omega \times Z$.

(H4) $m \in C^1([0, \infty))$ is a nonnegative function.

To interpret (1.1), we set

$$\begin{aligned} &u_t = v, \\ &v_t = -A^2 u - g(u, u_t) - m(\|B^{\frac{1}{2}} u\|^2) B u + \sigma(u, u_t)\dot{W} + \int_Z f(u, u_t, z)\tilde{N}(\mathrm{d}z, t). \end{aligned}$$

Then we set

$$\begin{aligned} &\mathscr{H} = \mathrm{Dom}(A) \times H, \qquad \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_{\mathscr{H}}^2 = \|Ax\|^2 + \|y\|^2, \\ &\mathrm{Dom}(U) = \mathrm{Dom}(A^2) \times \mathrm{Dom}(A), \qquad U = \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix}, \\ &G : \mathscr{H} \to \mathscr{H}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ g(x, y) - m(\|B^{\frac{1}{2}} x\|^2) B x \end{pmatrix}, \\ &\Sigma : \mathscr{H} \to \mathscr{L}_2(\mathrm{Rng} \mathrm{Q}^{\frac{1}{2}}, \mathscr{H}), \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ \sigma(x, y) \end{pmatrix}, \\ &F : \mathscr{H} \to \mathscr{H}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ f(x, y, z) \end{pmatrix}. \end{aligned}$$

Let $X = (u, v)$ and $X_0 = (u_0, u_1)$. Then (1.1) is rewritten as

$$\begin{cases} \mathrm{d}X = (UX + G(X))\mathrm{d}t + \Sigma(X)\mathrm{d}W + \int_Z F(X, z)\tilde{N}(\mathrm{d}z, t), \\ X(0) = X_0. \end{cases} \tag{2.1}$$

In order to obtain the existence and uniqueness of the mild solution to (1.1), we impose certain growth and global Lipschitz conditions on the coefficients g, σ and f of the system (1.1) in a Hilbert space. We introduce the subspace $U_0 = Q^{\frac{1}{2}} U$ with its natural Hilbert structure. Let $\mathscr{L}_2$ be the Hilbert-Schmidt operator space.

Suppose that

(H5) $\sigma : \mathscr{H} \to \mathscr{L}_2(U_0, H)$ is Lipschitz continuous on a bounded set in $\mathscr{H}$ and of a linear growth, that is,

$$\exists\, L_\sigma < \infty, \quad \text{s.t.} \ \ \forall\, x \in \mathscr{H}, \quad \|\sigma(x) Q^{\frac{1}{2}}\|_{HS} \leq L_\sigma (1 + \|x\|_{\mathscr{H}}),$$

$$\forall\ N \in \mathbf{N}, \quad \exists\ L_\sigma(N) < \infty, \quad \text{s.t.}\ \forall\ x,\ y \in \mathscr{H},\ \|x\|_{\mathscr{H}},\ \|y\|_{\mathscr{H}} \leq N,$$
$$\|[\sigma(x) - \sigma(y)]Q^{\frac{1}{2}}\|_{HS} \leq L_\sigma(N)\|x - y\|_{\mathscr{H}},$$

where $\|\cdot\|_{HS}$ denotes the norm of both $\mathscr{L}_2(U,\ H)$ and $\mathscr{L}_2(U,\ \mathscr{H})$.

(H6) $g : \mathscr{H} \to \mathscr{H}$ is Lipschitz continuous on a bounded set in $\mathscr{H}$ and there exists a $L_g \in [0, \infty]$ such that

$$\langle y,\ g(x, y)\rangle \geq -L_g(1 + \|\tau\|^2_{\mathscr{H}}), \qquad \tau = (x,\ y)^{\mathrm{T}} \in \mathscr{H}.$$

(H7) $f : \mathscr{H} \to \mathscr{H}$ is Lipschitz continuous on bounded sets in $\mathscr{H}$ and of a linear growth, that is,

$$\exists\ L_\sigma < \infty, \quad \text{s.t.}\ \forall\ x \in \mathscr{H}, \quad \int_Z \|f(x)\|^2_{HS}\nu(\mathrm{d}x) \leq L_f(1 + \|x\|^2_{\mathscr{H}}),$$
$$\forall\ N \in \mathbf{N}, \quad \exists\ L_\sigma(N) < \infty, \quad \text{s.t.}\ \forall\ x,\ y \in \mathscr{H},\ \|x\|_{\mathscr{H}},\ \|y\|_{\mathscr{H}} \leq N,$$
$$\int_Z \|f(x) - f(y)\|^2_{HS} \leq L_\sigma(N)\|x - y\|^2_{\mathscr{H}}.$$

