Communications in Mathematical Research
31(1)(2015), 1–14

A Split Least-squares Characteristic Procedure for Convection-dominated Parabolic Integro-differential Equations

Guo Hui and Fu Hong-fei
(*College of Science, China University of Petroleum, Qingdao, Shandong, 266580*)

Communicated by Ma Fu-ming

Abstract: In this paper, we combine a split least-squares procedure with the method of characteristics to treat convection-dominated parabolic integro-differential equations. By selecting the least-squares functional properly, the procedure can be split into two independent sub-procedures, one of which is for the primitive unknown and the other is for the flux. Choosing projections carefully, we get optimal order $H^1(\Omega)$ and $L^2(\Omega)$ norm error estimates for u and sub-optimal $(L^2(\Omega))^d$ norm error estimate for $\boldsymbol{\sigma}$. Numerical results are presented to substantiate the validity of the theoretical results.

Key words: split least-square, characteristic, convection-dominated, convergence analysis

2010 MR subject classification: 35L15, 65M15, 65M60

Document code: A

Article ID: 1674-5647(2015)01-0001-14

DOI: 10.13447/j.1674-5647.2015.01.01

1 Introduction

We consider the following convection-dominated parabolic integro-differential equations:

$$\begin{cases} c(x)\dfrac{\partial u}{\partial t} + \boldsymbol{d}(x)\cdot\nabla u - \nabla\cdot\Big(\boldsymbol{A}(x)\nabla u + \boldsymbol{B}(x)\displaystyle\int_0^t \nabla u(x,s)\mathrm{d}s\Big) = f(x,t), & (x,t)\in(\Omega\times I), \\ u(x,t) = 0, & (x,t)\in(\Gamma\times I), \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $I = (0, T]$ is the time interval, Ω is a bounded polygonal domain in $\mathbf{R}^d$, $d = 2, 3$, with a Lipschitz continuous boundary Γ, d is the space dimension. $\boldsymbol{d}(x) = (d_1(x), \cdots, d_d(x))^{\mathrm{T}}$.

Received date: May 17, 2012.
Foundation item: The NSF (11101431 and 11201485) of China and the NSF (ZR2010AL020) of Shandong Province.
E-mail address: sdugh@163.com (Guo H).

$\boldsymbol{A}(x) = (a_{ij}(x))_{d\times d}$, $\boldsymbol{B}(x) = (b_{ij}(x))_{d\times d}$ are bounded, symmetric and positive definite matrices, i.e., there exist positive constants a_*, a^*, b_* and b^* such that

$$a_*\|\boldsymbol{\xi}\|^2 \le (\boldsymbol{A}\boldsymbol{\xi}, \boldsymbol{\xi}) \le a^*\|\boldsymbol{\xi}\|^2, \quad b_*\|\boldsymbol{\xi}\|^2 \le (\boldsymbol{B}\boldsymbol{\xi}, \boldsymbol{\xi}) \le b^*\|\boldsymbol{\xi}\|^2, \qquad \boldsymbol{\xi} \in \mathbf{R}^d.$$

We make the following assumptions: there exist positive constants k_1, k_2 such that

$$0 < k_1 \le c(x) \le k_2, \qquad \|\boldsymbol{d}\|_{1,\infty} + \|c\|_{1,\infty} \le k_2.$$

We also assume that Ω is H^2-regular: for $f \in L^2(\Omega)$ the solution of the following problem

$$-\nabla \cdot (\boldsymbol{A}\nabla w) = f \quad \text{in } \Omega, \qquad w|_\Gamma = 0$$

exists and $\|w\|_2 \le K\|f\|$.

This model arises from many physical processes in which it is necessary to take into account the effects of memory due to the deficiency of the usual diffusion equations (see [1–2]). As we all know, the numerical simulation of convection-dominated problems requires special treatment. Generally, they either smear sharp physical fronts with excessive numerical diffusion, or introduce nonphysical oscillations into numerical solutions. The method of characteristic has proved effective in treating convection-dominated problems (see [3–4]).

We have introduced the least-squares method for such equations when A, B are proportional to a unit matrix in [5]. The least-squares finite element procedure has two typical advantages as follows: it is not subject to Ladyzhenskaya[6], Babuška[7], Brezzi[8] consistency condition, so the choice of approximation spaces becomes flexible, and it results in a symmetric positive definite system. However, it usually needs to solve a coupled system of equations for conventional least-squares finite element procedure, which brings to difficulties in some extent. We only get the optimal order $H^1(\Omega)$ norm error estimate for u in [5].

In [9–10], a kind of split least-squares Galerkin procedure was constructed for stationary diffusion reaction problems and parabolic problems. The purpose of this paper is to combine the split least-squares procedure with the method of characteristics for convection-dominated parabolic integro-differential equations. The most advantage of the scheme is: by selecting the least-squares functional properly, the resulting procedure can be split into two independent symmetric positive definite sub-schemes. One of sub-procedures is for the primitive unknown variable u, which is the same as a stand Galerkin characteristic finite element procedure and the other is for the introduced flux variable $\boldsymbol{\sigma}$. By carefully choosing projections, we see that the method leads to the optimal order $H^1(\Omega)$ and $L^2(\Omega)$ norm error estimates for u and sub-optimal $(L^2(\Omega))^d$ norm error estimate for $\boldsymbol{\sigma}$.

