Communications in Mathematical Research
30(1)(2014), 71–80

Variational Approach to Scattering by Inhomogeneous Layers Above Rough Surfaces

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Abstract: In this paper, we study, via variational methods, the problem of scattering of time harmonic acoustic waves by unbounded inhomogeneous layers above a sound soft rough surface. We first propose a variational formulation and exploit it as a theoretical tool to prove the well-posedness of this problem when the media is non-absorbing for arbitrary wave number and obtain an estimate about the solution, which exhibit explicitly dependence of bound on the wave number and on the geometry of the domain. Then, based on the non-absorbing results, we show that the variational problem remains uniquely solvable when the layer is absorbing by means of a priori estimate of the solution. Finally, we consider the finite element approximation of the problem and give an error estimate.

Key words: Helmholtz equation, rough surface, scattering problem
2000 MR subject classification: 47J30, 65N30
Document code: A
Article ID: 1674-5647(2014)01-0071-10

1 Introduction

This paper is concerned with the study of a boundary value problem for the Helmholtz equation modeling scattering of time harmonic waves by a layer above an unbounded rough surface on which the field vanishes. Such problems arise frequently in practical applications, such as in modeling outdoor noise propagation or sonar measurements in oceanography.

In this paper we focus on a particular, typical problem of the class, which models acoustic waves scattering by inhomogeneous layer above a sound soft unbounded rough surface. Since the unboundedness of the scattering object present a major challenge, mathematical methods to solve such scattering problem are often difficult to develop. Nevertheless, a variety of different methods and techniques have been introduced during the last years. Most of them were concerned with Drichlet boundary value problems for the Helmholtz equation with

Received date: Oct. 24, 2011.
Foundation item: The Education Department (12531136) of Heilongjiang, the NSF (10971083, 51178001) of China, Science and Technology Research Project (2014213) of Jilin Province Department of Education.
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constant real wave number (see [1–7]).

The idea of our argument is inspired by [5–6], in which a Rellich identity was used to prove the estimates for solutions of the Helmholtz equation posed on unbounded domains. Though the results and methods were closest to those of Chandler[5,7], who studied the similar problem tackled in those papers, and considered the homogeneous media only for non-absorbing case in [5], and obtained the well posed results just only for the wave number which is small enough in [7].

The main results of this paper are as follows: In Section 2, we introduce the boundary value problem considered in this paper. Then we propose the variational formulation, which is used as a theoretical tool to analyze the well-posedness of the problem. In Section 3, we consider the non-absorbing case. We first establish a Rellich-type identity, from which the inf-sup condition of the sesquilinear form follows. Then the existence and uniqueness of the solution to variational problem can be deduced by application of the generalized Lax-Milgram theory of Babuska (see [8]). In Section 4, we turn our interest to the absorbing scatterers, and establish the uniqueness via a priori estimate which also leads to an existence result based on the non-absorbing results. In Section 5, the finite element approximation of the problem is considered. Finally, we analyze the convergence and error estimate.

2 Boundary Value Problem and Variational Formulation

In this section, we first define some notations related to the problem. Then we introduce the boundary value problem and its equivalent variational formulation to be analyzed later. For $x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n$ $(n = 2, 3)$, let $\tilde{x} = (x_1, x_2, \cdots, x_{n-1})$ so that $x = (\tilde{x}, x_n)$. For $H \in \mathbf{R}$, let $U_H := \{x \mid x_n > H\}$ and $\varGamma_H := \{x \mid x_n = H\}$. Suppose that D is a connected open set with some constants $f_- < f_+$. Then it holds that

$$U_{f_+} \subset D \subset U_{f_-}, \tag{2.1}$$

and

$$x + s\boldsymbol{e}_n \in D, \qquad s > 0,\ x \in D, \tag{2.2}$$

where $\boldsymbol{e}_n$ denotes the unit vector in the direction of x_n. Let $\varGamma = \partial D$ and $S_H := D \backslash \overline{U}_H$ for some $H \geq f_+$. Moreover, we assume that the wave number k satisfies

$$\begin{cases} 0 \leq k \leq k_+, & x \in D; \\ k(x) = k_0 > 0, & x \in U_H; \\ \dfrac{\partial k^2(x)}{\partial x_n} \geq 0\ , & x \in S_H. \end{cases} \tag{2.3}$$

Next we introduce the main function spaces in which we set our problem. The Hilbert space V_H is defined by

$$V_H = \{\phi|_{S_H} : \phi \in H_0^1(D)\}, \qquad H \geq f_+,$$

on which we impose the wave number dependent scalar product

$$(u, v)_{V_H} := \int_{S_H} (\nabla u \cdot \nabla \bar{v} + k_0^2 u \bar{v}) \mathrm{d}x,$$

and the induced norm

$$\|u\|_{V_H} = \Big(\int_{S_H} (|\nabla u|^2 + k_0^2 |u|^2) \Big)^{\frac{1}{2}}.$$

