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Generalized Extended tanh-function Method for Traveling Wave Solutions of Nonlinear Physical Equations

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Abstract: In this paper, the generalized extended tanh-function method is used for constructing the traveling wave solutions of nonlinear evolution equations. We choose Fisher's equation, the nonlinear schrödinger equation to illustrate the validity and advantages of the method. Many new and more general traveling wave solutions are obtained. Furthermore, this method can also be applied to other nonlinear equations in physics.

Key words: generalized tanh-function method, nonlinear Schrödinger equation, Fisher's equation

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1 Introduction

It is well known that the nonlinear phenomena is very important in variety of the scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, capillary-gravity waves and chemical physics. Most of these phenomena are described by the nonlinear partial differential equations. So exact solutions of the nonlinear partial differential equations play an essential role in the nonlinear science. For this end, various methods,

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such as the inverse scattering method (see [1]), the Hirota's bilinear technique (see [2]), and truncated Painlv́e expansion (see [3]) have been developed to obtain exact solutions. The tanh method presented by Malfliet[4–6] is a powerful solution method to get the exact traveling wave solutions. Later, Fan *et al.*[7–8] proposed an extended tanh-function method and obtained the new traveling wave solutions which cannot be obtained by tanh-function method. Recently, El-Wakil and Abdou[9] modified the extended tanh-function method and obtained some new exact solutions. In this paper, we extended the modified tanh-function method to get the new exact traveling wave solutions. For illustration, we apply this method to Fisher's equation and the nonlinear Schrödinger equation with general nonlinearity.

2 The Generalized Extend tanh-function Method

In this section, we give a brief description of the generalized extended tanh method. Consider the following nonlinear partial differential equation (PDE):

$$F(u,\ u_t,\ u_x,\ u_{tt},\ u_{xt},\ u_{uu},\ \cdots)=0, \tag{2.1}$$

where $u=u(t,\,x)$ is an unknown function, F is a polynomial in $u=u(t,\,x)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

We first consider the traveling wave solutions of (2.1)

$$u(t,\,x)=U(\xi),\qquad \xi=\lambda(x-Vt),$$

and reduce (2.1) into the following ordinary differential equation (ODE):

$$F(U,\,-\lambda VU',\,\lambda U',\,V^2U'',\,-\lambda VU'',\,\lambda^2U'',\,\cdots)=0, \tag{2.2}$$

where $U'=\dfrac{\mathrm{d}U}{\mathrm{d}\xi}$. The solutions can be expressed as the polynomial form

$$U(\xi)=S(Y(\xi))=\sum_{k=0}^{M}a_kY^k, \tag{2.3}$$

where the positive integer M can be determined by balancing the highest order derivative term with the nonlinear terms in (2.2), and Y is the solution of the Riccati equation

$$Y'=Y^2+\alpha Y+b, \tag{2.4}$$

where α and b are constants to be determined. Substituting (2.3) and (2.4) into (2.2) and equating the coefficients of all powers Y^k to zero yield a system of algebraic equations for V, λ, a_0, a_i $(i=1,\,2,\,\cdots)$, from which the constants are obtained explicitly.

The Riccati equation (2.4) has general solutions as follows:

(I) If $\alpha=0$ and $b=-1$, then

$$Y=-\tanh(-\xi)\quad\text{or}\quad-\coth(-\xi). \tag{2.5}$$

This method is the traditional tanh method (see [4–6]).

(II) If $\alpha=0$ and b is an arbitrary constant, then

$$Y=\begin{cases}-\sqrt{-b}\tanh(-\sqrt{-b}\xi)\quad\text{or}\quad-\sqrt{-b}\coth(-\sqrt{-b}\xi), & b<0;\\ -\dfrac{1}{\xi}, & b=0;\\ \sqrt{b}\tan(\sqrt{b}\xi)\quad\text{or}\quad\sqrt{b}\cot(\sqrt{b}\xi), & b>0.\end{cases} \tag{2.6}$$

This method is the extended tanh-function method (see [7–9]).

(III) If $\alpha \neq 0$ and b is an arbitrary constant, we use the transformation $Y = Z - \dfrac{\alpha}{2}$ and write $B = \dfrac{4b-\alpha^2}{4}$, then (2.4) becomes the similar form as (II):

$$\frac{\mathrm{d}Z}{\mathrm{d}\xi} = Z^2 + B. \tag{2.7}$$

Thus the solutions of (2.4) are

$$Y = \begin{cases} -\dfrac{\sqrt{\alpha^2-4b}}{2}\tanh\Big(-\dfrac{\sqrt{\alpha^2-4b}}{2}\xi\Big) - \dfrac{\alpha}{2} \quad \text{or} & \\ \quad -\dfrac{\sqrt{\alpha^2-4b}}{2}\coth\Big(-\dfrac{\sqrt{\alpha^2-4b}}{2}\xi\Big) - \dfrac{\alpha}{2}, & \alpha^2-4b>0; \\ -\dfrac{1}{\xi} - \dfrac{\alpha}{2}, & \alpha^2-4b=0; \\ \dfrac{\sqrt{4b-\alpha^2}}{2}\tan\Big(\dfrac{\sqrt{4b-\alpha^2}}{2}\xi\Big) - \dfrac{\alpha}{2} \quad \text{or} & \\ \quad \dfrac{\sqrt{4b-\alpha^2}}{2}\cot\Big(\dfrac{\sqrt{4b-\alpha^2}}{2}\xi\Big) - \dfrac{\alpha}{2}, & \alpha^2-4b<0. \end{cases} \tag{2.8}$$

3 Applications

3.1 Fisher's Equation

We consider the generalized Fisher's equation

$$u_t = u_{xx} + u(1-u^2), \tag{3.1}$$

and look for the traveling wave solution

$$u(t,\, x) = U(\xi), \qquad \xi = \lambda(x - Vt).$$

Then (3.1) is transformed into the following ODE:

$$-\lambda V\frac{\mathrm{d}U}{\mathrm{d}\xi} = \lambda^2\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2} + U - U^3 = 0. \tag{3.2}$$

