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Some Notes on Normality Criteria of Meromorphic Functions

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Abstract: In this paper, we study the normality of families of meromorohic functions related to a Hayman conjecture. We prove that the conditions in Hayman conjecture and in other criterions can be relaxed. The results in this paper improve some previous results.

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1 Introduction and Main Results

We use $\mathbb{C}$ to denote the open complex plane, $\hat{\mathbb{C}}(= \mathbb{C} \cup \{\infty\})$ to denote the extended complex plane and D to denote a domain in $\mathbb{C}$. A family $\mathcal{F}$ of meromorphic functions defined in $D \subset \mathbb{C}$ is said to be normal, if any sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges spherically, and locally, uniformly in D to a meromorphic function or ∞. Clearly, $\mathcal{F}$ is said to be normal in D if and only if it is normal at every point in D (see [1]).

Let D be a domain in $\mathbb{C}$, f and g be two meromorphic functions, a and b be complex numbers. If $g(z) = b$ whenever $f(z) = a$, we write

$$f(z) = a \Rightarrow g(z) = b.$$

If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write

$$f(z) = a \Leftrightarrow g(z) = b.$$

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According to Bloch's principle (see [2]), every condition which reduces a meromorphic function in the plane $\mathbb{C}$ to a constant forces a family of meromorphic functions in a domain D normal. Although the principle is false in general (see [3]), many authors proved normality criterion for families of meromorphic functions by starting from Liouville-Picard type theorem (see [4]). Moreover, it is interesting to find normality criteria from the point of view of shared values. Schwick[5] first proved an interesting result that a family of meromorphic functions in a domain is normal if every function in that family shares three distinct finite complex numbers with its first derivative. And later, more results about normality criteria concerning shared values have emerged. In recent years, this subject has attracted the attention of many researchers worldwide.

In this paper, we use $\sigma(x, y)$ to denote the spherical distance between x and y and the definition of the spherical distance can be found in [6].

Theorem 1.1[7] *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc Δ, a and b be distinct complex numbers, and c be a nonzero complex number. If for every $f \in \mathcal{F}$, $f(z) = 0 \Leftrightarrow f'(z) = a$ and $f(z) = c \Leftrightarrow f'(z) = b$, then $\mathcal{F}$ is normal in Δ.*

In 2004, Singh A P and Singh A[8] proved that the condition for the constants in Theorem 1.1 to be the same for all $f \in \mathcal{F}$ can be relaxed to some extent, and they proved the following theorem.

Theorem 1.2[8] *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc Δ. For each $f \in \mathcal{F}$, suppose that there exist nonzero complex numbers b_f, c_f satisfying*:

(i) $\dfrac{b_f}{c_f}$ *is a constant*;

(ii) $\min\{\sigma(0, b_f), \sigma(0, c_f), \sigma(b_f, c_f)\} \geq m$ *for some* $m > 0$;

(iii) $f(z) = 0 \Leftrightarrow f'(z) = 0$ *and* $f(z) = c_f \Leftrightarrow f'(z) = b_f$.

Then $\mathcal{F}$ is normal in Δ.

Theorem 1.3[9] *Let $\mathcal{F}$ be a family of holomorphic (meromorphic) functions in a domain D, $n \in N$, $a \neq 0$, and $b \in \mathbb{C}$. If $f'(z) - af^n(z) \neq b$ for each function $f \in \mathcal{F}$ and $n \geq 2$ ($n \geq 3$), then $\mathcal{F}$ is normal in D.*

From the idea of Theorem 1.2, we generalize Theorem 1.3 as the following theorem.

Theorem 1.4 (Main Theorem I) *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc Δ, and $n (\geq 3)$ be a positive integer. For every $f \in \mathcal{F}$, there exist finite nonzero complex numbers b_f, c_f depending on f satisfying*:

(i) $\dfrac{b_f}{c_f}$ *is a constant*;

(ii) $\min\{\sigma(0, b_f), \sigma(0, c_f), \sigma(b_f, c_f)\} \geq m$ *for some* $m > 0$;

(iii) $f'(z) - \dfrac{1}{b_f^{n-1}} f^n(z) \neq c_f$.

Then $\mathcal{F}$ is normal in Δ.

Theorem 1.5[10] *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and $a(\neq 0)$, b be two finite constants. Define*

$$E_f = \Big\{z : z \in D,\ f'(z) + \frac{a}{f(z)} = b\Big\}.$$

If there exists a positive number M such that for every $f \in \mathcal{F}$, $|f(z)| \geq M$ whenever $z \in E_f$, then $\mathcal{F}$ is normal.