The paper is organized as follows. In Section 2, we formulate the procedure. The convergence theory on the algorithm is established in Section 3. In Section 4 we give the numerical experiment.

As in [11], we assume that the problem (1.1) is periodic with Ω. In this paper we use $W^{k,p}$ $(k \ge 0,\ 1 \le p \le \infty)$ to denote Sobolev spaces (see [12]) defined on Ω with a usual norm $\|\cdot\|_{W^{k,p}(\Omega)}$, and $H^k(\Omega)$, $L^2(\Omega)$ with norms $\|\cdot\|_k = \|\cdot\|_{H^k(\Omega)}$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, respectively. For simplicity we also use $L^s(H^k)$ to denote $L^s(0, T; H^k(\Omega))$. The inner product $(\,\cdot\,,\,\cdot\,)$ is both used for scalar-valued functions and vector-valued functions. Throughout this paper, the symbols K and δ are used to denote a generic constant and a generic small positive constant, respectively, which may appear differently at different occurrences.

2 Split Characteristics Least-squares Procedure

Introduce two function spaces

$$H = \{\boldsymbol{w} \in (L^2(\Omega))^d;\ \mathrm{div}\boldsymbol{w} \in L^2(\Omega)\}, \qquad S = \{v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma\}.$$

Letting the flux variable $\boldsymbol{\sigma} = -\Big(\boldsymbol{A}\nabla u + \boldsymbol{B}\int_0^t \nabla u(x,s)\mathrm{d}s\Big)$, we can rewrite the problem (1.1) as a first-order system

$$\begin{cases} c\dfrac{\partial u}{\partial t} + \boldsymbol{d}\cdot\nabla u + \mathrm{div}\boldsymbol{\sigma} = f(x,t), & x \in \Omega,\ 0 < t \le T, \\ \boldsymbol{\sigma} + \Big(\boldsymbol{A}\nabla u + \boldsymbol{B}\displaystyle\int_0^t \nabla u(x,s)\mathrm{d}s\Big) = 0, & x \in \Omega,\ 0 < t \le T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

Letting $\psi(x) = \sqrt{c(x)^2 + |\boldsymbol{d}(x)|^2}$, $|\boldsymbol{d}(x)|^2 = \sum\limits_{i=1}^d d_i^2$, we denote the characteristic direction associated with the operator $cu_t + \boldsymbol{d}\cdot\nabla u$ by $\tau = \tau(x)$, where

$$\psi\frac{\partial}{\partial \tau} = c(x)\frac{\partial}{\partial t} + \boldsymbol{d}(x)\cdot\nabla.$$

(2.1) can be put into the form

$$\begin{cases} \psi\dfrac{\partial u}{\partial \tau} + \mathrm{div}\boldsymbol{\sigma} = f(x,t), & x \in \Omega,\ 0 < t \le T, \\ \boldsymbol{\sigma} + \boldsymbol{A}\nabla u + \boldsymbol{B}\displaystyle\int_0^t \nabla u(x,s)\mathrm{d}s = 0, & x \in \Omega,\ 0 < t \le T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.2}$$

Given a time step $\Delta t = \dfrac{T}{N}$, where N is a positive integer, we approximate the solution at times $t^n = n\Delta t$, $n = 0,1,\cdots,N$. Let $g^n = g(x,t^n)$ and $D_t g^n = \dfrac{g^n - g^{n-1}}{\Delta t}$. The characteristic derivative is approximated basically in the following manner:

$$\psi(x)\Big(\frac{\partial u}{\partial \tau}\Big)^n = c(x)\frac{u^n - \bar{u}^{n-1}}{\Delta t} + e^n, \qquad e^n = O(\Delta t),$$

where $\bar{x} = x - \dfrac{\boldsymbol{d}(x)}{c(x)}\Delta t$, $\bar{u}^{n-1} = u^{n-1}(\bar{x})$.

We can rewrite the system (2.2) as the following semi-discrete system:

$$\begin{cases} c\dfrac{u^n - \bar{u}^{n-1}}{\Delta t} + \mathrm{div}\boldsymbol{\sigma}^n = f^n + R_1^n, & x \in \Omega, \\ \boldsymbol{\sigma}^n + \boldsymbol{A}\nabla u^n + \Delta t\boldsymbol{B}\displaystyle\sum_{i=1}^n \nabla u^i = \boldsymbol{R}_2^n, & x \in \Omega, \\ u^0(x) = u_0(x), & x \in \Omega, \end{cases} \tag{2.3}$$

where

$$R_1^n = c\frac{u^n - \bar{u}^{n-1}}{\Delta t} - \psi\Big(\frac{\partial u}{\partial \tau}\Big)^n = O(\Delta t),$$

$$\boldsymbol{R}_2^n = \Delta t\boldsymbol{B}\sum_{i=1}^n \nabla u^i - \boldsymbol{B}\int_0^{t^n} \nabla u(x,s)\mathrm{d}s = O(\Delta t),$$

$$D_t\boldsymbol{R}_2^n = \boldsymbol{B}\nabla u^n - \frac{\boldsymbol{B}}{\Delta t}\int_{t^{n-1}}^{t^n} \nabla u(x,s)\mathrm{d}s = O(\Delta t).$$