Substituting (2.3) and (2.4) into (3.2), we can get

$$\lambda V(Y^2+\alpha V+b)\frac{\mathrm{d}S}{\mathrm{d}Y} + \lambda^2\Big[(Y^2+\alpha Y+b)^2\frac{\mathrm{d}^2S}{\mathrm{d}Y^2} + (2Y+\alpha)(Y^2+\alpha Y+b)\frac{\mathrm{d}S}{\mathrm{d}Y}\Big] + S - S^3 = 0. \tag{3.3}$$

To determine the parameter M we usually balance the linear terms of highest order in the resulting equation (3.3) with the nonlinear terms of highest order. Thus we get

$$M - 2 + 4 = 3M \Rightarrow M = 1.$$

Write

$$U(\xi) = S(Y(\xi)) = a_0 + a_1Y(\xi). \tag{3.4}$$

Substituting (3.4) into (3.3), we have the algebraic equation with respect to V, λ, a_0, a_1, as follows:

$$\lambda Va_1(Y^2+\alpha V+b) + \lambda^2a_1(2Y+\alpha)(Y^2+\alpha Y+b) + (a_0+a_1Y) - (a_0+a_1Y)^3 = 0. \tag{3.5}$$

So we get

$$(a_0 - a_0^3 + a_1 bV\lambda + b\alpha\lambda^2 a_1) + Y(a_1 - 3a_0^2 a_1 + a_1 V\alpha\lambda 2b\lambda^2 a_1 + \alpha^2\lambda^2 a_1) + Y^2(-3a_0 a_1^2 + a_1 V\lambda + 3\alpha\lambda^2 a_1) + Y^3(-a_1^3 + 2\lambda^2 a_1) = 0.$$

Equating each coefficient of this polynomial to zero, we obtain the following system of the algebraic equations with respect to V, λ, a_0, a_1, α, b:

$$\begin{cases} Y^0: \ a_0 - a_0^3 + a_1 bV\lambda + b\alpha\lambda^2 a_1 = 0; \\ Y^1: \ a_1 - 3a_0^2 a_1 + 2b\lambda^2 a_1 + \alpha^2\lambda^2 a_1 = 0; \\ Y^2: \ -3a_0 a_1^2 + a_1 V\lambda + 3\alpha\lambda^2 a_1 = 0; \\ Y^3: \ -a_1^3 + 2\lambda^2 a_1 = 0. \end{cases} \tag{3.6}$$

(I) If $\alpha = 0$ and $b = -1$, with the aid of Mathematica, we get the solutions of (3.6):

$$V = 0, \quad a_0 = 0, \quad a_1 = \pm 1, \quad \lambda = \pm\frac{1}{\sqrt{2}};$$
$$V = \pm\frac{3}{\sqrt{2}}, \quad a_0 = \pm\frac{1}{2}, \quad a_1 = \pm\frac{1}{2}, \quad \lambda = \pm\frac{1}{2\sqrt{2}},$$

with

$$\mathrm{sgn}(a_0)\cdot\mathrm{sgn}(a_1)\cdot\mathrm{sgn}(\lambda)\cdot\mathrm{sgn}(V) = 1.$$

So, according to (2.5), we get the solutions of (3.1) as follows (see [10]):

$$\begin{aligned} u_{1-4}^{(\mathrm{I})} &= U(\lambda(x - Vt)) = S(Y(\xi)) = a_0 + a_1 Y(\xi) \\ &= \pm\tanh\Big(\pm\frac{1}{\sqrt{2}}x\Big) \quad \text{or} \quad \pm\coth\Big(\pm\frac{1}{\sqrt{2}}x\Big); \\ u_{5-12}^{(\mathrm{I})} &= \pm\frac{1}{2} \pm \frac{1}{2}\tanh\Big[\pm\frac{1}{2\sqrt{2}}\Big(x - \Big(\pm\frac{3}{\sqrt{2}}\Big)t\Big)\Big] \quad \text{or} \\ &\quad \pm\frac{1}{2} \pm \frac{1}{2}\coth\Big[\pm\frac{1}{2\sqrt{2}}\Big(x - \Big(\pm\frac{3}{\sqrt{2}}\Big)t\Big)\Big], \end{aligned}$$

with

$$\mathrm{sgn}(a_0)\cdot\mathrm{sgn}(a_1)\cdot\mathrm{sgn}(\lambda)\cdot\mathrm{sgn}(V) = 1.$$

(II) If $\alpha = 0$ and b is an arbitrary constant, then the method is the modified extended tanh-function method. With the aid of Mathematica, we get the solutions of (3.6):

$$V = 0, \quad b = -\frac{1}{2\lambda^2}, \quad a_0 = 0, \quad a_1 = \pm\sqrt{2}\lambda;$$
$$V = \pm\frac{3}{\sqrt{2}}, \quad b = -\frac{1}{8\lambda^2}, \quad a_0 = \pm\frac{1}{2}, \quad a_1 = \pm\sqrt{2}\lambda,$$

with

$$\mathrm{sgn}(a_0)\cdot\mathrm{sgn}(a_1)\cdot\mathrm{sgn}(\lambda)\cdot\mathrm{sgn}(V) = 1.$$

Obviously, when $b < 0$, according to (2.5), we get the solutions of (3.1) as follows:

$$\begin{aligned} u_{1-2}^{(\mathrm{II})} &= a_0 + a_1 Y(\lambda(x - Vt)) \\ &= \pm\tanh\Big(-\mathrm{sgn}(\lambda)\frac{1}{\sqrt{2}}x\Big) \quad \text{or} \\ &\quad \pm\coth\Big(-\mathrm{sgn}(\lambda)\frac{1}{\sqrt{2}}x\Big); \\ u_{3-4}^{(\mathrm{II})} &= \frac{1}{2} \pm \frac{\mathrm{sgn}(\lambda)}{2}\tanh\Big[\frac{-\mathrm{sgn}(\lambda)}{2\sqrt{2}}\Big(x \pm \frac{\sqrt{3}}{2}t\Big)\Big] \quad \text{or} \end{aligned}$$

$$\frac{1}{2} \pm \frac{\mathrm{sgn}(\lambda)}{2} \coth\Big[-\frac{\mathrm{sgn}(\lambda)}{2\sqrt{2}}\Big(x \pm \frac{\sqrt{3}}{2}t\Big)\Big];$$

$$u_{5-6}^{(\mathrm{II})} = -\frac{1}{2} \pm \frac{\mathrm{sgn}(\lambda)}{2} \tanh\Big[\frac{-\mathrm{sgn}(\lambda)}{2\sqrt{2}}\Big(x \mp \frac{\sqrt{3}}{2}t\Big)\Big] \quad \text{or}$$

$$-\frac{1}{2} \pm \frac{\mathrm{sgn}(\lambda)}{2} \coth\Big[-\frac{\mathrm{sgn}(\lambda)}{2\sqrt{2}}\Big(x \mp \frac{\sqrt{3}}{2}t\Big)\Big].$$