For $s \in \mathbf{R}$, denote by $H^s(\mathit{\Gamma}_H)$ the usual Sobolev space (see [9]) with norm

$$\|\phi\|_{H^s(\mathit{\Gamma}_H)} = \Big(\int_{\mathbf{R}^{n-1}} (k_0^2 + \xi^2)^s |\mathcal{F}\phi(\xi)|^2 \mathrm{d}\xi \Big)^{\frac{1}{2}},$$

where we identify $\mathit{\Gamma}_H$ with $\mathbf{R}^{n-1}$ and $\mathcal{F}$ is the Fourier transformation.

Then the problem of scattering by an inhomogeneous layer above a sound soft rough surface is formulated by the following boundary value problem: given a source $g \in L^2(D)$ supported in S_H for some $H \geq f_+$, we seek to find a scattered field $u : D \to \mathbf{C}$ such that $u|_{S_a} \in V_a$ for every $a > f_+$, and

$$\Delta u + k^2(x) u = g, \qquad \text{in } D, \tag{2.4}$$

$$u = 0, \qquad \text{on } \mathit{\Gamma}, \tag{2.5}$$

in a distributional sense. As part of the boundary value problem, we apply the radiation condition, which is often used in a formal manner in the rough surface scattering literature,

$$u(x) = \frac{1}{(2\pi)^{\frac{(n-1)}{2}}} \int_{\mathbf{R}^{n-1}} \exp\Big\{ \mathrm{i} \Big[(x_n - H)\sqrt{k_0^2 - \xi^2} + \tilde{x} \cdot \xi \Big] \Big\} \hat{F}_H(\xi) \mathrm{d}\xi, \tag{2.6}$$

where $\hat{F}_H(\xi)$ is the Fourier transformation of $F_H := u|_{\mathit{\Gamma}_H}$. This radiation condition shows that the solution can be represented in an integral form as a superposition of upward traveling and evanescent plane waves above the rough surface and the support of g.

Before giving the variational formulation related to the above problem, we introduce some operators. Recall from [9] that, for all $a > H \geq f_+$, there exist continuous embeddings, i.e., trace operators

$$\gamma_+ : H^1(U_H \backslash U_a) \to H^{\frac{1}{2}}(\mathit{\Gamma}_H), \qquad \gamma_- : V_H \to H^{\frac{1}{2}}(\mathit{\Gamma}_H).$$

The Dirichlet to Neumann map T on $\mathit{\Gamma}_H$ is defined by

$$T := \mathcal{F}^{-1} M_z \mathcal{F},$$

where M_z is the multiplying by

$$z(\xi) = \begin{cases} -\mathrm{i}\sqrt{k_0^2 - \xi^2}, & |\xi| \leq k_0; \\ \sqrt{\xi^2 - k_0^2}, & |\xi| > k_0. \end{cases}$$

From the definition of T and the Sobolev norm, we see that it is a map from $H^{\frac{1}{2}}(\mathit{\Gamma}_H)$ to $H^{-\frac{1}{2}}(\mathit{\Gamma}_H)$, and $\|T\| = \max\limits_{\xi \in \mathbf{R}^{n-1}} \dfrac{\sqrt{k_0^2 - \xi^2}}{\sqrt{k_0^2 + \xi^2}} = 1$. Next we recall some results needed about properties of the above operators.

Lemma 2.1[5] *For $\phi \in H^{\frac{1}{2}}(\mathit{\Gamma}_H)$, it holds that*

$$\mathrm{Re} \int_{\mathit{\Gamma}_H} \bar{\phi} T \phi \mathrm{d}s \geq 0, \qquad \mathrm{Im} \int_{\mathit{\Gamma}_H} \bar{\phi} T \phi \mathrm{d}s \leq 0.$$

Lemma 2.2[9] *If (2.6) holds, with $F_H \in H^{\frac{1}{2}}(\mathit{\Gamma}_H)$, then $\gamma_+ u = F_H$ and*

$$\int_{\mathit{\Gamma}_a} \Big(\Big| \frac{\partial u}{\partial x_n} \Big|^2 - |\nabla_{\tilde{x}} u|^2 + k_0^2 |u|^2 \Big) \mathrm{d}s \leq 2k_0 \mathrm{Im} \int_{\mathit{\Gamma}_a} \bar{u} \frac{\partial u}{\partial x_n} \mathrm{d}s, \tag{2.7}$$

with $a \geq H$.