Equivalently we have that

$$\begin{cases} c^{-\frac{1}{2}}[cu^n + \Delta t \mathrm{div}\boldsymbol{\sigma}^n - (c\bar{u}^{n-1} + \Delta t f^n + \Delta t R_1^n)] = 0, & x \in \Omega, \\ \tilde{\boldsymbol{A}}^{-\frac{1}{2}}[\boldsymbol{\sigma}^n + \tilde{\boldsymbol{A}}\nabla u^n + \boldsymbol{B}\sum_{i=1}^{n-1}\Delta t \nabla u^i - \boldsymbol{R}_2^n] = 0, & x \in \Omega, \\ u^0(x) = u_0(x), & x \in \Omega, \end{cases} \tag{2.4}$$

where $\tilde{\boldsymbol{A}} = \boldsymbol{A} + \Delta t \boldsymbol{B}$.

Define the minimization functional J_1^n as

$$\begin{aligned} J_1^n(\psi, v) = {} & \|c^{-\frac{1}{2}}[cv + \Delta t \mathrm{div}\psi - (c\bar{u}^{n-1} + \Delta t f^n + \Delta t R_1^n)]\|^2 \\ & + \Delta t \Big\| \tilde{\boldsymbol{A}}^{-\frac{1}{2}}\Big[\psi + \tilde{\boldsymbol{A}}\nabla v + \boldsymbol{B}\sum_{i=1}^{n-1}\Delta t\nabla u^i - \boldsymbol{R}_2^n\Big]\Big\|^2. \end{aligned} \tag{2.5}$$

Then the least-squares minimization corresponding to (2.4) is: find $(\boldsymbol{\sigma}^n,\ u^n) \in H \times S$ such that

$$J_1^n(\boldsymbol{\sigma}^n,\ u^n) = \min_{\boldsymbol{\psi}\in H, v\in S} J_1^n(\boldsymbol{\psi},\ v). \tag{2.6}$$

The weak formulation of (2.6) is: find $(\boldsymbol{\sigma}^n,\ u^n) \in H \times S$ such that

$$\begin{aligned} & a_n((\boldsymbol{\sigma}^n,\ u^n),\ (\boldsymbol{\psi},\ v)) \\ = {} & (c^{-1}(c\bar{u}^{n-1} + \Delta t f^n + \Delta t R_1^n),\ cv + \Delta t \mathrm{div}\boldsymbol{\psi}) \\ & - \Delta t\Big(\tilde{\boldsymbol{A}}^{-1}\Big(\boldsymbol{B}\sum_{i=1}^{n-1}\Delta t\nabla u^i - \boldsymbol{R}_2^n\Big),\ \boldsymbol{\psi} + \boldsymbol{A}\nabla v\Big), \qquad (\boldsymbol{\psi}, v) \in H \times S, \end{aligned} \tag{2.7}$$

where the bilinear form a_n is defined as

$$\begin{aligned} a_n((\boldsymbol{\sigma}, u),\ (\boldsymbol{\psi}, v)) = {} & (c^{-1}(cu + \Delta t \mathrm{div}\boldsymbol{\sigma}),\ cv + \Delta t \mathrm{div}\boldsymbol{\psi}) \\ & + \Delta t(\tilde{\boldsymbol{A}}^{-1}(\boldsymbol{\sigma} + \tilde{\boldsymbol{A}}\nabla u),\ \boldsymbol{\psi} + \tilde{\boldsymbol{A}}\nabla v). \end{aligned} \tag{2.8}$$

Let $T_{h_{\boldsymbol{\sigma}}}$ and T_{h_u} be two families of finite element partitions of the domain Ω, which are identical or not. Let $h_{\boldsymbol{\sigma}}$ and h_u be mesh parameters. Based on T_{h_σ} and T_{h_u}, we construct the finite element spaces $H_{h_\sigma} \subset H$ and $S_{h_u} \subset S$ with the following approximation properties: there exist integers $k \geq 0,\ l \geq 1$ such that

$$\begin{cases} \inf_{\boldsymbol{\omega_h}\in H_{h_\sigma}} \|\boldsymbol{\omega} - \boldsymbol{\omega_h}\| \leq K h_\sigma^{k+1}\|\boldsymbol{\omega}\|_{k+1}, \\ \inf_{\boldsymbol{\omega_h}\in H_{h_\sigma}} \|\mathrm{div}(\boldsymbol{\omega} - \boldsymbol{\omega_h})\| \leq K h_\sigma^{k_1}\|\boldsymbol{\omega}\|_{k_1+1}, \\ \inf_{v_h\in S_{h_u}} \{\|v - v_h\| + h_u\|\nabla(v - v_h)\|\} \leq K h_u^{l+1}\|v\|_{l+1} \end{cases} \tag{2.9}$$

for any $v \in S\bigcap H^{l+1}(\Omega)$, $\boldsymbol{\omega} \in H\bigcap(H^{k_1+1}(\Omega))^d$, where $k_1 = k+1$ in the case that the space H_{h_σ} is any one of Raviart-Thomas mixed elements (see [13]) and Nedelec mixed elements (see [14]) and $k_1 = k \geq 1$ in the case that the space H_{h_σ} is the C^0-elements (see [15]).