From [3] we know that the operators U and $-U$ are m-dissipative, G and F are Lipschitz continuous, and there exist nonexplosion mild solutions of (2.1). We define the mild solution of (2.1) as follows:

$$\begin{aligned} X(t) = \mathrm{e}^{Ut}X(0) &+ \int_0^t \mathrm{e}^{U(t-r)}G(X(r))\mathrm{d}r + \int_0^t \mathrm{e}^{U(t-r)}\Sigma(X(r))\mathrm{d}W(r) \\ &+ \int_0^t \int_Z \mathrm{e}^{U(t-r)}F(X(r))\tilde{N}(\mathrm{d}z, \mathrm{d}r), \qquad t \geq 0. \end{aligned} \tag{2.2}$$

Set

$$M(s) = \int_0^s m(r)\mathrm{d}r, \quad H(s) = \int_0^s h(r)\mathrm{d}r, \qquad s \geq 0.$$

In order to show the stability of the zero solution, the damping term g of (1.1) is of the form $g(u, u_t) = \beta u_t + h(|u|^2)u,\ \beta \geq 0$, where $h \in C^1([0,\ \infty))$ is a nonnegative function. In addition, assume that σ and f satisfy the linear growth conditions, respectively, as follows:

$$\exists\ R_\sigma < \infty, \quad \text{s.t.}\ \forall\ \tau \in \mathscr{H}, \qquad \|\sigma(\tau)Q^{\frac{1}{2}}\|_{HS} \leq R_\sigma\|\tau\|_{\mathscr{H}},$$
$$\exists\ R_f < \infty, \quad \text{s.t.}\ \forall\ \tau \in \mathscr{H}, \qquad \int_Z \|g(\tau)\|_{\mathscr{H}}\mu(\mathrm{d}x) \leq R_f\|\tau\|_{\mathscr{H}}.$$

3 Existence of Global Solutions

We first prove the existence of the stochastic beam differential equations in finite dimension, and then we extend the results to the infinite dimension by the Galerkin method. Define

$$P_n : H \to H_n P_n, \qquad y := \sum_{i=1}^n \langle y,\ e_i\rangle e_i, \quad y \in H.$$

Thus, $P_n|_H$ denotes the orthogonal projector from H onto H_n, and H_n is the space spanned by $\{e_k\}_{k=1}^n$. Define a linear operator A by

$$Au := -\Delta u, \qquad u \in \mathrm{Dom}(A).$$

Thus, there exists an orthomornal basis $\{e_k\}_{k=1,2,\cdots}$ of H, which consists of eigenvectors of A, such that for $k = 1,\ 2, \cdots,$

$$Ae_k = \lambda_k e_k.$$

Then the Galerkin equations associated with (2.1) can be expressed as

$$\begin{cases} \mathrm{d}X^{(n)} = UX^{(n)} + P_nG(X^{(n)})\mathrm{d}t + P_n\Sigma(X^{(n)})\mathrm{d}W^{(n)} \\ \qquad\qquad + \displaystyle\int_Z P_nF(X^{(n)}, z)\tilde{N}(\mathrm{d}z, t), \\ X^{(n)}(0) = P_nX_0, \end{cases} \tag{3.1}$$

with $X^{(n)} = (P_nu,\ P_nu_t)^{\mathrm{T}}$.

Theorem 3.1 *Suppose that* (H1)–(H5) *are given above, and* $X_0 : \Omega \to \mathscr{H}$. (2.1) *admits a unique global mild solution* X *on* $[0,\ \infty)$. *The process* $X \in L^\infty([0, \infty);\ \mathscr{H})$ *is adapted and càdlàg.*

Define an energy functional on $\mathscr{H}$ by

$$\Psi(X) = \frac{1}{2}\|X\|^2 + \frac{1}{2}M(\|B^{\frac{1}{2}}x\|^2), \qquad X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathscr{H}. \tag{3.2}$$