(III) (i) When $\alpha^2 - 4b = 0$, $\alpha \neq 0$, with the aid of Mathematica, we obtain the solutions of (3.6) as follows:

$$V = \frac{1}{6}(\alpha \pm \sqrt{216 + \alpha^2}), \quad a_0 = \pm\frac{\sqrt{2}}{3}, \quad a_1 = \frac{1}{72}(-6Va_0 + \alpha a_0), \quad \lambda = -\frac{1}{18};$$

$$V = -\frac{\alpha}{6}, \quad a_0 = 0, \quad a_1 = \pm\frac{1}{9\sqrt{2}}, \quad \lambda = \frac{1}{18};$$

$$V = \frac{-\alpha + 216\alpha\lambda^2 - 23328\alpha\lambda^3}{2}, \quad a_0 = \frac{\pm\sqrt{1 - 324\lambda^2}}{\sqrt{3}},$$

$$a_1 = \frac{\alpha a_0(13 - 108\lambda - 5184\lambda^2 + 244944\lambda^3)}{432};$$

$$V = -\frac{27(16\alpha\lambda^2 + 72\alpha\lambda^3 - \alpha^3\lambda^3 + 36\alpha^3\lambda^4)}{2(1 + 18\lambda)}, \quad a_0 = \pm\frac{\sqrt{2 + 3\alpha^2\lambda^2}}{\sqrt{6}}, \quad 1 + 18\lambda \neq 0,$$

$$a_1 = \frac{a_0(-12V - 4\alpha + 36\alpha\lambda - 1944\alpha\lambda^2 - 9\alpha^3\lambda^2 - 11664\alpha\lambda^3 + 486\alpha^3\lambda^3 - 5832\alpha^3\lambda^4)}{216 + \alpha^2}.$$

According to (2.8), we obtain the solutions of (3.1):

$$u_{1-4}^{(\mathrm{i})} = \frac{\pm\sqrt{2}}{3} + \frac{-6Va_0 + \alpha a_0}{72}\Big(\frac{18}{x - Vt} - \frac{\alpha}{2}\Big), \quad V = \frac{1}{6}(\alpha \pm \sqrt{216 + \alpha^2});$$

$$u_{5-6}^{(\mathrm{i})} = \pm\frac{1}{9\sqrt{2}}\Big(-\frac{18}{x + \frac{\alpha}{6}t} - \frac{\alpha}{2}\Big);$$

$$u_{7-8}^{(\mathrm{i})} = \frac{\pm\sqrt{1 - 324\lambda^2}}{\sqrt{3}} + \frac{-a_1}{\lambda\Big[x - \frac{1}{2}(-\alpha + 216\alpha\lambda^2 - 23328\alpha\lambda^3)t\Big]} - \frac{\alpha}{2};$$

$$u_{9-10}^{(\mathrm{i})} = \pm\frac{\sqrt{2 + 3\alpha^2\lambda^2}}{\sqrt{6}} - \frac{a_1}{\lambda\Big[x + \frac{27\left(16\alpha\lambda^2 + 72\alpha\lambda^3 - \alpha^3\lambda^3 + 36\alpha^3\lambda^4\right)}{2(1 + 18\lambda)}t\Big]} - \frac{\alpha a_1}{2}.$$

(ii) When $\alpha^2 - 4b \neq 0$ and $\alpha \neq 0$, with the aid of Mathematica, we get the solutions as follows:

$$V = -3\alpha\lambda, \quad a_0 = 0, \quad a_1 = \pm\sqrt{2}\lambda, \quad \alpha = \frac{\pm 1}{\sqrt{-\lambda^2 - 9\lambda^3}}, \quad b = \frac{9\alpha^2\lambda}{2}; \tag{3.7a}$$

$$V = 0, \quad a_0 = \pm\frac{\alpha\lambda}{\sqrt{2}}, \quad a_1 = \frac{2a_0}{\alpha}, \quad b = \frac{-2 + \alpha^2\lambda^2}{4\lambda^2}; \tag{3.7b}$$

$$V = \frac{3}{8}(-9\alpha\lambda - 18\alpha\lambda^2 \pm \sqrt{32 + 33\alpha^2\lambda^2 + 324\alpha^2\lambda^3 + 324\alpha^2\lambda^4}), \quad a_1 = \frac{6\lambda a_0}{V + 3\alpha\lambda},$$

$$a_0 = \frac{\pm\sqrt{V^2+6V\alpha\lambda+9\alpha^2\lambda^2}}{3\sqrt{2}}, \quad b = \frac{-6+V^2+6V\alpha\lambda+3\alpha^2\lambda^2}{12\lambda^2}. \tag{3.7c}$$

Obviously, from (3.7a) we have

$$\lambda < -\frac{1}{9}, \qquad \alpha^2 - 4b = \alpha^2(1-18\lambda) > 0.$$

So, according to (2.8), we have the exact traveling wave solutions of (3.1) as follows:

$$\begin{aligned} u^{(\mathrm{ii})}_{1-4} = &\pm \sqrt{2}\lambda\Big\{-\frac{1}{2}\Big(\pm\frac{1}{\sqrt{-\lambda^2-9\lambda^3}}\Big) \\ &+\frac{1}{2}\sqrt{\frac{1-18\lambda}{-\lambda^2-9\lambda^3}}\tanh\Big[\frac{1}{2}\lambda\sqrt{\frac{1-18\lambda}{-\lambda^2-9\lambda^3}}\Big(x\pm\frac{3\lambda}{\sqrt{-\lambda^2-9\lambda^3}}\Big)t\Big]\Big\} \quad \text{or} \\ &\pm \sqrt{2}\lambda\Big\{-\frac{1}{2}\Big(\pm\frac{1}{\sqrt{-\lambda^2-9\lambda^3}}\Big) \\ &+\frac{1}{2}\sqrt{\frac{1-18\lambda}{-\lambda^2-9\lambda^3}}\coth\Big[\frac{1}{2}\lambda\sqrt{\frac{1-18\lambda}{-\lambda^2-9\lambda^3}}\Big(x\pm\frac{3\lambda}{\sqrt{-\lambda^2-9\lambda^3}}\Big)t\Big]\Big\} \end{aligned}$$

with $\lambda < -\frac{1}{9}$.