Omitting the time truncation error terms in (2.7), we define the following least-squares procedure with the method of characteristics.

Scheme I. Given an initial approximation $(\boldsymbol{\sigma}_h^0,\ u_h^0) = (\mathrm{Q}\boldsymbol{\sigma}^0,\ Ru^0) \in (H_{h_\sigma} \times S_{h_u})$, which is defined in Section 3, we seek $(\boldsymbol{\sigma}_h^n,\ u_h^n) \in H_{h_\sigma} \times S_{h_u}$ such that

$$
\begin{aligned}
&a_n((\boldsymbol{\sigma}_h^n,\ u_h^n),\ (\boldsymbol{\psi}_h,\ v_h))\\
=\ &(c^{-1}(c\bar{u}_h^{n-1} + \Delta t f^n),\ cv_h + \Delta t\mathrm{div}\boldsymbol{\psi}_h)\\
&- \Delta t(\tilde{\boldsymbol{A}}^{-1}\boldsymbol{B}\sum_{i=1}^{n-1}\Delta t\nabla u_h^i,\ \boldsymbol{\psi}_h + \tilde{\boldsymbol{A}}\nabla v_h), \qquad (\boldsymbol{\psi}_h,\ v_h) \in H_{h_\sigma} \times S_{h_u}.
\end{aligned}
\tag{2.10}
$$

Remark 2.1 Scheme I is still a conventional least-squares characteristic mixed finite element scheme. Now we discuss the bilinear form a_n in the following lemma, which leads to a decoupled system.

Lemma 2.1 *For any* $(\boldsymbol{\sigma}, u),\ (\boldsymbol{\psi}, v) \in H \times S$, *we have that*

$$
\begin{aligned}
&a_n((\boldsymbol{\sigma},\ u),\ (\boldsymbol{\psi},\ v))\\
=\ &(cu,\ v) + \Delta t^2(c^{-1}\mathrm{div}\boldsymbol{\sigma},\ \mathrm{div}\boldsymbol{\psi}) + \Delta t(\tilde{\boldsymbol{A}}^{-1}\boldsymbol{\sigma},\ \boldsymbol{\psi}) + \Delta t(\tilde{\boldsymbol{A}}\nabla u,\ \nabla v).
\end{aligned}
\tag{2.11}
$$

Proof. A direct calculation shows that

$$
\begin{aligned}
&a_n((\boldsymbol{\sigma},\ u),\ (\boldsymbol{\psi},\ v))\\
=\ &(cu,\ v) + \Delta t^2(c^{-1}\mathrm{div}\boldsymbol{\sigma},\ \mathrm{div}\boldsymbol{\psi}) + \Delta t(\tilde{\boldsymbol{A}}^{-1}\boldsymbol{\sigma},\ \boldsymbol{\psi}) + \Delta t(\tilde{\boldsymbol{A}}\nabla u,\ \nabla v)\\
&+ \Delta t[(u,\ \mathrm{div}\boldsymbol{\psi}) + (\mathrm{div}\boldsymbol{\sigma},\ v) + (\boldsymbol{\sigma},\ \nabla v) + (\nabla u,\ \boldsymbol{\psi})].
\end{aligned}
$$

By applying Green's formula we have

$$
(u,\ \mathrm{div}\boldsymbol{\psi}) + (\mathrm{div}\boldsymbol{\sigma},\ v) + (\boldsymbol{\sigma},\ \nabla v) + (\nabla u,\ \boldsymbol{\psi}) = 0,
$$

which completes the proof of (2.11).

Now, we can derive the split least-squares characteristic finite element method.

Scheme I$'$. Let $\boldsymbol{\psi}_h = 0$ and $v_h = 0$ in (2.10), alternatively. By using Lemma 2.1, we have the equivalent form of Scheme I

$$
\begin{aligned}
&(cu_h^n,\ v_h) + \Delta t(\tilde{\boldsymbol{A}}\nabla u_h^n,\ \nabla v_h)\\
=\ &(c\bar{u}_h^{n-1},\ v_h) + \Delta t(f^n,\ v_h) - \Delta t\Big(\boldsymbol{B}\sum_{i=1}^{n-1}\Delta t\nabla u_h^i,\ \nabla v_h\Big), \qquad v_h \in H_{h_u},
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
&(\tilde{\boldsymbol{A}}^{-1}\boldsymbol{\sigma}_h^n,\ \boldsymbol{\psi}_h) + \Delta t(c^{-1}\mathrm{div}\boldsymbol{\sigma}_h^n,\ \mathrm{div}\boldsymbol{\psi}_h)\\
=\ &(\bar{u}_h^{n-1},\ \mathrm{div}\boldsymbol{\psi}_h) + \Delta t(c^{-1}f^n,\ \mathrm{div}\boldsymbol{\psi}_h) - \Big(\tilde{\boldsymbol{A}}^{-1}\boldsymbol{B}\sum_{i=1}^{n-1}\Delta t\nabla u_h^i,\ \boldsymbol{\psi}_h\Big), \qquad \boldsymbol{\psi}_h \in H_{h_\sigma}.
\end{aligned}
\tag{2.13}
$$

Remark 2.2 It is obviously that Scheme I$'$ is a process in which u_h and $\boldsymbol{\sigma}_h$ can be solved separately. Moreover, the sub-procedure (2.12) is the same as a fully discrete Galerkin Characteristic finite element procedure for the problem (1.1).