Lemma 3.1 *Under the assumptions in Theorem* 3.1, *let* $X^{(n)} = (u^{(n)},\ u_t^{(n)})$. *Then there exist* $\lambda > 0$ *and* $C > 0$ *such that*

$$\sup_{n\in\mathbf{N}} E\,\Psi(X^{(n)}(t)) \le \mathrm{e}^{Ct}(1 + E\,\Psi(X^{(n)}(0))).$$

Proof. Applying Itô formula to $\Psi(X^{(n)}(t))$, we have

$$\begin{aligned} \Psi(X^{(n)}(t)) = {} & \Psi(X^{(n)}(0)) + \int_0^t \{\langle D\,\Psi(X^{(n)}(s)),\ UX^{(n)}(s) + P_nG(X^{(n)}(s))\rangle \\ & + Tr(P_n\Sigma(X^{(n)}(s))^* D^2\,\Psi(X^{(n)}(s))P_n\Sigma(X^{(n)}(s)))\}\mathrm{d}s + M_1^{(n)}(t) \\ & + M_2^{(n)}(t) + \int_0^t \int_Z \|D^2\,\Psi(X^{(n)}(s))P_nF(X^{(n)}(s), z))\|^2 \mathrm{d}s\nu(\mathrm{d}x), \end{aligned} \tag{3.3}$$

where $D\,\Psi$ and $D^2\,\Psi$ denote the first and the second Fréchet derivatives of the functional Ψ, respectively,

$$D\,\Psi(X) = X + m(\|B^{\frac{1}{2}}x\|^2)\begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix}, \tag{3.4}$$

$$\begin{aligned} D^2\,\Psi(X) = {} & I_{\mathscr{H}} + 2m'(\|B^{\frac{1}{2}}x\|^2)\begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix} \otimes \begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix} \\ & + m(\|B^{\frac{1}{2}}x\|^2)\begin{pmatrix} A^{-2}B & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{3.5}$$

and

$$M_1^{(n)}(t) = \int_0^t \langle D\,\Psi(X^{(n)}(s)),\ P_n\Sigma(X^{(n)}(s))\mathrm{d}W^{(n)}(s)\rangle_H, \tag{3.6}$$

$$\begin{aligned} M_2^{(n)}(t) = {} & \int_0^t \int_Z (\langle D\,\Psi(X^{(n)}(s)),\ P_nF(X^{(n)}(s), z)\rangle_H \\ & + \|D^2\,\Psi(X^{(n)}(s))P_nF(X^{(n)}(s), z))\|^2)\tilde{N}(\mathrm{d}z, \mathrm{d}s). \end{aligned} \tag{3.7}$$

From (3.4)-(3.5) we get

$$\langle D\,\Psi(X^{(n)}(s),\ UX^{(n)}(s))\rangle = m(\|B^{\frac{1}{2}}x^{(n)}\|^2)\langle Bx^{(n)},\ y^{(n)}\rangle,$$

$$\langle D\,\Psi(X^{(n)}(s),\ P_nG(X^{(n)}(s))\rangle = -m(\|B^{\frac{1}{2}}x^{(n)}\|^2)\langle y^{(n)},\ Bx^{(n)}\rangle - \langle y^{(n)},\ G(x^{(n)}, y^{(n)})\rangle,$$

and

$$\mathrm{Tr}(P_n\Sigma(X^{(n)}(s))^* D^2\Psi(X^{(n)}(s))P_n\Sigma(X^{(n)}(s))) = \|P_n\Sigma(s,\ X^n(s))\|_H^2,$$

where

$$X^{(n)} = (x^n,\ y^n)^{\mathrm{T}} \in \mathscr{H}.$$

Taking expectation on both sides of (3.3), we have

$$\begin{aligned}
E\Psi(X^{(n)}(t)) &= E\Psi(X^{(n)}(0)) + E\int_0^t \{-\langle u_t^{(n)}(s),\ g(X^{(n)}(s))\rangle + \frac{1}{2}\|\Sigma(X^n(s))\|_{HS}^2\}\mathrm{d}s \\
&\quad + \frac{1}{2}E\int_0^t\int_Z \|f(X^n(s),\ z)\|^2\mathrm{d}s\nu(\mathrm{d}x) \\
&\le E\Psi(X^{(n)}(0)) + (L_g + L_\sigma^2 + L_f^2)\int_0^t (1 + E\|X^{(n)}(s)\|_{\mathscr{H}}^2)\mathrm{d}s \\
&\le E\Psi(X^{(n)}(0)) + (L_g + L_\sigma^2 + L_f^2)\int_0^t (1 + E\Psi^{(n)}(s))\mathrm{d}s. \qquad (3.8)
\end{aligned}$$