Lemma 2.3[5] *For $u \in V_H$,*

$$\|\gamma u\|_{H^{\frac{1}{2}}(\Gamma_H)} \le \|u\|_{V_H}, \qquad \|u\| \le \frac{H-f_-}{\sqrt{2}}\Big\|\frac{\partial u}{\partial x_n}\Big\|, \tag{2.8}$$

where $\|\cdot\|$ denotes the induced norm of the scalar product $(\,\cdot\,,\,\cdot\,)$ on $L^2(S_H)$.

Now we are ready to introduce the sesquilinear form $b : V_H \times V_H \to \mathbf{C}$ defined by

$$b(u, v) = (\nabla u,\, \nabla v) - (k^2 u,\, v) + \int_{\Gamma_H} \gamma_- \bar{v}\; T\; \gamma_- u \mathrm{d}s. \tag{2.9}$$

Problem V We formulate the variational problem: For $g \in L^2(S_H)$, find $u \in V_H$ such that

$$b(u, v) = -(g, v), \qquad v \in V_H. \tag{2.10}$$

We have already known that the boundary value problem (2.4)–(2.6) is equivalent to the related variational problem, which is stated in the theorem as follows.

Theorem 2.1[9] *If u is a solution of the boundary value problem (2.4)–(2.6), then $u|_{S_H}$ satisfies the variational problem (2.10). Conversely, set $F_H := \gamma_- u$ and the definition of u is extended to D by (2.6), for $x \in U_H$. If u solves the variational problem (2.10), then the extended function satisfies the boundary value problem (2.4)–(2.6), with g extended by zero and k extended by taking the value k_0 from S_H to D.*

3 Existence and Uniqueness for Non-absorbing Medium

In this section, we prove the equivalent variational problem. Thus the boundary value problem is uniquely solvable by establishing the inf-sup condition of the sesquilinear form, via application of the generalized Lax-Milgram theory of Babuska. Our methods of argument depend on a priori estimate established by means of a Rellich type identity and the results on approximation of nonsmooth by smooth domains.

Lemma 3.1[10] *If the bounded sesquilinear form b satisfies the inf-sup condition*

$$\beta := \inf_{0 \ne u \in V_H} \sup_{0 \ne v \in V_H} \frac{|b(u, v)|}{\|u\|_{V_H} \|v\|_{V_H}} > 0 \tag{3.1}$$

and the transposed inf-sup condition

$$\sup_{0 \ne u \in V_H} \frac{|b(u, v)|}{\|u\|_{V_H}} > 0, \qquad v \in V_H \backslash \{0\}, \tag{3.2}$$

then the variational problem (2.10) has exactly one solution $u \in V_H$ such that

$$\|u\|_{V_H} \le \beta^{-1} \|g\|. \tag{3.3}$$

In terms of the lemmas above, we could deduce the following result.

Theorem 3.1 *Suppose that D satisfies the assumptions (2.1)–(2.2), and the wave number k satisfies (2.3). Then the variational problem (2.10) has a unique solution $u \in V_H$ such that*

$$\|w\|_{V_H} \le \frac{H - f_-}{\sqrt{2}} (K_+ + 1)(K_0 + 3) \|g\|, \tag{3.4}$$

where $K_+ = k_+(H - f_-)$ and $K_0 = k_0(H - f_-)$.

Since b is bounded (see [7]) and the transposed inf-sup condition can be deduced by the inf-sup condition (see [5]), it suffice to establish (3.1). And we know that Lemmas 4.4–4.5, and Lemmas 4.10–4.11 in [5] reduce the problem of showing (3.1) to that of establishing an a priori bound for the solutions of the boundary problem when the boundary Γ is sufficiently smooth. Thus we need first to establish a Rellich identity for the solution of Helmholtz equation.

Theorem 3.2 *Suppose that* $\Gamma = \{(\tilde{x}, x_n) \mid x_n = f(\tilde{x}),\ \tilde{x} \in \mathbf{R}^{n-1}\}$ *with* $f \in C^\infty(\mathbf{R}^{n-1})$. *If* $u \in V_H$ *is a solution of the variational problem* (2.10), *then*

$$\begin{aligned}&\int_{S_H} \frac{\partial k^2(x)}{\partial x_n}(x_n - f_-)|w|^2 + \int_{S_H} 2\Big|\frac{\partial w}{\partial x_n}\Big|^2 \mathrm{d}x - \int_\Gamma (x_n - f_-)\nu_n \Big|\frac{\partial w}{\partial \nu}\Big|^2 \mathrm{d}s \\ &= (H - f_-)\int_{\Gamma_H} \Big(\Big|\frac{\partial w}{\partial x_n}\Big|^2 - |\nabla_{\tilde{x}} w| + k_0^2|w|^2\Big)\mathrm{d}s \\ &\quad + \int_{S_H} \Big\{|\nabla w|^2 - k^2(x)|w|^2 - 2\mathrm{Re}(x_n - f_-)g\frac{\partial \bar{w}}{\partial x_n}\Big\}\mathrm{d}x. \end{aligned} \tag{3.5}$$