Remark 2.3 Lemma 2.1 also tells us that the bilinear form $a_n(\,\cdot\,,\,\cdot\,)$ is continuous and positive definite in $H \times S$. So it follows from Lax-Milgram theorem that Scheme I$'$ has a unique solution.

3 Convergence Analysis

In this section, we analyze the convergence of the procedure. For this purpose we introduce some projection operators first.

From the approximate property of finite element spaces we know that there exists a vector-valued function $\mathrm{Q}\boldsymbol{\sigma} \in H_{h_\sigma}$ such that

$$\begin{cases} \|\boldsymbol{\sigma} - \mathrm{Q}\boldsymbol{\sigma}\| \le K h_\sigma^{k+1} \|\boldsymbol{\sigma}\|_{k+1}, \\ \|\mathrm{div}(\boldsymbol{\sigma} - \mathrm{Q}\boldsymbol{\sigma})\| \le K h_\sigma^{k_1} \|\boldsymbol{\sigma}\|_{k_1+1}. \end{cases} \tag{3.1}$$

For $u^n \in S$, we define its elliptic projection $Ru^n \in H_{h_u}$ such that

$$(\boldsymbol{A}\nabla(Ru^n - u^n),\ \nabla v_h) + \sum_{i=1}^{n} \Delta t(\boldsymbol{B}\nabla(Ru^i - u^i),\ \nabla v_h) = 0, \qquad v_h \in S_{h_u}. \tag{3.2}$$

Lemma 3.1 *There exists a positive constant K independent of the discretization parameters Δt, h_σ and h_u such that for $s = 0, 1$*

$$\begin{cases} \|u^n - Ru^n\|_s \le K h_u^{l+1-s} \|u\|_{L^\infty(H^{l+1}(\Omega))}, \\ \|D_t(u^n - Ru^n)\|_s \le K h_u^{l+1-s} [\|u\|_{L^\infty(H^{l+1}(\Omega))} + \|u_t\|_{L^\infty(H^{l+1}(\Omega))}]. \end{cases} \tag{3.3}$$

Lemma 3.2[3] *Let $q \in L^\infty(L^2)$. Then*

$$\|\bar{q}\|_c^2 \le (1 + K\Delta t)\|q\|_c^2, \tag{3.4}$$

where $\|q\|_c^2 = (cq, q)$, the constant K depends only on k_1 and k_2.

Lemma 3.3[3] *If $\eta \in L^\infty(L^2)$ and $\bar{\eta}(x) = \eta(x - g(x)\Delta t)$, where g and g' are bounded, then*

$$\|\eta - \bar{\eta}\|_{-1} \le K\|\eta\|\Delta t. \tag{3.5}$$

We are able to demonstrate our main result for the scheme.

Theorem 3.1 *Let $(\boldsymbol{\sigma}, u)$ be the solution of (2.1) and $(\boldsymbol{\sigma}_h^n, u_h^n)$ be the solution of Scheme I′. Suppose that the solution of (2.1) has regular properties that $u, u_t \in L^\infty(H^{k+1})$, $\boldsymbol{\sigma} \in L^\infty((H^{k_1+1})^d)$. Then we have the priori error estimates for $s = 0, 1$,*

$$\|u^n - u_h^n\|_s \le K(\Delta t + h_u^{l+1-s}), \tag{3.6}$$

$$\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\| \le K(\Delta t + h_u^l + h_\sigma^{k+1} + \Delta t^{\frac{1}{2}} h_\sigma^{k_1}), \tag{3.7}$$

where the constant K is dependent upon T and some norms of the solution $(\boldsymbol{\sigma}, u)$, and independent of the mesh parameters h_u, h_σ and Δt.

Proof. We prove (3.6) first. Letting $\boldsymbol{\psi} = 0$ in (2.7) and using Lemma 2.1, we have

$$\begin{aligned} (cu^n, v) + \Delta t(\tilde{\boldsymbol{A}}\nabla u^n, \nabla v) &= (c\bar{u}^{n-1}, v) + \Delta t(f^n, v) - \Delta t\Big(\boldsymbol{B}\sum_{i=1}^{n-1} \Delta t\nabla u^i, \nabla v\Big) \\ &\quad + \Delta t(R_1^n, v) + \Delta t(\boldsymbol{R}_2^n, \nabla v), \qquad v \in S. \end{aligned} \tag{3.8}$$

Subtracting (3.8) from (2.12), we obtain

$$\begin{aligned} &(c(u^n - u_h^n),\ v_h) + \Delta t(\tilde{\boldsymbol{A}}\nabla(u^n - u_h^n),\ \nabla v_h) \\ &= (c(\bar{u}^{n-1} - \bar{u}_h^{n-1}),\ v_h) - \Delta t\Big(\boldsymbol{B}\sum_{i=1}^{n-1} \Delta t\nabla(u^i - u_h^i),\ \nabla v_h\Big) \\ &\quad + \Delta t(R_1^n,\ v_h) + \Delta t(\boldsymbol{R}_2^n,\ \nabla v_h), \qquad v_h \in S_{h_u}. \end{aligned} \tag{3.9}$$