From (3.8) we get

$$E\Psi(X^{(n)}(t)) \le \mathrm{e}^{Ct}(1 + E\Psi(X^{(n)}(0))), \qquad t \ge 0,$$

where

$$C = L_g + L_\sigma^2 + L_f^2 < \infty.$$

Proof of Theorem 3.1 From Lemma 3.1 it follows that $\{(u^{(n)}, u_t^{(n)})\}_{n\ge1}$ is bounded in $L^\infty(0,T;\mathscr{H})$. Then we choose a subsequence of $\{(u^{(n)}, u_t^{(n)})\}_{n\ge1}$, and denote it by $\{(u^{(n_k)}, u_t^{(n_k)})\}_{n\ge1}$, such that

$$X^{(n_k)} \rightharpoonup X \quad \text{weakly in } L^\infty(0,T;V\times H) \qquad \text{as } n_k \to \infty. \qquad (3.9)$$

Now we check that X in (3.9) is a weak solution of (2.1). To attain this, we proceed the following steps:

(i) We claim $\frac{\mathrm{d}}{\mathrm{d}t}X^{(n_k)} \rightharpoonup \frac{\mathrm{d}}{\mathrm{d}t}X$ in $L^2(0,T,V^*)$ by (3.9).

(ii) We claim $A^2u^{(n_k)} \rightharpoonup A^2u$ in $L^2(0,T,V^*)$. Actually, for each $\eta \in L^2(0,T,V)$,

$$\begin{aligned}
\int_0^T \langle A^2u^{(n_k)}(t),\ \eta(t)\rangle &= \int_0^T \sum_{k=1}^\infty \lambda_k\langle u^{(n_k)},\ e_k\rangle\langle\eta,\ e_k\rangle\mathrm{d}t \\
&\to \int_0^T \sum_{k=1}^\infty \lambda_k\langle u,\ e_k\rangle\langle\eta,\ e_k\rangle\mathrm{d}t \\
&= \int_0^T \langle A^2u(t),\ \eta(t)\rangle\mathrm{d}t \qquad \text{as } n_k\to\infty,
\end{aligned}$$

where λ_k and e_k are the eigenvalues and the eigenvectors of the operator A^2, respectively, so that $A^2e_k = \lambda_ke_k$ and $\langle u,\ v\rangle_{2s} := \sum_{k=1}^\infty \lambda_k^{2s}\langle u,\ e_k\rangle\langle v,\ e_k\rangle$.

(iii) We claim $P_ng(X^{(n_k)}(t)) \rightharpoonup g(X(t))$ in $L^2(0,T,V^*)$. Actually, for each $\eta\in C(0,T;V)$,

$$\begin{aligned}
&\int_0^T \langle P_ng(u^{n_k}(t)) - g(u(t)),\ \eta(t)\rangle\mathrm{d}t \\
&\le \int_0^T \langle P_ng(u^{n_k}(t)) - g(u^{n_k}(t)),\ \eta(t)\rangle\mathrm{d}t + \int_0^T \langle g(u^{n_k}(t)) - g(u(t)),\ \eta(t)\rangle\mathrm{d}t
\end{aligned}$$

$$\leq \sum_{i=1}^{m} \|(P_n - I)\eta_i\|(l|u^{n_k}(t)|^2) \sup_{0\leq t\leq T} \|\beta_i(t)\| + C|\eta| \cdot \int_0^T [r(\|u - u^{n_k}\|_{L^\infty}^2)]\mathrm{d}t$$

$$\to 0 \qquad \text{as } n_k \to +\infty,$$

where

$$\eta := \Big\{\eta(t) = \sum_{i=1}^{m} \eta_i\beta_i(t) : [0,T] \to V;\ \eta_i \in V,\ \beta_i \in C(0,T),\ m \in \mathbf{N}\Big\}.$$

(iv) We claim $P_n m(\|B^{\frac{1}{2}}u^{(n_k)}\|^2)Bu^{(n_k)} \rightharpoonup m(\|B^{\frac{1}{2}}u\|^2)Bu$ in $L^2(0,T,V^*)$. Actually, by a proof similar to that of (ii) and (iii) we can get the claim.