From (3.7b) we have $\alpha^2 - 4b = 2 > 0$. The traveling wave solutions of (3.1) are

$$\begin{aligned} u^{(\mathrm{ii})}_{5-6} = &\pm\frac{\sqrt{\alpha^2\lambda^2}\sqrt{\alpha^2-\frac{-6+3\alpha^2\lambda^2}{3\lambda^2}}}{2\sqrt{2}\alpha}\Big\{\tanh\Big[\frac{1}{2}x\lambda\sqrt{\alpha^2-\frac{-6+3\alpha^2\lambda^2}{3\lambda^2}}\Big]\Big\} \quad \text{or} \\ &\pm\frac{\sqrt{\alpha^2\lambda^2}\sqrt{\alpha^2-\frac{-6+3\alpha^2\lambda^2}{3\lambda^2}}}{2\sqrt{2}\alpha}\Big\{\coth\Big[\frac{1}{2}x\lambda\sqrt{\alpha^2-\frac{-6+3\alpha^2\lambda^2}{3\lambda^2}}\Big]\Big\}. \end{aligned}$$

From (3.7c) we have $\alpha^2 - 4b = \frac{6-V^2-6V\alpha\lambda}{3\lambda^2}$.

So, if $6 - V^2 - 6V\alpha\lambda > 0$, then we get the solutions of (3.1) as follows:

$$\begin{aligned} u^{(\mathrm{ii})}_{7-10} = &\frac{\sqrt{V^2+6V\alpha\lambda+9\alpha^2\lambda^2}}{3\sqrt{2}} + \frac{\sqrt{2}\lambda\sqrt{V^2+6V\alpha\lambda+9\alpha^2\lambda^2}}{V+3\alpha\lambda} \\ &\cdot\Big\{\frac{\sqrt{\alpha^2-4b}}{2}\tanh\Big[\frac{\sqrt{\alpha^2-4b}\lambda(x-tV)}{2}\Big]-\frac{\alpha}{2}\Big\} \quad \text{or} \\ &\pm\frac{\sqrt{V^2+6V\alpha\lambda+9\alpha^2\lambda^2}}{3\sqrt{2}} + \frac{\sqrt{2}\lambda\sqrt{V^2+6V\alpha\lambda+9\alpha^2\lambda^2}}{V+3\alpha\lambda} \\ &\cdot\Big\{\frac{\sqrt{\alpha^2-4b}}{2}\coth\Big[\frac{\sqrt{\alpha^2-4b}\lambda(x-tV)}{2}\Big]-\frac{\alpha}{2}\Big\} \end{aligned}$$

with

$$V = \frac{3}{8}(\pm\sqrt{32+33\alpha^2\lambda^2+324\alpha^2\lambda^3+324\alpha^2\lambda^4}-9\alpha\lambda-18\alpha\lambda^2);$$

if $6 - V^2 - 6V\alpha\lambda < 0$, then the traveling wave solutions of (3.1) are

$$\begin{aligned} u^{(\mathrm{ii})}_{11-14} = &\frac{\sqrt{V^2+6V\alpha\lambda+9\alpha^2\lambda^2}}{3\sqrt{2}} + \frac{\sqrt{2}\lambda\sqrt{V^2+6V\alpha\lambda+9\alpha^2\lambda^2}}{V+3\alpha\lambda} \\ &\cdot\Big\{\frac{\sqrt{4b-\alpha^2}}{2}\tanh\Big[\frac{\sqrt{4b-\alpha^2}\lambda(x-tV)}{2}\Big]-\frac{\alpha}{2}\Big\} \quad \text{or} \\ &\pm\frac{\sqrt{V^2+6V\alpha\lambda+9\alpha^2\lambda^2}}{3\sqrt{2}} + \frac{\sqrt{2}\lambda\sqrt{V^2+6V\alpha\lambda+9\alpha^2\lambda^2}}{V+3\alpha\lambda} \end{aligned}$$

$$\cdot\Big\{\frac{\sqrt{4b-\alpha^2}}{2}\coth\Big[\frac{\sqrt{4b-\alpha^2}\lambda(x-tV)}{2}\Big]-\frac{\alpha}{2}\Big\}$$

with the same velocity as above.

3.2 The Nonlinear Schrödinger Equation

We consider the nonlinear Schrödinger equation

$$\mathrm{i}u_t=v_{xx}+mu+|u^{2n}|u-\epsilon u_{xt}=0,\qquad n\in\mathbf{Z}^+,\tag{3.8}$$

where $u(t,\,x)$ is a complex function, and $m,\,\epsilon\in\mathbf{R}$ are constants. We assume that

$$u(t,\,x)=U(t,\,x)\mathrm{e}^{\mathrm{i}(\mu x+\nu t)},\tag{3.9}$$

where $U(t,\,x)$ is a real function, μ and ν are constants to be determined. Substituting (3.9) into (3.8), removing the common factor $\mathrm{e}^{\mathrm{i}(\mu x+\nu t)}$ and separating the real and imaginary parts, we have the following PDEs of $U(t,\,x)$:

$$\begin{cases}(2\mu+\epsilon\nu)U_x-(1-\epsilon\mu)U_t=0,\\ U_{xx}+\epsilon U_{x,t}+(\nu-\mu^2+m-\epsilon\nu\mu)U+U^{2n+1}=0.\end{cases}\tag{3.10}$$