Theorem 1.6[11] *Let $\mathcal{F}$ be a family of meromorphic functions in D, n be a positive integer, and a, b be two finite complex numbers such that $a \neq 0$. If for each function $f \in \mathcal{F}$, $f' - af^{-n} \neq b$, then $\mathcal{F}$ is normal in D.*

Theorem 1.7 (Main Theorem II) *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc Δ, and n be a positive integer. For every $f \in \mathcal{F}$, there exist finite nonzero complex numbers b_f, c_f depending on f satisfying*:

(i) $\dfrac{b_f}{c_f}$ *is a constant*;

(ii) $\min\{\sigma(0, b_f), \sigma(0, c_f), \sigma(b_f, c_f)\} \geq m$ *for some* $m > 0$;

(iii) $z \in E_f = \Big\{z \in \Delta : f'(z) - \dfrac{b_f^{n+1}}{f^n(z)} = c_f\Big\} \Rightarrow |f(z)| \geq |b_f|$.

Then $\mathcal{F}$ is normal in Δ.

In 2009, Charak and Rieppo[12] generalized Theorem 1.5 and obtained two normality criteria of Lahiri's type.

Theorem 1.8[12] *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. Let $a, b \in \mathbb{C}$ with $a \neq 0$. Let m_1, m_2, n_1, n_2 be nonnegative integers such that*

$$m_1 n_2 - m_2 n_1 > 0, \qquad m_1 + m_2 \geq 1, \qquad n_1 + n_2 \geq 2.$$

Put

$$E_f = \Big\{z : z \in D,\ (f(z))^{n_1}(f'(z))^{m_1} + \frac{a}{(f(z))^{n_2}(f'(z))^{m_2}} = b\Big\}.$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

Theorem 1.9[12] *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. Let $a, b \in \mathbb{C}$ with $a \neq 0$. Let m_1, m_2, n_1, n_2 be nonnegative integers such that $m_1 n_2 = m_2 n_1 > 0$, and put*

$$E_f = \Big\{z : z \in D,\ (f(z))^{n_1}(f'(z))^{m_1} + \frac{a}{(f(z))^{n_2}(f'(z))^{m_2}} = b\Big\}.$$

If there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E_f$, then $\mathcal{F}$ is a normal family.

In this paper, we also obtain the following results.

Theorem 1.10 (Main Theorem III) *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc Δ, and m_1, m_2, n_1, n_2 be nonnegative integers such that $m_1n_2 - m_2n_1 > 0$, $m_1 + m_2 \geq 1$, and $n_1 + n_2 \geq 2$. For every $f \in \mathcal{F}$, there exist finite nonzero complex numbers b_f, c_f depending on f satisfying*:

(i) $\dfrac{b_f}{c_f}$ *is a constant*;

(ii) $\min\{\sigma(0, b_f),\ \sigma(0, c_f),\ \sigma(b_f, c_f)\} \geq m$ *for some* $m > 0$;

(iii) $z \in E_f = \left\{z \in \Delta : (f(z))^{n_1}(f'(z))^{m_1} + \dfrac{b_f^{s+t}}{(f(z))^{n_2}(f'(z))^{m_2}} = c_f^s \Rightarrow |f(z)| \geq |b_f|\right\}$,

where $s = n_1 + m_1$, $t = n_2 + m_2$.
Then $\mathcal{F}$ is normal in Δ.

Theorem 1.11 (Main Theorem IV) *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and m_1, m_2, n_1, n_2 be nonnegative integers such that $m_1n_2 = m_2n_1 > 0$. For every $f \in \mathcal{F}$, there exist finite nonzero complex numbers b_f, c_f depending on f satisfying*:

(i) $\dfrac{b_f}{c_f}$ *is a constant*;

(ii) $\min\{\sigma(0, b_f),\ \sigma(0, c_f),\ \sigma(b_f, c_f)\} \geq m$ *for some* $m > 0$;

(iii) $z \in E_f = \left\{z \in \Delta : (f(z))^{n_1}(f'(z))^{m_1} + \dfrac{b_f^{s+t}}{(f(z))^{n_2}(f'(z))^{m_2}} = c_f^s \Rightarrow |f(z)| \geq |b_f|\right\}$,

where $s = n_1 + m_1, t = n_2 + m_2$.
Then $\mathcal{F}$ is normal in Δ.

2 Some Lemmas

In order to prove our theorems, we require the following results.