(v) We claim $P_n\sigma(X^{(n_k)})\dot{W} \rightharpoonup \sigma(X)\dot{W}$ in $L^2(0,T,V^*)$. Actually,

$$\begin{aligned}\sup_{t\in[0,T]} \Big\{E\Big\| \int_0^t P_n\sigma(X^{(n_k)})\mathrm{d}W\Big\|_{\mathscr{H}}^2\Big\}^{\frac{1}{2}} &\leq cE\Big[\int_0^T \|P_n\sigma(X^{(n_k)})\|_{\mathscr{H}}^2\mathrm{d}t\Big]^{\frac{1}{2}}\\ &\leq cE\int_0^T L_\sigma(1 + \|X^{(n_k)}\|_{\mathscr{H}}^2)\mathrm{d}t\\ &\leq \text{Constant}.\end{aligned}$$

Thus

$$\int_0^{\cdot} P_n\sigma(X^{(n_k)})\mathrm{d}W \to \int_0^{\cdot} \sigma(X)\mathrm{d}W.$$

(vi) We claim $P_n \int_Z f(X^{(n_k)})\tilde{N}(\mathrm{d}z,t) \rightharpoonup \int_Z f(X)\tilde{N}(\mathrm{d}z,t)$ in $L^2(0,T,V^*)$. Actually, by a proof similar to that of (v) we can get that

$$P_n\int_0^{\cdot}\int_Z f(X^{(n_k)})\tilde{N}(\mathrm{d}z,\mathrm{d}t) \to \int_0^{\cdot}\int_Z f(X)\tilde{N}(\mathrm{d}z,\mathrm{d}t).$$

According to (i)–(v), we take weak limit on both sides of (3.1) as $n \to \infty$ and we find that $X \in L^\infty(0,T;\mathscr{H})$ is a solution of (2.1). This completes the proof.

4 Stability

In this section, we prove the stability of solutions to (1.1), which are given as follows:

Theorem 4.1 *Suppose that* (H1)–(H5) *are given above,*

$$R^2 < \beta, \quad \forall\, \gamma \in \mathscr{H}, \quad \|\sigma(\gamma)Q^{\frac{1}{2}}\|_{HS} \leq R_\sigma\|\gamma\|_{\mathscr{H}}, \quad \|f(\gamma)\|_{\mathscr{H}} \leq R_f\|\gamma\|_{\mathscr{H}}, \tag{4.1}$$

and there exists an $\alpha > 0$ *such that*

$$ym(y) \geq \alpha M(y), \quad yh(y) \geq \alpha H(y), \qquad y \geq 0. \tag{4.2}$$

Then the zero solution of (1.1) *is exponentially mean-square stable with probability one, i.e., there exists constants* $C < \infty$, $\lambda > 0$ *such that if* X *is a solution to* (1.1) *satisfying* $\Phi(X_0) < \infty$, *then we have*

$$E\|X(t)\|^2 \leq C\mathrm{e}^{-\lambda t}\Phi(X_0), \qquad t \geq 0. \tag{4.3}$$

We define an operator P (see [3]) by

$$P : \mathscr{H} \mapsto \mathscr{H}, \qquad \begin{pmatrix} x\\ y\end{pmatrix} \mapsto \begin{pmatrix}\beta^2A^{-2}x + 2x + \beta A^{-2}y\\ \beta x + 2y\end{pmatrix},$$

that is,

$$P = \begin{pmatrix} \beta^2 A^{-2} + 2I & \beta A^{-2} \\ \beta I & 2I \end{pmatrix}.$$

The operator P is a positive, self-adjoint isomorphism of $\mathscr{H}$, which has proved in [3]. So we can construct a Lyapunov function $\Phi : \mathscr{H} \to \mathbf{R}_+$ by

$$\Phi(m) = \frac{1}{2}\langle \gamma, P\gamma\rangle + M(\|B^{\frac{1}{2}}x\|^2) + \frac{1}{2}H(|x|^2), \qquad \gamma = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathscr{H}. \tag{4.4}$$