(I) When $n=1$, we look for the traveling wave solutions

$$U(t,\,x)=\varPhi(t,x)=\varPhi(\xi),\qquad \xi=(1-\epsilon\mu)x+(2\mu+\epsilon\nu)t.$$

Then (3.10) becomes

$$(1-\epsilon\mu)(1+\epsilon\mu+\epsilon^2\nu)\varPhi_{\xi\xi}+(\nu-\mu^2+m-\epsilon\nu\mu)\varPhi+\varPhi^3=0.\tag{3.11}$$

Substituting $\varPhi(\xi)=S(Y(\xi))=\sum\limits_{k=0}^{M}a_kY^k$ into (3.11), according to (2.3)–(2.4), we get

$$(1-\epsilon\mu)(1+\epsilon\mu+\epsilon^2\nu)\Big[(Y^2+\alpha Y+b)^2\frac{\mathrm{d}^2S(Y)}{\mathrm{d}Y^2}+(2Y+\alpha)(Y^2+\alpha Y+b)\frac{\mathrm{d}S(Y)}{\mathrm{d}Y}\Big]$$
$$+(\nu-\mu^2+m-\epsilon\nu\mu)S(Y)+S^3(Y)=0.$$

Balancing the linear term of the highest order with the nonlinear term yields $M=1$. Therefore, we get $\varPhi(\xi)=a_0+a_1Y$. Substituting it into the above equation, we get the system of algebraic equations with respect to $a_0,\,a_1,\,\mu,\,\nu,\,\alpha,\,b$:

$$\begin{cases}Y^0:\ (m-\mu^2+\nu-\epsilon\mu\nu)a_0+a_0^3+b\alpha(1-\epsilon\mu)(1+\epsilon\mu+\epsilon^2\nu)a_1=0;\\ Y^1:\ 2b(1-\epsilon\mu)\left(1+\epsilon\mu+\epsilon^2\nu\right)a_1+3a_0^2a_1\\ \qquad +\alpha^2a_1(1-\epsilon\mu)(1+\epsilon\mu+\epsilon^2\nu)+(m-\mu^2+\nu-\epsilon\mu\nu)a_1=0;\\ Y^2:\ 3\alpha(1-\epsilon\mu)\left(1+\epsilon\mu+\epsilon^2\nu\right)a_1+3a_0a_1^2=0;\\ Y^3:\ 2(1-\epsilon\mu)\left(1+\epsilon\mu+\epsilon^2\nu\right)a_1+a_1^3=0.\end{cases}\tag{3.12}$$

(i) When $\alpha=0$ and $b=-1$, with the aid of Mathematica, we get the solutions of (3.12):

$$a_0=0,\quad a_1=\pm\frac{\sqrt{2}\sqrt{1-m\epsilon^2}}{\sqrt{-1+2\epsilon^2}},\quad \mu=-\frac{1}{2}\epsilon\nu\pm\frac{1}{2}\sqrt{4\nu+\epsilon^2\nu^2+\frac{8-4m}{-1+2\epsilon^2}}.$$

So, by the traditional tanh method, we can get the traveling wave solutions of (3.8):

$$u_{1-4}^{(\mathrm{i})}=\Big\{\pm\frac{\sqrt{2}\sqrt{1-m\epsilon^2}}{\sqrt{-1+2\epsilon^2}}\tanh\Big[\Big(1+\frac{1}{2}\epsilon^2\nu\mp\frac{1}{2}\epsilon\sqrt{4\nu+\epsilon^2\nu^2+\frac{8-4m}{-1+2\epsilon^2}}\Big)x+\Big(\nu-\epsilon\nu$$
$$\pm\sqrt{4\nu+\epsilon^2\nu^2+\frac{8-4m}{-1+2\epsilon^2}}\Big)t\Big]\Big\}\mathrm{e}^{\mathrm{i}\Big[\Big(-\frac{1}{2}\epsilon\nu\pm\frac{1}{2}\sqrt{4\nu+\epsilon^2\nu^2+\frac{8-4m}{-1+2\epsilon^2}}\Big)x+\nu t\Big]}\quad\text{or}$$
$$\Big\{\pm\frac{\sqrt{2}\sqrt{1-m\epsilon^2}}{\sqrt{-1+2\epsilon^2}}\coth\Big[\Big(1+\frac{1}{2}\epsilon^2\nu\mp\frac{1}{2}\epsilon\sqrt{4\nu+\epsilon^2\nu^2+\frac{8-4m}{-1+2\epsilon^2}}\Big)x+\Big(\nu-\epsilon\nu$$

$$\pm\sqrt{4\nu+\epsilon^2\nu^2+\frac{8-4m}{-1+2\epsilon^2}}\Big)t\Big]\Big\}\mathrm{e}^{\mathrm{i}\Big[\Big(-\frac{1}{2}\epsilon\nu\pm\frac{1}{2}\sqrt{4\nu+\epsilon^2\nu^2+\frac{8-4m}{-1+2\epsilon^2}}\Big)x+\nu t\Big]}.$$

(ii) When $\alpha=0$ and b is an arbitrary constant, we can obtain the solutions of (3.12):

$$a_0=0,\quad a_1=\pm\frac{\sqrt{2}\sqrt{-1+m\epsilon^2}}{\sqrt{1+2b\epsilon^2}},\quad \mu=\frac{-\epsilon\nu}{2}\pm\frac{\sqrt{8b+4m+(4+8b\epsilon^2)\nu+(\epsilon^2+2b\epsilon^4)\nu^2}}{2\sqrt{1+2b\epsilon^2}}.$$