Lemma 2.1[7,13] *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and k be a positive integer, such that each function $f \in \mathcal{F}$ has only zeros of multiplicity at least k, and suppose there exists an $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then for each $0 \leq \alpha \leq k$, there exist a sequence of points $z_n \in D$ with $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0^+$, and a subsequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \varsigma)}{\rho_n^\alpha} \to g(\zeta)$$

locally uniformly with respect to the spherical metric in $\mathbb{C}$, where g is a nonconstant meromorphic function, all of whose zeros have multiplicity at least k, such that $g^\sharp(\zeta) \leq g^\sharp(0) = kA+1$. Moreover, g has order at most 2.

Here as usual, $g^\sharp(\zeta) = \dfrac{|g'(\zeta)|}{1 + |g(\zeta)|^2}$ is the spherical derivative.

Lemma 2.2[6] *Let m be any positive number. Then, a Möbius transformation g satisfying $\sigma(g(a), g(b)) \geq m$, $\sigma(g(b), g(c)) \geq m$, $\sigma(g(c), g(a)) \geq m$ for some constants a, b and c, also satisfies the uniform Lipschitz condition*

$$\sigma(g(z),\ g(w)) \leq k_m \sigma(z, w),$$

where k_m is a constant depending on m.

Lemma 2.3[14] *Let f be a meromorphic function in $\mathbb{C}$, n be a positive integer, and b be a non-zero constant. If $f^n f' \neq b$, then f is a constant. Moreover, if f is a transcendental function, then $f^n f'$ assumes every finite non-zero value infinitely often.*

Lemma 2.4 *Let f be a nonconstant meromorphic function in $\mathbb{C}$, and $a \neq 0$ be a finite complex number. Then for any positive integer $n \geq 3$, $f' - af^n$ has least one zero.*

Proof. If $f' - af^n \equiv 0$, then $\dfrac{-1}{n-1} \cdot \dfrac{1}{f^{n-1}} \equiv a\xi + c$, where c is a constant. This contradicts that f is a meromorphic function since $n \geq 3$.

If $f' - af^n \neq 0$, then $\dfrac{f'}{f^n} \neq a$. Set $f = \dfrac{1}{\varphi}$. Then $\varphi^{n-2}\varphi' \neq -a$. By Lemma 2.3, φ is a constant, and so f is a constant which contradicts that f is a nonconstant. Hence, $f' - af^n$ has least one zero.

Lemma 2.5[15] *Take nonnegative integers n, n_1, $\cdots$, n_k with $n \geq 1$, $n_1 + n_2 + \cdots + n_k \geq 1$ and define $d = n + n_1 + n_2 + \cdots + n_k$. Let f be a transcendental meromorphic function with the deficiency $\Delta(0, f) > \dfrac{3}{3d+1}$. Then for any nonzero value c, the function $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} - c$ has infinitely many zeros.*

Lemma 2.6[12] *Let f be a nonconstant rational function, and m, n be natural numbers. Then, the function $f^n(f')^m$ takes every finite nonzero value $a \in \mathbb{C}$.*

Lemma 2.7[12] *Let $a, b \in \mathbb{C}$ with $a \neq 0$, and f be a nonconstant meromorphic function. If m_1, m_2, n_1, n_2 are positive integers such that $m_1 n_2 = m_2 n_1$, then*

$$F = (f(z))^{n_1}(f'(z))^{m_1} + \frac{a}{(f(z))^{n_2}(f'(z))^{m_2}} - b$$

has a finite zero.

3 Proof of the Theorems

Proof of Theorem 1.4 Let $M = \dfrac{b_f}{c_f}$. We can find nonzero constants b and c satisfying $M = \dfrac{b}{c}$. For each $f \in \mathcal{F}$, define a Möbius map g_f by $g_f = \dfrac{c_f z}{c}$. Then $g_f^{-1} = \dfrac{cz}{c_f}$.

Next, we show that $G = \{(g_f^{-1} \circ f) \mid f \in \mathcal{F}\}$ is normal in Δ. Suppose to the contrary, G were not normal in Δ. Then by Lemma 2.1, we could find $g_j \in G$, $z_j \in \Delta$ and $\rho_j \to 0^+$ such that $T_j(\xi) = g_j(z_j + \rho_j \xi)\rho_j^{\frac{1}{n-1}}$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function $T(\xi)$ with bounded spherical derivative and T has order at most 2.