Write $u^n - u_h^n = u^n - Ru^n + Ru^n - u_h^n = \rho^n + \theta^n$. Since the estimates of ρ^n are known, we need to find only the estimates of θ^n. From (3.9) we see that θ^n satisfies the following error equation

$$\begin{aligned}&(c(\theta^n - \bar{\theta}^{n-1}),\ v_h) + \Delta t(\boldsymbol{A}\nabla\theta^n,\ \nabla v_h)\\ =\ &(c(\bar{\rho}^{n-1} - \rho^n),\ v_h) - \Delta t^2(\boldsymbol{B}\sum_{i=1}^{n}\nabla\theta^i,\ \nabla v_h) + \Delta t(R_1^n,\ v_h) + \Delta t(\boldsymbol{R}_2^n,\ \nabla v_h),\end{aligned} \tag{3.10}$$

where we have used the equation

$$(\boldsymbol{A}\nabla\rho^n,\ \nabla v_h) + \sum_{i=1}^{n}\Delta t(\boldsymbol{B}\nabla\rho^i,\ \nabla v_h) = 0.$$

Setting $v_h = \theta^n - \theta^{n-1} = \Delta t D_t\theta^n$ in (3.10), we have

$$\begin{aligned}&\Delta t(cD_t\theta^n,\, D_t\theta^n) + \frac{1}{2}[(\boldsymbol{A}\nabla\theta^n,\, \nabla\theta^n) - (\boldsymbol{A}\nabla\theta^{n-1},\, \nabla\theta^{n-1}) + (\boldsymbol{A}\nabla(\theta^n - \theta^{n-1}),\\ &\qquad \nabla(\theta^n - \theta^{n-1})]\\ =\ &-(c(\theta^{n-1} - \bar{\theta}^{n-1},\, D_t\theta^n) + (c(\bar{\rho}^{n-1} - \rho^n),\, D_t\theta^n) - \Delta t\Big(\boldsymbol{B}\sum_{i=1}^{n}\nabla\theta^i,\, \nabla(\theta^n - \theta^{n-1})\Big)\\ &+ \Delta t(R_1^n,\, D_t\theta^n) + (\boldsymbol{R}_2^n,\, \nabla(\theta^n - \theta^{n-1}))\\ =\ &T_1 + T_2 + \cdots + T_5.\end{aligned} \tag{3.11}$$

Using Cauchy's inequality, we estimate (3.11) term by term.

$$\begin{aligned}T_1 =\ &-(c(\theta^{n-1} - \bar{\theta}^{n-1},\, D_t\theta^n)\\ \le\ &K\Delta t\|\nabla\theta^{n-1}\|\|D_t\theta^n\|\\ \le\ &K\Delta t\|\boldsymbol{A}^{\frac{1}{2}}\nabla\theta^{n-1}\|^2 + \delta\Delta t\|c^{\frac{1}{2}}D_t\theta^n\|^2,\end{aligned} \tag{3.12}$$

$$\begin{aligned}T_2 &= (c(\bar{\rho}^{n-1} - \rho^{n-1}),\, D_t\theta^n) + (c(\rho^{n-1} - \rho^n),\, D_t\theta^n)\\ &= T_{21} + T_{22}.\end{aligned} \tag{3.13}$$

From Lemmas 3.2 and 3.3, we have

$$\begin{aligned}T_{21} &= \frac{1}{\Delta t}[(c(\bar{\rho}^n - \rho^n),\, \theta^n) - (c(\bar{\rho}^{n-1} - \rho^{n-1}),\, \theta^{n-1})] - (c(D_t\bar{\rho}^n - D_t\rho^n),\, \theta^n)\\ &\le \frac{1}{\Delta t}[(c(\bar{\rho}^n - \rho^n),\, \theta^n) - (c(\bar{\rho}^{n-1} - \rho^{n-1}),\, \theta^{n-1})] + K\Delta t\|D_t\rho^n\|^2\\ &\quad + K\Delta t(\|c^{\frac{1}{2}}\theta^n\|^2 + \|\boldsymbol{A}^{\frac{1}{2}}\nabla\theta^n\|^2)\\ &\le \frac{1}{\Delta t}[(c(\bar{\rho}^n - \rho^n),\, \theta^n) - (c(\bar{\rho}^{n-1} - \rho^{n-1}),\, \theta^{n-1})] + K\Delta t h_u^{2(l+1)}\\ &\quad + K\Delta t(\|c^{\frac{1}{2}}\theta^n\|^2 + \|\boldsymbol{A}^{\frac{1}{2}}\nabla\theta^n\|^2),\end{aligned} \tag{3.14}$$

$$\begin{aligned}T_{22} &= -\Delta t(D_t\rho^n,\, D_t\theta^n)\\ &\le K\Delta t\|D_t\rho^n\|^2 + \delta\Delta t\|c^{\frac{1}{2}}D_t\theta^n\|^2\\ &\le K\Delta t h^{2(l+1)} + \delta\Delta t\|c^{\frac{1}{2}}D_t\theta^n\|^2,\end{aligned} \tag{3.15}$$