Lemma 4.1 *Suppose that the hypotheses of Theorem* 1.1 *are satisfied. Then there exists a constant* $\lambda > 0$ *such that if* X *is a solution of* (2.1) *with* $\Phi(X(0)) < \infty$, *then the process* $(\mathrm{e}^{\lambda t}\Phi(X(t)))_{t\geq 0}$ *is a nonnegative continuous supermartingale.*

Proof. Obviously, $\Phi \in C^2(\mathscr{H})$, and we have the first and the second Fréchet derivatives of the function Φ, respectively, as follows: for $\gamma = (x, y)^{\mathrm{T}}$,

$$D\Phi(\gamma) = P\gamma + 2m(\|B^{\frac{1}{2}}x\|^2)\begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix} + h(|x|^2)\begin{pmatrix} A^{-2}x \\ 0 \end{pmatrix}, \tag{4.5}$$

$$\begin{aligned} D^2\Phi(\gamma) = {} & P + 4m'(\|B^{\frac{1}{2}}x\|^2)\begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix} \otimes \begin{pmatrix} A^{-2}Bx \\ 0 \end{pmatrix} + h(|x|^2)\begin{pmatrix} A^{-2} & 0 \\ 0 & 0 \end{pmatrix} \\ & + 2m(\|B^{\frac{1}{2}}x\|^2)\begin{pmatrix} A^{-2}B & 0 \\ 0 & 0 \end{pmatrix} + h'(|x|^2)\begin{pmatrix} A^{-2}x \\ 0 \end{pmatrix} \otimes \begin{pmatrix} A^{-2}x \\ 0 \end{pmatrix}. \end{aligned} \tag{4.6}$$

Now, we compute the terms that would appear in the Itô formula for $\Phi(X(t))$. By using (4.5), we have

$$\begin{aligned} \langle D\Phi(\gamma), U\gamma\rangle &= -\beta\|Ax\|^2 + \beta^2\langle y, x\rangle + \beta\|y\|^2 + 2m(\|B^{\frac{1}{2}}x\|^2)\langle y, Bx\rangle + h(|x|^2)\langle y, x\rangle, \\ \langle D\phi(\gamma), G(\gamma)\rangle &= -\beta^2\langle x, y\rangle - 2\beta\|y\|^2 - \beta m(\|B^{\frac{1}{2}}(x)\|^2)\langle Bx, x\rangle - 2m(\|B^{\frac{1}{2}}x\|^2)\langle Bx, y\rangle \\ &\quad - \beta h(\|x\|^2)\|x\|^2 - h(|x|^2)\langle x, y\rangle. \end{aligned} \tag{4.7}$$

By using (4.6), we can compute the second derivative

$$\mathrm{Tr}(Q^{\frac{1}{2}}\Sigma(\gamma)^*D^2\Phi(\gamma)\Sigma(\gamma)Q^{\frac{1}{2}}) = 2\|\Sigma(\gamma)Q^{\frac{1}{2}}\|_{HS}^2.$$

For the stochastic term, we have

$$\Sigma(\gamma)^*D\Phi(\gamma) = \sigma(\gamma)^*(\beta x + 2y). \tag{4.8}$$

For the jump diffusion term, we have

$$F(\gamma, z)^*D\Phi(\gamma) = f(\gamma, z)^*(\beta x + 2y). \tag{4.9}$$

Brzeźniak *et al.*[3] proved that X is a global strong solution to (2.1). As the Itô formula cannot be applied directly to $\Phi(X(t))$, we consider the approximating strong solution X_n of (2.1). Let us consider the Yosida approximations U_n $(n \geq 1)$ to V:

$$U_n = nU(nI - U)^{-1} = n^2(nI - U)^{-1} - nI,$$

and introduce the canonical projections

$$\pi_1 : \mathscr{H} \to \mathrm{Dom(A)} \hookrightarrow H, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x, \qquad \pi_2 : \mathscr{H} \to \mathrm{Dom(A)} \hookrightarrow H, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto y.$$

For $n \in \mathbf{N}$, we consider

$$\mathrm{d}x(t) = (Ux(t) + G(t))\mathrm{d}t + \Sigma(t)\mathrm{d}W(t) + \int_Z F(t)\tilde{N}(\mathrm{d}z, t), \qquad x(0) = X(0),$$

$$\mathrm{d}x_n(t) = (U_n x_n(t) + G(t))\mathrm{d}t + \Sigma(t)\mathrm{d}W(t) + \int_Z F(t)\tilde{N}(\mathrm{d}z, t), \qquad x_n(0) = X_n(0).$$