By Lemma 2.4, there exists a ξ_0 such that

$$T'(\xi_0) - \frac{1}{b^{n-1}} T^n(\xi_0) = 0.$$

Note that

$$T_j'(\xi) - \frac{1}{b^{n-1}} T_j^n(\xi) - \rho_j^{\frac{n}{n-1}} c \to T'(\xi) - \frac{1}{b^{n-1}} T^n(\xi).$$

By Hurwitz's Theorem, there exists a sequence of points $\xi_j \to \xi_0$ such that (for j large enough)

$$\begin{aligned} 0 &= T_j'(\xi_j) - \frac{1}{b^{n-1}} T_j^n(\xi_j) - \rho_j^{\frac{n}{n-1}} c \\ &= \rho_j^{\frac{n}{n-1}} \Big(g_j'(z_j + \rho_j \xi_j) - \frac{1}{b^{n-1}} g_j^n(z_j + \rho_j \xi_j) - c \Big). \end{aligned}$$

Hence

$$f_j'(z_j + \rho_j \xi_j) - \frac{1}{b_{f_j}^{n-1}} f_j^n(z_j + \rho_j \xi_j) = c_{f_j}.$$

We get a contradiction. Hence $G = \{(g_f^{-1} \circ f) \mid f \in \mathcal{F}\}$ is normal and equicontinuous in $\varDelta$. Then given $\dfrac{\varepsilon}{k_m} > 0$, where k_m is the constant of Lemma 2.2, there exists a $\varDelta > 0$ such that for the spherical distance $\sigma(x, y) < \varDelta$, one has

$$\sigma((g_f^{-1} \circ f)(x),\ (g_f^{-1})(y)) < \frac{\varepsilon}{k_m}$$

for each $f \in \mathcal{F}$. By Lemma 2.2, we get

$$\begin{aligned} \sigma(f(x),\ f(y)) &= \sigma((g_f \circ g_f^{-1} \circ f)(x),\ (g_f \circ g_f^{-1} \circ f)(y)) \\ &= k_m \sigma((g_f^{-1} \circ f)(x),\ (g_f^{-1} \circ f)(y)) \\ &< \varepsilon. \end{aligned}$$

Therefore, the family is equicontinuous in $\varDelta$. This completes the proof of Theorem 1.4.

Proof of Theorem 1.7 Let $M = \dfrac{b_f}{c_f}$. We can find nonzero constants b and c satisfying $M = \dfrac{b}{c}$. For each $f \in \mathcal{F}$, define a Möbius map g_f by $g_f = \dfrac{c_f z}{c}$. Then $g_f^{-1} = \dfrac{cz}{c_f}$.

Next, we show that $G = \{(g_f^{-1} \circ f) \mid f \in \mathcal{F}\}$ is normal in $\varDelta$. Suppose to the contrary, G were not normal in $\varDelta$. Then by Lemma 2.1, we could find $g_j \in G$, $z_j \in \varDelta$ and $\rho_n \to 0^+$, such that

$$T_j(\xi) = \frac{g_j(z_j + \rho_j \xi)}{\rho_j^{\frac{1}{n+1}}}$$

converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function $T(\xi)$ whose spherical derivative is limited and T has order at most 2.

By Lemma 2.3, we get

$$T^n(\xi_0) T'(\xi_0) - b^{n+1} = 0, \qquad T'(\xi_0) - \frac{b^{n+1}}{T^n(\xi_0)} = 0$$

for some $\xi_0 \in \mathbb{C}$. Clearly, ξ_0 is neither a zero nor a pole of $T(\xi)$. Note that

$$T_j'(\xi) - \frac{b^{n+1}}{T_j^n(\xi)} - \rho_j^{\frac{n}{n+1}} c \to T'(\xi) - \frac{b^{n+1}}{T^n(\xi)}.$$

By Hurwitz's Theorem, there exists a sequence of points $\xi_j \to \xi_0$ such that (for j large enough)

$$\begin{aligned} 0 &= T_j'(\xi_j) - \frac{b^{n+1}}{T_j^n(\xi_j)} - \rho_j^{\frac{n}{n+1}} c \\ &= \rho_j^{\frac{n}{n+1}} \Big(g_j'(z_j + \rho_j \xi_j) - \frac{b^{n+1}}{g_j^n(z_j + \rho_j \xi_j)} - c \Big). \end{aligned}$$

Hence

$$f_j'(z_j + \rho_j \xi_j) - \frac{b_{f_j}^{n+1}}{f_j^n(z_j + \rho_j \xi_j)} = c_{f_j}.$$