According to (2.6), if $-\frac{1}{\epsilon^2}<b<0$, we can get the traveling wave solutions of (3.8):

$$\begin{aligned}u_{1-4}^{(\mathrm{ii})}=&\pm\frac{\sqrt{-2b}\sqrt{-1+m\epsilon^2}}{\sqrt{1+2b\epsilon^2}}\tanh\Big\{\sqrt{-b}\Big[\Big(\frac{-\epsilon\nu}{2}\pm\frac{\sqrt{8b+4m+(4+8b\epsilon^2)\nu+(\epsilon^2+2b\epsilon^4)\nu^2}}{2\sqrt{1+2b\epsilon^2}}\Big)t\\&+\Big(1+\frac{\epsilon^2\nu}{2}\pm\frac{\epsilon\sqrt{8b+4m+(4+8b\epsilon^2)\nu+(\epsilon^2+2b\epsilon^4)\nu^2}}{2\sqrt{1+2b\epsilon^2}}\Big)x\Big]\Big\}\mathrm{e}^{\mathrm{i}(\mu x+\nu t)}\quad\text{or}\\&\pm\frac{\sqrt{-2b}\sqrt{-1+m\epsilon^2}}{\sqrt{1+2b\epsilon^2}}\coth\Big\{\sqrt{-b}\Big[\Big(\frac{-\epsilon\nu}{2}\pm\frac{\sqrt{8b+4m+(4+8b\epsilon^2)\nu+(\epsilon^2+2b\epsilon^4)\nu^2}}{2\sqrt{1+2b\epsilon^2}}\Big)t\\&+\Big(1+\frac{\epsilon^2\nu}{2}\pm\frac{\epsilon\sqrt{8b+4m+(4+8b\epsilon^2)\nu+(\epsilon^2+2b\epsilon^4)\nu^2}}{2\sqrt{1+2b\epsilon^2}}\Big)x\Big]\Big\}\mathrm{e}^{\mathrm{i}(\mu x+\nu t)}.\end{aligned}$$

If $b=0$, then the solutions of (3.8) are

$$u_{5-8}^{(\mathrm{ii})}=-\frac{\pm\sqrt{2}\sqrt{-1+m\epsilon^2}}{\Big[(\pm\sqrt{4m+4\nu+\epsilon^2\nu^2}+\nu-\epsilon\nu)t+\Big(1+\frac{1}{2}\epsilon^2\nu\pm\epsilon\sqrt{4m+4\nu+\epsilon^2\nu^2}\Big)x\Big]}\mathrm{e}^{\mathrm{i}(\mu x+\nu t)}.$$

If $b>0$, then the traveling wave solutions of (3.8) are

$$\begin{aligned}u_{9-12}^{(\mathrm{ii})}=&\frac{\pm\sqrt{2b}\sqrt{-1+m\epsilon^2}}{\sqrt{1+2b\epsilon^2}}\tanh\Big\{\sqrt{b}\Big[\Big((1-\epsilon)\nu\pm\frac{\sqrt{8b+4m+(4+8b\epsilon^2)\nu+(\epsilon^2+2b\epsilon^4)\nu^2}}{\sqrt{1+2b\epsilon^2}}\Big)t\\&+\Big(1+\frac{\epsilon^2\nu}{2}\pm\frac{\epsilon\sqrt{8b+4m+(4+8b\epsilon^2)\nu+(\epsilon^2+2b\epsilon^4)\nu^2}}{2\sqrt{1+2b\epsilon^2}}\Big)x\Big]\Big\}\mathrm{e}^{\mathrm{i}(\mu x+\nu t)}\quad\text{or}\\&\frac{\pm\sqrt{2b}\sqrt{-1+m\epsilon^2}}{\sqrt{1+2b\epsilon^2}}\coth\Big\{\sqrt{b}\Big[\Big((1-\epsilon)\nu\pm\frac{\sqrt{8b+4m+(4+8b\epsilon^2)\nu+(\epsilon^2+2b\epsilon^4)\nu^2}}{\sqrt{1+2b\epsilon^2}}\Big)t\\&+\Big(1\pm\frac{\epsilon\sqrt{8b+4m+(4+8b\epsilon^2)\nu+(\epsilon^2+2b\epsilon^4)\nu^2}}{2\sqrt{1+2b\epsilon^2}}+\frac{\epsilon^2\nu}{2}\Big)x\Big]\Big\}\mathrm{e}^{\mathrm{i}(\mu x+\nu t)}.\end{aligned}$$

(iii) When $\alpha\neq0$ and $\alpha^2-4b=0$, the solutions of (3.12) are

$$a_0=\pm\frac{\sqrt{-\alpha^2+m\alpha^2\epsilon^2}}{\sqrt{2}},\quad a_1=\frac{2a_0}{\alpha},\quad \mu=\frac{1}{2}\Big(-\epsilon\nu+\sqrt{4m+4\nu+\epsilon^2\nu^2+12a_0^2-6\alpha a_0a_1}\Big).$$

According to (2.8), we get the traveling wave solutions of (3.8):

$$u_{1-4}^{(\mathrm{iii})}=\pm\sqrt{\frac{-\alpha^2+m\alpha^2\epsilon^2}{2}}\cdot\frac{-\frac{2}{\alpha}}{t\left(\nu-\epsilon\nu+\sqrt{4m+4\nu+\epsilon^2\nu^2}\right)+x\Big(1+\frac{1}{2}\epsilon^2\nu-\frac{1}{2}\epsilon\sqrt{4m+4\nu+\epsilon^2\nu^2}\Big)}\mathrm{e}^{\mathrm{i}(\mu x+\nu t)}.$$

If $\alpha\neq0$ and $\alpha^2-4b\neq0$, then we obtain two sets of solutions of (3.12) as follows:

(1)

$$a_0=\pm\frac{\sqrt{\alpha^2-m\alpha^2\epsilon^2}}{\sqrt{-2-4b\epsilon^2+\alpha^2\epsilon^2}},\qquad a_1=\frac{2a_0}{\alpha},$$

$$\mu=\frac{1}{2}\Big(-\epsilon\nu\pm\sqrt{4m+4\nu+\epsilon^2\nu^2+12a_0^2-4\alpha a_0a_1-4ba_1^2}\Big);$$

(2)

$$a_0 \neq 0, \quad a_1 = \frac{2a_0}{\alpha}, \quad b = \frac{1}{4}(-2m + \alpha^2),$$
$$\mu = \frac{1}{2}\Big(- \epsilon\nu \pm \sqrt{4m + 4\nu + \epsilon^2\nu^2 + 2ma_1^2}\Big).$$