$$T_3 = -\Delta t\Big(\boldsymbol{B}\ \sum_{i=1}^{n}\nabla\theta^i,\, \nabla(\theta^n - \theta^{n-1})\Big)$$

$$
\begin{aligned}
&= -\Delta t\Big[\Big(\boldsymbol{B}\sum_{i=1}^{n}\nabla\theta^i,\,\nabla\theta^n\Big) - \Big(\boldsymbol{B}\sum_{i=1}^{n-1}\nabla\theta^i,\,\nabla\theta^{n-1}\Big)\\
&\qquad - \Big(\boldsymbol{B}\sum_{i=1}^{n}\nabla\theta^i - \boldsymbol{B}\sum_{i=1}^{n-1}\nabla\theta^i, \nabla\theta^{n-1}\Big)\Big]\\
&\le -\Delta t\Big[\Big(\boldsymbol{B}\sum_{i=1}^{n}\nabla\theta^i, \nabla\theta^n\Big) - \Big(\boldsymbol{B}\sum_{i=1}^{n-1}\nabla\theta^i, \nabla\theta^{n-1}\Big)\Big]\\
&\qquad + K\Delta t(\|\boldsymbol{A}^{\frac12}\nabla\theta^n\|^2 + \|\boldsymbol{A}^{\frac12}\nabla\theta^{n-1}\|^2),
\end{aligned} \tag{3.16}
$$

$$
\begin{aligned}
T_4 &= \Delta t(R_1^n,\, D_t\theta^n)\\
&\le K\Delta t\|R_1^n\|^2 + \delta\Delta t\|c^{\frac12}D_t\theta^n\|^2,
\end{aligned} \tag{3.17}
$$

$$
\begin{aligned}
T_5 &= (\boldsymbol{R}_2^n,\, \nabla(\theta^n - \theta^{n-1}))\\
&= [(\boldsymbol{R}_2^n,\, \nabla\theta^n) - (\boldsymbol{R}_2^{n-1},\, \nabla\theta^{n-1}) - \Delta t(D_t\boldsymbol{R}_2^n,\, \nabla\theta^{n-1})]\\
&\le [(\boldsymbol{R}_2^n,\, \nabla\theta^n) - (\boldsymbol{R}_2^{n-1},\, \nabla\theta^{n-1})] + K\Delta t(\|D_t\boldsymbol{R}_2^n\|^2 + \|\boldsymbol{A}^{\frac12}\nabla\theta^{n-1}\|^2).
\end{aligned} \tag{3.18}
$$

Therefore, combining these estimates, we can obtain

$$
\begin{aligned}
&\|\boldsymbol{A}^{\frac12}\nabla\theta^n\|^2 + 2\Delta t\|c^{\frac12}D_t\theta^n\|^2\\
\le{}& \|\boldsymbol{A}^{\frac12}\nabla\theta^{n-1}\|^2 + K\Delta t(\|c^{\frac12}\theta^n\|^2 + \|\boldsymbol{A}^{\frac12}\nabla\theta^{n-1}\|^2 + \|\boldsymbol{A}^{\frac12}\nabla\theta^n\|^2)\\
&+ K\Delta t(h_u^{2(l+1)} + \|R_1^n\|^2 + \|D_t\boldsymbol{R}_2^n\|^2) + \delta\Delta t\|c^{\frac12}D_t\theta^n\|^2\\
&+ \frac{1}{\Delta t}[(c(\bar\rho^n - \rho^n),\, \theta^n) - (c(\bar\rho^{n-1} - \rho^{n-1}),\, \theta^{n-1})]\\
&- \Delta t\Big[\Big(\boldsymbol{B}\sum_{i=1}^{n}\nabla\theta^i,\, \nabla\theta^n\Big) - \Big(\boldsymbol{B}\sum_{i=1}^{n-1}\nabla\theta^i,\, \nabla\theta^{n-1}\Big)\Big]\\
&+ [(\boldsymbol{R}_2^n,\, \nabla\theta^n) - (\boldsymbol{R}_2^{n-1},\, \nabla\theta^{n-1})].
\end{aligned} \tag{3.19}
$$

Summing (3.19) for n from 1 to J, we have

$$
\begin{aligned}
&\|\boldsymbol{A}^{\frac12}\nabla\theta^J\|^2 + 2\Delta t\sum_{n=1}^{J}\|c^{\frac12}D_t\theta^n\|^2\\
\le{}& K\Delta t\sum_{n=1}^{J}(\|c^{\frac12}\theta^n\|^2 + \|\boldsymbol{A}^{\frac12}\nabla\theta^{n-1}\|^2 + \|\boldsymbol{A}^{\frac12}\nabla\theta^n\|^2)\\
&+ K\Delta t\sum_{n=1}^{J}(h_u^{2(l+1)} + \Delta t^2) + \delta\Delta t\sum_{n=1}^{J}\|c^{\frac12}D_t\theta^n\|^2\\
&+ \frac{1}{\Delta t}(c(\bar\rho^J - \rho^J),\, \theta^J) - \Delta t\Big(\boldsymbol{B}\sum_{i=1}^{J}\nabla\theta^i,\, \nabla\theta^J\Big) + (\boldsymbol{R}_2^n,\, \nabla\theta^J),
\end{aligned} \tag{3.20}
$$

where we have applied the initial approximation $u_h^0 = Ru^0$, $\theta^0 = 0$.