So we can apply the Itô formula to the process $\mathrm{e}^{\lambda r}\Phi(x_n(r))$ on an interval $[s,\, t]$ to obtain

$$\begin{aligned}\Phi(x_n(t))\mathrm{e}^{\lambda t} = {} & \Phi(x_n(s))\mathrm{e}^{\lambda s} + \int_s^t \mathrm{e}^{\lambda r}\{\lambda\Phi(x_n(r)) + \langle D\Phi(x_n(r)),\ U_n x_n(r)\rangle_U \\ & + \langle D\Phi(x_n(r)),\ G(r)\rangle + \frac{1}{2}\mathrm{Tr}(Q^{\frac{1}{2}}\Sigma(x_n(r))^* D^2\Phi(x_n(r))\Sigma(x_n(r))Q^{\frac{1}{2}})\}\mathrm{d}r \\ & + \int_0^t\int_Z \frac{1}{2}\mathrm{e}^{\lambda r}\|D^2\Phi(x_n(s))F(x_n(r))\|^2\mathrm{d}r\nu(\mathrm{d}x) \\ & + \int_s^t \mathrm{e}^{\lambda r}\Sigma(x_n(r))^* D\Phi(x_n(r))\mathrm{d}W(r) + \int_0^t\int_Z \mathrm{e}^{\lambda r}(\langle D\Phi(x_n(s)),\ F(x_n(s))\rangle_H \\ & + \|D^2\Phi(x_n(s))F(x_n(s))\|^2)\tilde{N}(\mathrm{d}z,\mathrm{d}s).\end{aligned}$$

We have

$$\begin{aligned}\Phi(x_n(t))\mathrm{e}^{\lambda t} = {} & \Phi(x_n(s))\mathrm{e}^{\lambda s} + \int_s^t \mathrm{e}^{\lambda r}\{\lambda\Phi(x_n(r)) + \langle Px_n(r),\ U_n x_n(r)\rangle + 2m(\|B^{\frac{1}{2}}\pi_1 x_n(r)\|^2) \\ & \times\langle \pi_1 U_n x_n(r),\ B\pi_1 x_n(r)\rangle + h(\|\pi_1 x_n(r)\|^2)\langle\pi_1 U_n x_n(r),\ x_n(r)\rangle \\ & + \langle\beta\pi_1 x_n(r) + 2\pi_2 x_n(r),\ \pi_2 f(r)\rangle_U + \|\Sigma(r)Q^{\frac{1}{2}}\|_{HS}^2\}\mathrm{d}r \\ & + \int_0^t\int_Z \|F(r)\|_{HS}^2\mathrm{d}r\nu(\mathrm{d}x) + \int_s^t \mathrm{e}^{\lambda r}\Sigma(x_n(r))^* D\Phi(x_n(r))\mathrm{d}W(r) \\ & + \int_0^t\int_Z \mathrm{e}^{\lambda r}(\langle D\Phi(x_n(s)),\ F(x_n(s))\rangle)_H + \|D^2\Phi(x_n(s))F(x_n(s))\|^2)\tilde{N}(\mathrm{d}z,\mathrm{d}s).\end{aligned}$$

By Fatou's lemma, taking the limit as $n\to\infty$, we have

$$\begin{aligned}& \limsup_{n\to\infty}\int_s^t\int_s^t \mathrm{e}^{\lambda t}\langle U_n x_n,\ Px_n\rangle\mathrm{d}r \\ & \le \int_s^t \mathrm{e}^{\lambda r}(-\beta\|A\pi_1 x(r)\|^2 + \beta^2\langle\pi_2 x(r),\ \pi_1 x(r)\rangle + \beta\|\pi_2 x(r)\|^2)\mathrm{d}r.\end{aligned}$$