Proof. Let $r = |\tilde{x}|$. For $A \geq 1$, let $\phi_A \in C_0^\infty(\mathbf{R})$ such that $0 \leq \phi_A \leq 1$ with $\phi_A(r) = 1$, if $r \leq A$; $\phi_A(r) = \phi_A$, if $A < r < A+1$; and $\phi_A(r) = 0$, if $r \geq A+1$. Finally $\|\phi'\|_\infty \leq M$ for some fixed $M > 0$ independent of A.

It follows from Theorem 2.1 that, when extended to D by (2.6) with $F_H := \gamma_- w$, w satisfies the boundary value problem with g extended by zero and k by k_0 from S_H to D. Since the boundary Γ is smooth, by standard local regularity results (see [11]), we have $u \in H^2_{\mathrm{loc}}(D)$. In view of this regularity, one has

$$\begin{aligned}&2\mathrm{Re}\int_{S_H} \phi_A(r)(x_n - f_-)(\Delta w + k^2(x)w)\frac{\partial \bar{w}}{\partial x_n}\mathrm{d}x \\ &= 2\mathrm{Re}\int_{S_H} \nabla\cdot\Big(\phi_A(r)(x_n - f_-)\frac{\partial \bar{w}}{\partial x_n}\nabla w\Big)\mathrm{d}x - \int_{S_H} 2\phi_A(r)\Big|\frac{\partial w}{\partial x_n}\Big|^2 \\ &\quad - \int_{S_H} (x_n - f_-)\phi_A(r)\frac{\partial |\nabla w|^2}{\partial x_n} - 2\phi_A'(r)(x_n - f_-)\frac{\tilde{x}}{|\tilde{x}|}\mathrm{Re}\Big(\nabla_{\tilde{x}} w\frac{\partial \bar{w}}{\partial x_n}\Big)\mathrm{d}x \\ &\quad + \int_{S_H} k^2(x)(x_n - f_-)\phi_A(r)\frac{\partial |w|^2}{\partial x_n}\mathrm{d}x.\end{aligned}$$

Furthermore, we use divergence theorem and integration by parts to obtain

$$\begin{aligned}&2\mathrm{Re}\int_{S_H} \phi_A(r)(x_n - f_-)(\Delta w + k^2(x)w)\frac{\partial \bar{w}}{\partial x_n}\mathrm{d}x \\ &= (H - f_-)\int_{\Gamma_H} \phi_A(r)\Big(\Big|\frac{\partial w}{\partial x_n}\Big|^2 - |\nabla_{\tilde{x}} w| + k_0^2|w|^2\Big)\mathrm{d}s \\ &\quad - \int_\Gamma (x_n - f_-)\phi_A(r)\Big\{\nu_n|\nabla w|^2 - 2\mathrm{Re}\Big(\frac{\partial \bar{w}}{\partial x_n}\cdot\frac{\partial w}{\partial \nu}\Big)\Big\}\mathrm{d}s \\ &\quad + \int_{S_H} \phi_A(r)\Big(|\nabla w|^2 - k^2(x)|w|^2 - 2\Big|\frac{\partial w}{\partial x_n}\Big|^2\Big)\mathrm{d}x \\ &\quad - \int_{S_H} \Big\{2\phi_A'(r)(x_n - f_-)\mathrm{Re}\Big(\frac{\partial \bar{w}}{\partial x_n}\cdot\frac{\partial w}{\partial r}\Big) - \frac{\partial k^2(x)}{\partial x_n}(x_n - f_-)\phi_A(r)|w|^2\Big\}\mathrm{d}x \\ &= 2\mathrm{Re}\int_{S_H} \phi_A(r)(x_n - f_-)g\frac{\partial \bar{w}}{\partial x_n}\mathrm{d}x,\end{aligned} \tag{3.6}$$

where ν is the unit outward normal derivative to Γ. Since $w = 0$ on Γ, it holds that $\nabla w = \dfrac{\partial w}{\partial \nu}\nu$, and hence

$$\frac{\partial w}{\partial x_n} = \boldsymbol{e}_n \cdot \nabla w = \boldsymbol{e}_n \cdot \nu \frac{\partial w}{\partial \nu} = \nu_n \frac{\partial w}{\partial \nu} \tag{3.7}$$

with $\nu_n = \boldsymbol{e}_n \cdot \nu$. Substituting (3.7) into (3.6) and rearranging terms yield