So we have $|f_j(z_j + \rho_j \xi_j)| \ge |b_{f_j}|$, by the condition

$$f'(z) - \frac{b_f^{n+1}}{f^n(z)} \Rightarrow |f(z)| \ge |b_f|.$$

Thus

$$|T(\xi_0)| = \lim_{j\to\infty} \Big| \frac{g_j(z_j + \rho_j \xi_j)}{\rho_j^{\frac{1}{n+1}}} \Big| = \lim_{j\to\infty} \Big| \frac{c f_j(z_j + \rho_j \xi_j)}{c_{f_j} \rho_j^{\frac{1}{n+1}}} \Big| \ge \lim_{j\to\infty} \frac{|b|}{\rho_j^{\frac{1}{n+1}}} = \infty.$$

So ξ_0 is a pole of $T(\xi)$, a contradiction. Hence $G = \{(g_f^{-1} \circ f) \mid f \in \mathcal{F}\}$ is normal and equicontinuous in Δ. Then given $\dfrac{\varepsilon}{k_m} > 0$, where k_m is the constant of Lemma 2.2, there exists a $\Delta > 0$ such that for the spherical distance $\sigma(x, y) < \Delta$,

$$\sigma((g_f^{-1} \circ f)(x),\ (g_f^{-1})(y)) < \frac{\varepsilon}{k_m}$$

for each $f \in \mathcal{F}$. Hence, by Lemma 2.2,

$$\begin{aligned} \sigma(f(x),\ f(y)) &= \sigma((g_f \circ g_f^{-1} \circ f)(x),\ (g_f \circ g_f^{-1} \circ f)(y)) \\ &= k_m \sigma((g_f^{-1} \circ f)(x),\ (g_f^{-1} \circ f)(y)) \\ &< \varepsilon. \end{aligned}$$

Therefore, the family is equicontinuous in Δ. This completes the proof of Theorem 1.7.

Proof of Theorem 1.10 Let $M = \dfrac{b_f}{c_f}$. We can find nonzero constants b and c satisfying $M = \dfrac{b}{c}$. For each $f \in F$, define a Möbius map g_f by $g_f = \dfrac{c_f z}{c}$, and thus $g_f^{-1} = \dfrac{cz}{c_f}$.

Next, we show that $G = \{(g_f^{-1} \circ f) \mid f \in \mathcal{F}\}$ is normal in Δ. Suppose to the contrary, G were not normal in Δ. By Lemma 2.1, we could find $g_j \in G$, $z_j \in \Delta$ and $\rho_j \to 0^+$ such that

$$T_j(\xi) = g_j(z_j + \rho_j \xi) \rho_j^{-\frac{m_1+m_2}{s+t}}$$

converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function $T(\xi)$ whose spherical derivate is limited and T has order at most 2. By Lemmas 2.5 and 2.6, we obtain

$$(T(\xi_0))^{n_1+n_2} (T'(\xi_0))^{m_1+m_2} + b^{s+t} = 0,$$

$$(T(\xi_0))^{n_1} (T'(\xi_0))^{m_1} + \frac{b^{s+t}}{(T(\xi_0))^{n_2} (T'(\xi_0))^{m_2}} = 0$$

for some $\xi_0 \in \mathbb{C}$. Clearly, ξ_0 is neither a zero nor a pole of $T(\xi)$. Note

$$
\begin{aligned}
&(T_j(\xi))^{n_1}(T_j'(\xi))^{m_1} + \frac{b^{s+t}}{(T_j(\xi))^{n_2}(T_j'(\xi))^{m_2}} - \rho_j^{\frac{m_1n_2-m_2n_1}{s+t}}c^s\\
&\to (T(\xi))^{n_1}(T'(\xi))^{m_1} + \frac{b^{s+t}}{(T(\xi))^{n_2}(T'(\xi))^{m_2}}.
\end{aligned}
$$

By Hurwitz's Theorem, there exists a sequence of points $\xi_j \to \xi_0$ such that (for j large enough)