Let $x(r) = X(r)$. We obtain

$$\begin{aligned}\Phi(X(t))\mathrm{e}^{\lambda t} \le {} & \Phi(X(s))\mathrm{e}^{\lambda s} + \int_s^t \mathrm{e}^{\lambda r}\{\lambda\Phi(X(r)) - \beta\|X(r)\|^2 - \beta m(\|B^{\frac{1}{2}}X(r)\|^2) \\ & \times\langle BX(r), X(r)\rangle - \beta h(\|X(r)\|^2)\langle X(r), X(r)\rangle + \|\sigma(X(r))Q^{\frac{1}{2}}\|^2\}\mathrm{d}r \\ & + \int_s^t\int_Z \mathrm{e}^{\lambda r}\|f(X(r))\|^2\mathrm{d}r\nu(\mathrm{d}x) + \int_s^t \mathrm{e}^{\lambda r}(\sigma(X(r))^*(\beta X(r) + 2X_t(r)))\mathrm{d}W(r) \\ & + \int_s^t\int_Z \mathrm{e}^{\lambda r}((f(X(r))^*(\beta X(r) + 2X_t(r)) + \|f(X(r))\|^2)\tilde{N}(\mathrm{d}z,\mathrm{d}r). \qquad (4.10)\end{aligned}$$

By (4.1)–(4.15), we have

$$\begin{aligned}\Phi(X(t))\mathrm{e}^{\lambda t} \le {} & \Phi(X(s))\mathrm{e}^{\lambda s} + \int_s^t \mathrm{e}^{\lambda r}\Big\{\frac{1}{2}(\lambda c + R_\sigma^2 + R_f^2 - \beta)\|X(r)\|^2 \\ & + \Big(\frac{\lambda}{\alpha} - \beta\Big)[m(\|B^{\frac{1}{2}}X(r)\|^2)\|B^{\frac{1}{2}}X(r)\|^2 + h(\|X(r)\|^2)\|X(r)\|^2]\Big\}\mathrm{d}r \\ & + \int_s^t \mathrm{e}^{\lambda r}(\sigma(X(r))^*(\beta X(r) + 2X_t(r)))\mathrm{d}W(r) \\ & + \int_s^t\int_Z \mathrm{e}^{\lambda r}((f(X(r))^*(\beta X(r) + 2X_t(r)) + \|f(X(r))\|^2)\tilde{N}(\mathrm{d}z,\mathrm{d}r).\end{aligned}$$

Choosing $0 < \lambda < 2c^{-1}(\beta - R_\sigma^2 + R_f^2) \wedge \alpha\beta$, from (3.13) we get

$$\begin{aligned}\Phi(X(t))\mathrm{e}^{\lambda(t)} \leq\ & \Phi(X(s)\mathrm{e}^{\lambda s} + \int_s^t \mathrm{e}^{\lambda r}\sigma(X(r))^*(\beta X(r) + 2X_t(r))\mathrm{d}W(r) \\ & + \int_s^t \int_Z \mathrm{e}^{\lambda r}((f(X(r))^*(\beta X(r) + 2X_t(r)) + \|f(X(r))\|^2)\tilde{N}(\mathrm{d}z, \mathrm{d}r).\end{aligned}$$

Choosing $k \to \infty$, we have

$$\begin{aligned}\Phi(X(t))\mathrm{e}^{\lambda t} \leq\ & \Phi(X(s))\mathrm{e}^{\lambda s} + \int_s^t \mathrm{e}^{\lambda r}\sigma(X(r))^*(\beta X(r) + 2X_t(r))\mathrm{d}W(r) \\ & + \int_s^t \int_Z \mathrm{e}^{\lambda r}((f(X(r))^*(\beta X(r) + 2X_t(r)) + \|f(X(r))\|^2)\tilde{N}(\mathrm{d}z, \mathrm{d}r).\end{aligned}$$

This completes the proof.

Proof of Theorem 4.1 In Lemma 4.1, we have proved that the process $(\mathrm{e}^{\lambda t}\Phi(X(t)))_{t\geq 0}$ is a nonnegative continuous supermartingale. We can directly prove our results from the proof of Theorem 1.4 in [3]. By Lemma 4.1, we have

$$E\Phi(X(t)) \leq \mathrm{e}^{-\lambda t}E\Phi(X(0)), \qquad t \geq 0. \tag{4.11}$$

By using (4.4) and the nonnegativity of M, we obtain

$$\frac{1}{2}E\|X(t)\|_{\mathscr{H}}^2 \leq \mathrm{e}^{-\lambda t}E\Phi(X(0)), \qquad t \geq 0. \tag{4.12}$$

This completes the proof.

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