$$\begin{aligned}
&\int_{S_H} \frac{\partial k^2(x)}{\partial x_n}(x_n - f_-)\phi_A(r)|w|^2 + 2\phi_A(r)\Big|\frac{\partial w}{\partial x_n}\Big|^2 \mathrm{d}x - \int_{\Gamma} \phi_A(r)(x_n - f_-)\nu_n \Big|\frac{\partial w}{\partial \nu}\Big|^2 \mathrm{d}s \\
&= (H - f_-)\int_{\Gamma_H} \phi_A(r)\Big(\Big|\frac{\partial w}{\partial x_n}\Big|^2 - |\nabla_{\tilde{x}} w| + k_0^2|w|^2\Big)\mathrm{d}s \\
&\quad + \int_{S_H} \Big\{\phi_A(r)[|\nabla w|^2 - k^2(x)|w|^2] - 2\phi_A'(r)(x_n - f_-)\mathrm{Re}\Big(\frac{\partial \bar{w}}{\partial x_n}\cdot\frac{\partial w}{\partial r}\Big)\Big\}\mathrm{d}x \\
&\quad - 2\mathrm{Re}\int_{S_H} \phi_A(r)(x_n - f_-)g\frac{\partial \bar{w}}{\partial x_n}\mathrm{d}x.
\end{aligned} \tag{3.8}$$

Next we try to estimate the terms in the above equality. Let $S_H^b = \{x \in S_H : |\tilde{x}| < b\}$. Then

$$\Big|\int_{S_H} 2\phi_A'(r)(x_n - f_-)\mathrm{Re}\Big(\frac{\partial \bar{w}}{\partial x_n}\cdot\frac{\partial w}{\partial r}\Big)\mathrm{d}x\Big| \le 2M(H - f_-)\int_{S_H^{A+1}\backslash \bar{S}_H^A} |\nabla w|^2 \mathrm{d}x \to 0, \quad A \to \infty,$$

with $w \in H^1(S_H)$. On the other hand, since $w \in H^2(U_H \backslash U_{f_+})$ (see [6]), one has $\nabla w \in H^{\frac{1}{2}}(\Gamma_H)$. Thus taking the limit as $A \to \infty$ in (3.8), we have

$$\begin{aligned}
&\int_{S_H} \frac{\partial k^2(x)}{\partial x_n}(x_n - f_-)|w|^2 + \int_{S_H} 2\Big|\frac{\partial w}{\partial x_n}\Big|^2 \mathrm{d}x - \int_{\Gamma} (x_n - f_-)\nu_n \Big|\frac{\partial w}{\partial \nu}\Big|^2 \mathrm{d}s \\
&= (H - f_-)\int_{\Gamma_H} \Big\{\Big|\frac{\partial w}{\partial x_n}\Big|^2 - |\nabla_{\tilde{x}} w| + k_0^2|w|^2\Big\}\mathrm{d}s \\
&\quad + \int_{S_H} \Big\{|\nabla w|^2 - k^2(x)|w|^2 - 2\mathrm{Re}(x_n - f_-)g\frac{\partial \bar{w}}{\partial x_n}\Big\}\mathrm{d}x.
\end{aligned}$$

The Rellich identity above is our main tool to derive a priori estimate for a solution of the variational problem, which allows us to show an inf-sup condition for the sesquilinear form and thereby prove the well-posedness of the scattering problem.

Theorem 3.3 *Suppose that* $\Gamma = \{(\tilde{x}, x_n) \mid x_n = f(\tilde{x}),\ \tilde{x} \in \mathbf{R}^{n-1}\}$ *with* $f \in C^\infty(\mathbf{R}^{n-1})$. *The domain* D *and the wave number* k *satisfy the assumption of Theorem* 3.1. *Let* $g \in L^2(S_H)$ *and* $w \in V_H$ *with* $H > f_+$ *satisfy*

$$b(w, \phi) = -(g, \phi), \qquad \phi \in V_H.$$

Then

$$\|w\|_{V_H} \le \frac{H - f_-}{\sqrt{2}}(K_+ + 1)(K_0 + 3)\|g\|. \tag{3.9}$$

Proof. By Theorem 3.2, we directly have (3.5) for w. On the other hand, it follows from Lemma 2.2 that

$$\begin{aligned}
\int_{\Gamma_H} \Big(\Big|\frac{\partial w}{\partial x_n}\Big|^2 - |\nabla_{\tilde{x}} w|^2 + k_0^2|w|^2\Big)\mathrm{d}s &\le 2k_0 \mathrm{Im}\int_{\Gamma_H} \bar{w}\frac{\partial w}{\partial x_n}\mathrm{d}s \\
&= -2k_0 \mathrm{Im}\int_{\Gamma_H} \gamma_- \bar{w} T \gamma_- w \mathrm{d}s.
\end{aligned} \tag{3.10}$$