Obviously, in (1) we have $\alpha^2 - 4b > \frac{2}{\epsilon^2} > 0$. So the traveling wave solutions of (3.8) are

$$u_{5-8}^{(\mathrm{iii})} = \pm\sqrt{\frac{\alpha^2 - m\alpha^2\epsilon^2}{-2 - 4b\epsilon^2 + \alpha^2\epsilon^2}}\Big\{\frac{\sqrt{\alpha^2 - 4b}}{\alpha}\tanh\Big[\frac{\sqrt{\alpha^2 - 4b}}{2}$$
$$\cdot\Big(\pm\sqrt{\frac{(\alpha^2 - 4b)(\epsilon\nu^2 + 2)^2 + 8(m + \nu)}{2 + 4b\epsilon^2 - \alpha^2\epsilon^2}} + \nu - \epsilon\nu\Big)t$$
$$+ \Big(1 + \frac{1}{2}\epsilon^2\nu \pm \frac{1}{2}\epsilon\sqrt{\frac{(\alpha^2 - 4b)(\epsilon\nu^2 + 2)^2 + 8(m + \nu)}{2 + 4b\epsilon^2 - \alpha^2\epsilon^2}}\Big)x\Big]\Big\}\mathrm{e}^{\mathrm{i}(\mu x + \nu t)} \quad \text{or}$$
$$\pm\sqrt{\frac{\alpha^2 - m\alpha^2\epsilon^2}{-2 - 4b\epsilon^2 + \alpha^2\epsilon^2}}\Big\{\frac{\sqrt{-4b + \alpha^2}}{\alpha}\coth\Big[\frac{\sqrt{\alpha^2 - 4b}}{2}$$
$$\cdot\Big(\pm\sqrt{\frac{(\alpha^2 - 4b)(\epsilon\nu^2 + 2)^2 + 8(m + \nu)}{2 + 4b\epsilon^2 - \alpha^2\epsilon^2}} + \nu - \epsilon\nu\Big)t$$
$$+ \Big(1 + \frac{1}{2}\epsilon^2\nu \pm \frac{1}{2}\epsilon\sqrt{\frac{(\alpha^2 - 4b)(\epsilon\nu^2 + 2)^2 + 8(m + \nu)}{2 + 4b\epsilon^2 - \alpha^2\epsilon^2}}\Big)x\Big]\Big\}\mathrm{e}^{\mathrm{i}(\mu x + \nu t)}.$$

In (2), we have $\alpha^2 - 4b = 2m$. So, if $m > 0$, we can get the solutions of (3.8)

$$u_{9-12}^{(\mathrm{iii})} = \frac{\sqrt{-4b + \alpha^2}}{\alpha}\tanh\Big\{\frac{\sqrt{-4b + \alpha^2}}{2}\Big[\Big(\nu - \epsilon\nu \pm \sqrt{4m + 4\nu + \epsilon^2\nu^2 + \frac{8ma_0^2}{\alpha^2}}\Big)t$$
$$+ \Big(1 + \frac{1}{2}\epsilon^2\nu \pm \frac{1}{2}\epsilon\sqrt{4m + 4\nu + \epsilon^2\nu^2 + \frac{8ma_0^2}{\alpha^2}}\Big)x\Big]\Big\}\mathrm{e}^{\mathrm{i}(\mu x + \nu t)} \quad \text{or}$$
$$\frac{\sqrt{-4b + \alpha^2}}{\alpha}\tanh\Big\{\frac{\sqrt{-4b + \alpha^2}}{2}\Big[\Big(\nu - \epsilon\nu \pm \sqrt{4m + 4\nu + \epsilon^2\nu^2 + \frac{8ma_0^2}{\alpha^2}}\Big)t$$
$$+ \Big(1 + \frac{1}{2}\epsilon^2\nu \pm \frac{1}{2}\epsilon\sqrt{4m + 4\nu + \epsilon^2\nu^2 + \frac{8ma_0^2}{\alpha^2}}\Big)x\Big]\Big\}\mathrm{e}^{\mathrm{i}(\mu x + \nu t)}.$$

If $m < 0$, according to (2.8), the traveling wave solutions of (3.8) are

$$u_{13-16}^{(\mathrm{iii})} = \frac{\sqrt{\alpha^2 - 4b}}{\alpha}\tanh\Big\{\frac{\sqrt{\alpha^2 - 4b}}{2}\Big[\Big(\nu - \epsilon\nu \pm \sqrt{4m + 4\nu + \epsilon^2\nu^2 + \frac{8ma_0^2}{\alpha^2}}\Big)t$$
$$+ \Big(1 + \frac{1}{2}\epsilon^2\nu \pm \frac{1}{2}\epsilon\sqrt{4m + 4\nu + \epsilon^2\nu^2 + \frac{8ma_0^2}{\alpha^2}}\Big)x\Big]\Big\}\mathrm{e}^{\mathrm{i}(\mu x + \nu t)} \quad \text{or}$$
$$\frac{\sqrt{\alpha^2 - 4b}}{\alpha}\tanh\Big\{\frac{\sqrt{\alpha^2 - 4b}}{2}\Big[\Big(\nu - \epsilon\nu \pm \sqrt{4m + 4\nu + \epsilon^2\nu^2 + \frac{8ma_0^2}{\alpha^2}}\Big)t$$
$$+ \Big(1 + \frac{1}{2}\epsilon^2\nu \pm \frac{1}{2}\epsilon\sqrt{4m + 4\nu + \epsilon^2\nu^2 + \frac{8ma_0^2}{\alpha^2}}\Big)x\Big]\Big\}\mathrm{e}^{\mathrm{i}(\mu x + \nu t)}.$$

(II) When $n > 1$, if we proceed as presented above, we find $M = \frac{1}{n}$. This means that the tanh method is not appropriate for any positive integer $n \geq 2$. In order to use the method, we make the following transformation as that in [10]:

$$U(t, x) = \Phi(t, x)^{\frac{1}{n}}, \qquad n \in \mathbf{Z}^+.$$

Then (3.10) is changed into

$$\begin{cases} (2\mu+\epsilon\nu)\varPhi_x-(1-\epsilon\mu)\varPhi_t=0, \\ \varPhi\varPhi_{xx}+\Big(\dfrac{1}{n}-1\Big)\varPhi_x^2+\epsilon\Big[\Big(\dfrac{1}{n}-1\Big)\varPhi_t\varPhi_x+\varPhi\varPhi_{x,t}\Big] \\ +n(\nu-\mu^2+m-\epsilon\nu\mu)\varPhi+n\varPhi^4=0. \end{cases} \tag{3.13}$$