$$
\begin{aligned}
0 &= (T_j(\xi))^{n_1}(T_j'(\xi))^{m_1} + \frac{b^{s+t}}{(T_j(\xi))^{n_2}(T_j'(\xi))^{m_2}} - \rho_j^{\frac{m_1n_2-m_2n_1}{s+t}}c^s\\
&= \rho_j^{\frac{m_1n_2-m_2n_1}{s+t}}\Big((g_j(z_j+\rho_j\xi_j))^{n_1}(g_j'(z_j+\rho_j\xi_j))^{m_1}\\
&\qquad + \frac{b^{s+t}}{(g_j(z_j+\rho_j\xi_j))^{n_2}(g_j'(z_j+\rho_j\xi_j))^{m_2}} - c^s\Big)\\
&= \rho_j^{\frac{m_1n_2-m_2n_1}{s+t}}((f_j(z_j+\rho_j\xi_j))^{n_1}(f_j'(z_j+\rho_j\xi_j))^{m_1})\frac{c^s}{c^s_{f_j}}\\
&\quad + b^{s+t}\frac{c^t_{f_j}}{c^t}\frac{1}{(f_j(z_j+\rho_j\xi_j))^{n_2}(f_j'(z_j+\xi_j))^{m_2}} - c^s.
\end{aligned}
$$

Hence

$$
(f_j(z_j+\rho_j\xi_j))^{n_1}(f_j'(z_j+\rho_j\xi_j))^{m_1} + \frac{b^{s+t}_{f_j}}{(f_j(z_j+\rho_j\xi_j))^{n_2}(f_j'(z_j+\xi_j))^{m_2}} = c^s_{f_j}.
$$

So we have $|f_j(z_j+\rho_j\xi_j)| \geq |b_{f_j}|$ by the condition

$$
(f(z))^{n_1}(f'(z))^{m_1}) + \frac{b_f^{s+t}}{(f(z))^{n_2}(f'(z))^{m_2}} = c_f^s \Rightarrow |f(z)| \geq |b_f|.
$$

Thus

$$
|T(\xi_0)| = \lim_{j\to\infty}\left|\frac{g_j(z_j+\rho_j\xi_j)}{\rho_j^{\frac{m_1+m_2}{s+t}}}\right| = \lim_{j\to\infty}\left|\frac{cf_j(z_j+\rho_j\xi_j)}{c_{f_j}\rho_j^{\frac{m_1+m_2}{s+t}}}\right| \geq \lim_{j\to\infty}\frac{|b|}{\rho_j^{\frac{m_1+m_2}{s+t}}} = \infty.
$$

So ξ_0 is a pole of $T(\xi)$, a contradiction. Hence $G = \{(g_f^{-1}\circ f) \mid f \in \mathcal{F}\}$ is normal and equicontinuous in $\varDelta$. Then given $\dfrac{\varepsilon}{k_m} > 0$, where k_m is the constant of Lemma 2.2, there exists a $\varDelta > 0$ such that for the spherical distance $\sigma(x,y) < \varDelta$, one has

$$
\sigma((g_f^{-1}\circ f)(x),\ (g_f^{-1})(y)) < \frac{\varepsilon}{k_m}
$$

for each $f \in F$. Hence by Lemma 2.2.

$$
\begin{aligned}
\sigma(f(x),\ f(y)) &= \sigma((g_f\circ g_f^{-1}\circ f)(x),\ (g_f\circ g_f^{-1}\circ f)(y))\\
&= k_m\sigma((g_f^{-1}\circ f)(x),\ (g_f^{-1}\circ f)(y))\\
&< \varepsilon.
\end{aligned}
$$

Therefore, the family is equicontinuous in $\varDelta$. This completes the proof of Theorem 1.10.

Proof of Theorem 1.11 Let $M = \dfrac{b_f}{c_f}$. We can find nonzero constants b and c satisfying $M = \dfrac{b}{c}$. For each $f \in F$, define a Möbius map g_f by $g_f = \dfrac{c_f z}{c}$, and thus $g_f^{-1} = \dfrac{cz}{c_f}$.

Next, we show that $G = \{(g_f^{-1}\circ f) \mid f \in \mathcal{F}\}$ is normal in $\varDelta$. Suppose to the contrary, G were not normal in $\varDelta$. Then by Lemma 2.1, we could find $g_j \in G$, $z_j \in \varDelta$, and $\rho_j \to 0^+$

such that

$$T_j(\xi) = g_j(z_j + \rho_j \xi)\rho_j^{-\frac{m_1+m_2}{s+t}}$$

converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function $T(\xi)$ whose spherical derivate is limited and T has order at most 2. By Lemmas 2.5 and 2.6, we get

$$(T(\xi_0))^{n_1}(T'(\xi_0))^{m_1} + \frac{b^{s+t}}{(T(\xi_0))^{n_2}(T'(\xi_0))^{m_2}} = c^s.$$

Finally, we get a contradiction by using Lemma 2.7, and in a similar way to the proof of Theorem 1.10, we can prove the Theorem 1.11 easily. This completes the proof of the theorem.

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