Moreover, since w satisfies the variational problem, we conclude

$$\int_{S_H}[|\nabla w|^2 - k^2(x)|w|^2]\mathrm{d}x = \int_{\Gamma_H}\gamma_-\bar{w}T\gamma_- w\mathrm{d}s - \int_{S_H} g\bar{w}\mathrm{d}x.$$

According to Lemma 2.1, it yields

$$\int_{S_H}(|\nabla w|^2 - k^2(x)|w|^2)\mathrm{d}x \le -\mathrm{Re}\int_{S_H} g\bar{w}\mathrm{d}x \tag{3.11}$$

and

$$-2k_0\mathrm{Im}\int_{\Gamma_H}\gamma_-\bar{w}T\gamma_- w\mathrm{d}s = 2k_0\mathrm{Im}\int_{S_H} g\bar{w}\mathrm{d}x.$$

By using (3.5) and the inequalities (3.10)–(3.11), we obtain

$$\int_{S_H}\frac{\partial k^2(x)}{\partial x_n}(x_n - f_-)|w|^2 + \int_{S_H} 2\Big|\frac{\partial w}{\partial x_n}\Big|^2\mathrm{d}x - \int_{\Gamma}(x_n - f_-)\nu_n\Big|\frac{\partial w}{\partial \nu}\Big|^2\mathrm{d}s$$
$$\le 2(H - f_-)k_0\mathrm{Im}\int_{S_H} g\bar{w}\mathrm{d}x - \mathrm{Re}\int_{S_H}\Big\{g\bar{w} + 2(x_n - f_-)g\frac{\partial \bar{w}}{\partial x_n}\Big\}\mathrm{d}x.$$

By the assumption $\dfrac{\partial k^2(x)}{\partial x_n} \ge 0$ in S_H and $\nu_n < 0$, it follows by the Cauchy-Schwarz inequality that

$$2\Big\|\frac{\partial w}{\partial x_n}\Big\|^2 \le \Big(2K_0\|w\| + \|w\| + 2(H - f_-)\Big\|\frac{\partial w}{\partial x_n}\Big\|\Big)\|g\|.$$

Thus, from Lemma 2.3, we have

$$\Big\|\frac{\partial w}{\partial x_n}\Big\| \le (H - f_-)\Big(\frac{1}{\sqrt{2}}K_0 + \frac{1}{2\sqrt{2}} + 1\Big)\|g\|,$$

and furthermore,

$$\|w\| \le (H - f_-)^2\Big(\frac{1}{2}K_0 + \frac{1}{4} + \frac{1}{\sqrt{2}}\Big)\|g\|.$$

Then we deduce from (3.11) that

$$\|w\|_{V_H}^2 \le 2k_+^2\|w\|^2 + \|g\|\|w\| \le \frac{(H - f_-)^2}{2}(K_+^2 + 1)(K_0 + 3)^2\|g\|^2.$$

Hence, it holds that

$$\|w\|_{V_H} \le \frac{H - f_-}{\sqrt{2}}(K_+ + 1)(K_0 + 3)\|g\|.$$

4 Existence and Uniqueness for Absorbing Media

After our study on non-absorbing layers, we now turn our interests to absorbing scatterers. We prefer to establish the uniqueness via an a priori bound which also leads to an existence result. We assume in this section that $\mathrm{Re}(k^2) \ge 0$, $\mathrm{Im}(k^2) \ge 0$ and $\mathrm{Re}(k^2) \le k_+^2$, $\mathrm{Im}(k^2) \le k_-^2$ with $k_+ > 0$ in D, $k(x) = k_0 > 0$ for $x \in \bar{U}_H$, and $\dfrac{\partial \mathrm{Re}(k^2)}{\partial x_n} \ge 0$ in S_H.

Theorem 4.1 *Suppose that the wave number $k \in \mathbf{C}$ satisfies the assumption above. Then there exists a unique solution $u \in V_H$ of the variational problem* (2.10) *such that*

$$\|u\|_{V_H} \le (H - f_-)(K_+ + 1)(K_0 + 3)\Big(\sqrt{2} + (K_+ + 1)(K_0 + 3)K_-\frac{k_-}{k_0}\Big)\|g\|. \tag{4.1}$$

Proof. We rewrite the variational formulation (2.10) as

$$(\nabla u,\, \nabla v) - (\mathrm{Re}(k^2)u,\, v) + \int_{\Gamma_H} \gamma_- \bar{v}\ T\gamma_- u \mathrm{d}s = (\mathrm{i}\,\mathrm{Im}(k^2)u - g,\, v), \qquad v \in V_H.$$