Assume that the traveling wave solutions of (3.13) have the form

$$\varPhi(t,x)=\varPhi(\xi),\qquad \xi=(1-\epsilon\mu)x+(2\mu+\epsilon\nu)t. \tag{3.14}$$

Substituting (3.14) into (3.13), we get

$$\begin{aligned}&(1-\epsilon\mu)(1+\epsilon\mu+\epsilon^2\nu)\varPhi\varPhi_{\xi\xi}+\Big(\frac{1}{n}-1\Big)(1-\epsilon\mu)(1+\epsilon\mu+\epsilon^2\nu)\varPhi_\xi^2\\&+n(\nu-\mu^2+m-\epsilon\nu\mu)\varPhi+n\varPhi^4=0.\end{aligned} \tag{3.15}$$

According to (2.3), we assume that

$$\varPhi(\xi)=S(Y(\xi))=\sum_{k=0}^{M}a_kY^k.$$

Substituting (2.4) into (3.15), we can get

$$\begin{aligned}&(1-\epsilon\mu)(1+\epsilon\mu+\epsilon^2\nu)S\Big[(Y^2+\alpha Y+b)^2\frac{\mathrm{d}^2S}{\mathrm{d}Y^2}+(2Y+\alpha)(Y^2+\alpha Y+b)\frac{\mathrm{d}S}{\mathrm{d}Y}\Big]\\&+\Big(\frac{1}{n}-1\Big)\times(1-\epsilon\mu)(1+\epsilon\mu+\epsilon^2\nu)(Y^2+\alpha Y+b)^2\Big(\frac{\mathrm{d}S}{\mathrm{d}Y}\Big)^2\\&+n(\nu-\mu^2+m-\epsilon\nu\mu)S+nS^4=0.\end{aligned} \tag{3.16}$$

Balancing the term $(1-\epsilon\mu)(1+\epsilon\mu+\epsilon^2\nu)(Y^2+\alpha Y+b)^2\Big(\dfrac{\mathrm{d}S}{\mathrm{d}Y}\Big)^2$ with S^4 yields $M=1$. This gives the solution in the form

$$\varPhi(\xi)=S(Y(\xi))=a_0+a_1Y(\xi). \tag{3.17}$$

Substituting (3.17) into (3.16), we can get the system of the algebraic equation with respect to a_0, a_1, ν, μ:

$$\begin{aligned}&Aa_1(a_0+a_1Y)(2Y+\alpha)(Y^2+\alpha Y+b)+\Big(\frac{1}{n}-1\Big)Aa_1^2(Y^2+\alpha Y+b)^2\\&+nB(a_0+a_1Y)+n(a_0+a_1Y)^4=0,\end{aligned}$$

with

$$A=(1-\epsilon\mu)(1+\epsilon\mu+\epsilon^2\nu),\qquad B=\nu-\mu^2+m-\epsilon\nu\mu.$$

Equating each coefficient of this polynomial in Y to zero, we obtain the following system of the algebraic equations:

$$\begin{cases} Y^0: Bna_0+na_0^4+Ab\alpha a_0a_1+Ab^2\Big(\dfrac{1}{n}-1\Big)a_1^2=0;\\ Y^1: Bna_1+2Aba_0a_1+A\alpha^2a_0a_1+4na_0^3a_1+Ab\alpha a_1^2+2Ab\Big(\dfrac{1}{n}-1\Big)\alpha a_1^2=0;\\ Y^2: 3A\alpha a_0a_1+2Aba_1^2+2Ab\Big(\dfrac{1}{n-1}\Big)a_1^2+A\alpha^2a_1^2+A\Big(\dfrac{1}{n}-1\Big)\alpha^2a_1^2+6na_0^2a_1^2=0;\\ Y^3: 2Aa_0a_1+3A\alpha a_1^2+2A\Big(\dfrac{1}{n}-1\Big)\alpha a_1^2+4na_0a_1^3=0;\\ Y^4: 2Aa_1^2+A\Big(\dfrac{1}{n}-1\Big)a_1^2+na_1^4=0.\end{cases} \tag{3.18}$$

(i) When $\alpha=0$ and $b=-1$, there are no non-trivial solutions of (3.18).

(ii) When $\alpha = 0$ and b is an arbitrary constant. If $b = 0$, with the aid of Mathematica, we get the solutions of (3.18)

$$a_0 = 0, \quad a_1 = \pm\frac{\sqrt{-1-n+m\epsilon^2+mn\epsilon^2}}{n}, \quad \mu = \frac{1}{2}\left(-\epsilon\nu \pm \sqrt{4m+4\nu+\epsilon^2\nu^2}\right).$$

So, the traveling wave solutions of (3.8) are

$$u_{1-4}^{(\mathrm{ii})} = \frac{\mp\sqrt{(m\epsilon^2-1)(n+1)}}{n\left[\left(\nu-\epsilon\nu\pm\sqrt{4m+4\nu+\epsilon^2\nu^2}\right)t+\left(1+\frac{1}{2}\epsilon^2\epsilon\nu\mp\frac{1}{2}\epsilon\sqrt{4m+4\nu+\epsilon^2\nu^2}\right)x\right]}\mathrm{e}^{\mathrm{i}(\mu x+\nu t)}.$$

If $b \neq 0$, then there are no non-trivial solutions of (3.18).

(iii) When $\alpha \neq 0$, with the help of Mathematica, there are no non-trivial solutions of (3.18).

4 Conclusion

In this paper, we have applied the generalized tanh method to construct a series of traveling wave solutions for some special types of equations: Fisher's equation and the nonlinear Schrödinger equation. These traveling wave solutions are expressed in terms of hyperbolic tangent (cotangent), trigonometric and rational functions depending on different parameters. The performance of the generalized tanh method is direct, concise and effective. This method will be used in further works to establish more and new solutions of many other nonlinear evolution equations.

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