Because $\mathrm{Re}(k^2)$ satisfies the assumptions of Theorem 3.1, we directly obtain an a priori estimate for u,

$$\|u\|_{V_H} \le \frac{H - f_-}{\sqrt{2}}(K_+ + 1)(K_0 + 3)\|g\| + \frac{H - f_-}{\sqrt{2}}(K_+ + 1)(K_0 + 3)\|\mathrm{Im}(k^2)u\|.$$

Taking the imaginary part of the variational formulation with $v = u$, we get

$$\int_{S_H} \mathrm{Im}(k^2)|u|^2 \mathrm{d}x + \mathrm{Im}\int_{\Gamma_H} \bar{u}Tu \mathrm{d}s = \mathrm{Im}\int_{S_H} g\bar{u}\mathrm{d}x.$$

By Lemma 2.1, we have

$$\begin{aligned}\|\mathrm{Im}(k^2)u\|^2 &\le k_-^2 \int_{S_H} \mathrm{Im}(k^2)|u|^2 \mathrm{d}x \\ &\le k_-^2\, \mathrm{Im}\int_{S_H} g\bar{u}\mathrm{d}x \\ &\le k_-^2\, \|g\|\|u\|. \end{aligned} \tag{4.2}$$

Since $(ab)^{\frac{1}{2}} \le ca + \dfrac{b}{2c}$ for $a,\, b,\, c \ge 0$, letting

$$a = \frac{k_-^2}{k_0}\|g\|, \qquad b = k_0\|u\|, \qquad c = \frac{H - f_-}{\sqrt{2}}(K_+ + 1)(K_0 + 3),$$

we have

$$\begin{aligned}\|\mathrm{Im}(k^2)u\| \le\ & \frac{H - f_-}{\sqrt{2}}(K_+ + 1)(K_0 + 3)\frac{k_-^2}{k_0}\|g\| \\ & + \frac{1}{\sqrt{2}(H - f_-)(K_+ + 1)(K_0 + 3)}\|u\|_{V_H}. \end{aligned} \tag{4.3}$$

This allows us to conclude that

$$\|u\|_{V_H} \le \sqrt{2}(H - f_-)(K_+ + 1)(K_0 + 3)\|g\| + (K_+ + 1)^2(K_0 + 3)^2\frac{K_-^2}{k_0}\|g\|, \tag{4.4}$$

yielding an a priori estimate for u from which the existence and uniqueness of the solution to the variational problem follow.

5 Finite Element Approximation

In this section, we consider the numerical approach to solve the Problem V. We use the finite element method (FEM for short) to get the solution of the variational formulation (2.10). This is a classical approach for numerical treatment of the bounded domain problem. Thus a necessary first step towards solving the Problem V numerically is to approximate it by a variational formulation on a domain of finite size, in which standard FEM can then be applied. This approximation consists simply in replacing S_H by a finite region S_H^R defined by: $S_H^R = \{X = (\tilde{x},\, x_n) \in S_H : |\tilde{x}| < R\}$ and D by $D^R = \{X = (\tilde{x},\, x_n) \in D : |\tilde{x}| < R\}$.

For $R > 0$, we approximate the problem (2.10) by a corresponding variational equation on S_H^R. Let V_H^R denote the Hilbert space V_H. In the case that we replace D by D^R, explicitly V_H^R denotes the completion of $\{u|_{S_H^R} : u \in C_0^\infty(D^R)\}$ with the norm

$$\|u\|_{V_H^R} = \Big(\int_{S_H^R} (|u|^2 + k_0^2|\nabla u|^2)\mathrm{d}x\Big)^{\frac{1}{2}}.$$

Problem AV The approximating variational problem is as follows: Find $u^R \in V_H^R$ such that

$$b_R(u^R, v^R) = -(g^R, v^R), \qquad v^R \in V_H^R \tag{5.1}$$

with $g^R = g|_{S_H^R}$. Here, b_R is the continuous sesquilinear form on $V_H^R \times V_H^R$, which is defined by (2.9), with S_H replaced by S_H^R, i.e.,

$$b_R(u, v) = \int_{S_H^R} (\nabla u \cdot \nabla \bar{v} - k^2 u \bar{v}) \mathrm{d}x + \int_{\Gamma_H^R} \gamma_- \bar{v} \; T \; \gamma_- u \mathrm{d}s,$$

where $\Gamma_H^R = \Gamma_H \bigcap S_H^R$.

Let T_h be a regular triangulation partition of the computational domain S_H^R into elements K, and h_K be the diameter of K. Then we can define the step $h = \max\limits_K h_K$. We denote the finite element space corresponding to T_h by V_h constructed by piecewise polynomials of degree p. If u_I is the interpolation of u^R in V_h, then there is a well-known approximation estimation (see [10])

$$\|u^R - u_I\|_m \le ch^{p+1-m} |u^R|_{p+1}, \qquad 0 \le m \le p. \tag{5.2}$$

Here and in the sequel, c denotes a generic constant, which may have different values at different places.

Problem GAV We consider the Galerkin approximation problem: Find $u_h^R \in V_h$ satisfying

$$b_R(u_h^R, v) = -(g^R, v), \qquad v \in V_h \tag{5.3}$$

with $g^R = g|_{S_H^R}$.

Since the inf-sup condition for the Problem V has been established by Theorem 3.1, it still holds for the Problem AV, which means that the Problem AV is well-posed. Meanwhile, we can also obtain the existence and the uniqueness of the solution for the Problem GAV by using the similar way used in Theorem 3.1. Furthermore, if $e_h = u^R - u_h^R$, we obtain by the definition of b_R that

$$b_R(e_h, e_h) = \int_{S_H^R} (|\nabla e_h|^2 - k^2 |e_h|^2) \mathrm{d}x + \int_{\Gamma_H^R} (T e_h) \bar{e}_h \mathrm{d}s. \tag{5.4}$$

Then taking the real part of (5.4) yields

$$\mathrm{Re} b_R(e_h, e_h) = \int_{S_H^R} (|\nabla e_h|^2 - \mathrm{Re}(k^2) |e_h|^2) \mathrm{d}x + \mathrm{Re} \int_{\Gamma_H^R} (T e_h) \bar{e}_h \mathrm{d}s.$$

After rearranging terms, we have

$$|e_h|_1^2 + \mathrm{Re} \int_{\Gamma_H^R} (T e_h) \bar{e}_h \mathrm{d}s = \int_{S_H^R} \mathrm{Re}(k^2) |e_h|^2 \mathrm{d}x + \mathrm{Re} b_R(e_h, e_h). \tag{5.5}$$

It is obvious that for any $\phi \in V_h$, it holds that

$$b_R(u^R - u_h^R, u^R - u_h^R) = b_R(u^R - u_h^R, u^R - \phi).$$

Thus

$$|b_R(e_h, e_h)| = \Big| \int_{S_H^R} (\nabla e_h \cdot \nabla \overline{(u^R - \phi)} - k^2 e_h \overline{(u^R - \phi)}) \mathrm{d}x + \int_{\Gamma_H^R} \overline{(u^R - \phi)} \; T e_h \mathrm{d}s \Big|.$$

By the continuity of the operator T and the trace theorem, it holds that

$$\Big|\int_{\Gamma_H^R} \overline{(u^R-\phi)}\ Te_h \mathrm{d}s\Big| \le c\|e_h\|_{V_H^R}^2 \|u^R-\phi\|_{V_H^R}.$$

Together with the inequalities above, we get by means of the ε inequality that

$$|b_R(e_h,\, e_h)| \le 2\varepsilon|\nabla e_h|_1^2 + 2\varepsilon|e_h|^2 + c\|u^R-u_I\|_{V_H^R}^2.$$

Combining with (5.5) and selecting a sufficient small value of ε, we finally obtain

$$|e_h|_1^2 + \mathrm{Re}\int_{\Gamma_H^R} (Te_h)\bar{e}_h \mathrm{d}s \le c(\|e_h\|^2 + \|u^R-u_I\|_{V_H^R}).$$

By the interpolation estimation (5.2) again, we conclude that

$$|e_h|_1^2 \le c(k_+)(\|e_h\|_{S_H^R}^2 + \|u^R-u_I\|_{V_H^R}). \tag{5.6}$$

By the Aubin-Nitsche technique and the interpolation approximation property (5.2), one has

$$\|e_h\|_{L^2(S_H^R)} \le Ch\|e_h\|_{V_H^R}.$$

When h is small enough, it holds that

$$\|e_h\|_{L^2(S_H^R)} \le Ch\|e_h\|_1.$$

We further obtain from (5.6) that

$$\|e_h\|_1 \le c\|u^R-u_I\|_{V_H^R}.$$

So we finally get the conclusion as follows.

Theorem 5.1 *Assume that $u^R \in H^s(S_H^R)$ $(s \ge 2)$ is the solution of the Problem AV. Then there exists an $h_0 \in (0,1]$ such that for $h \in (0,\, h_0)$ the Problem GAV has a unique solution $u_h^R \in V_h$, and*

$$\|u^R-u_h^R\|_{L^2(S_H^R)} \le ch^{p+1}\|u^R\|_{H^s}, \qquad \|u^R-u_h^R\|_{V_H^R} \le ch^p\|u^R\|_{H^s}. \tag{5.7}$$

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