From Lemmas 3.2 and 3.3, we have

$$
\frac{1}{\Delta t}(c(\bar\rho^J - \rho^J),\, \theta^J) \le K\frac{1}{\Delta t}\|\bar\rho^J - \rho^J\|_{-1}\|\theta^J\|_1
$$

$$\begin{aligned}&\leq K\|\rho^J\|\|\theta^J\|_1\\&\leq Kh_u^{2(l+1)}+\delta(\|c^{\frac{1}{2}}\theta^J\|+\|\boldsymbol{A}^{\frac{1}{2}}\nabla\theta^J\|^2),\end{aligned}\tag{3.21}$$

$$\begin{aligned}-\Delta t\Big(\boldsymbol{B}\sum_{i=1}^{J}\nabla\theta^i,\,\nabla\theta^J\Big)&\leq K\Delta t^2\Big\|\sum_{i=1}^{J}\nabla\theta^i\Big\|^2+\delta\|\boldsymbol{A}^{\frac{1}{2}}\nabla\theta^J\|^2\\&\leq K\Delta t\sum_{i=1}^{J}\|\nabla\theta^i\|^2+\delta\|\boldsymbol{A}^{\frac{1}{2}}\nabla\theta^J\|^2,\end{aligned}\tag{3.22}$$

$$(\boldsymbol{R}_2^J,\nabla\theta^J)\leq K\|\boldsymbol{R}_2^J\|^2+\delta\|\boldsymbol{A}^{\frac{1}{2}}\nabla\theta^J\|^2\leq K\Delta t^2+\delta\|\boldsymbol{A}^{\frac{1}{2}}\nabla\theta^J\|^2.\tag{3.23}$$

Applying (3.21)–(3.23), we obtain

$$\begin{aligned}&\|\boldsymbol{A}^{\frac{1}{2}}\nabla\theta^J\|^2+\Delta t\sum_{n=1}^{J}\|c^{\frac{1}{2}}D_t\theta^n\|^2\\&\leq K\Delta t\sum_{n=1}^{J}(\|c^{\frac{1}{2}}\theta^n\|^2+\|\boldsymbol{A}^{\frac{1}{2}}\nabla\theta^{n-1}\|^2+\|\boldsymbol{A}^{\frac{1}{2}}\nabla\theta^n\|^2)\\&\quad+K(h_u^{2(l+1)}+\Delta t^2)+\delta\|c^{\frac{1}{2}}\theta^J\|^2.\end{aligned}\tag{3.24}$$

Applying a known inequality

$$\|c^{\frac{1}{2}}\theta^J\|^2\leq\|c^{\frac{1}{2}}\theta^0\|^2+\delta\sum_{n=1}^{J}\Delta t\|c^{\frac{1}{2}}D_t\theta^n\|^2+K\sum_{n=1}^{J}\Delta t\|c^{\frac{1}{2}}\theta^n\|^2,\tag{3.25}$$

and the discrete Gronwall's Lemma to (3.24), we derive the estimate

$$\|\theta^J\|^2+\|\nabla\theta^J\|^2+\Delta t\sum_{n=1}^{J}\|D_t\theta^n\|^2\leq K(h_u^{2(l+1)}+\Delta t^2),\qquad J\leq N.\tag{3.26}$$

Noticing $u^n-u_h^n=\rho^n+\theta^n$, we are able to demonstrate (3.6). Now we analyze the error estimate for $\boldsymbol{\sigma}$. Letting $v=0$ in (2.10) and using Lemma 2.1, we have

$$\begin{aligned}&(\tilde{\boldsymbol{A}}^{-1}\boldsymbol{\sigma}^n,\,\boldsymbol{\psi})+\Delta t(c^{-1}\mathrm{div}\boldsymbol{\sigma}^n,\,\mathrm{div}\boldsymbol{\psi})\\&=(\bar{u}^{n-1},\,\mathrm{div}\boldsymbol{\psi})+\Delta t(c^{-1}f^n,\,\mathrm{div}\boldsymbol{\psi})+\Delta t(c^{-1}\boldsymbol{R}_1^n,\,\mathrm{div}\boldsymbol{\psi})\\&\quad-(\tilde{\boldsymbol{A}}^{-1}\boldsymbol{B}\sum_{i=1}^{n-1}\Delta t\nabla u^i,\,\boldsymbol{\psi})+(\tilde{\boldsymbol{A}}^{-1}\boldsymbol{R}_2^n,\,\boldsymbol{\psi}),\qquad\forall\,\boldsymbol{\psi}\in H.\end{aligned}\tag{3.27}$$

Let $\boldsymbol{\pi}^n=\mathrm{Q}\boldsymbol{\sigma}^n-\boldsymbol{\sigma}_h^n$, $\boldsymbol{\epsilon}^n=\boldsymbol{\sigma}^n-\mathrm{Q}\boldsymbol{\sigma}^n$. Subtracting (3.27) from (2.13), we see that $\boldsymbol{\pi}^n$ satisfies the following error equation

$$\begin{aligned}&(\tilde{\boldsymbol{A}}^{-1}\boldsymbol{\pi}^n,\,\boldsymbol{\psi}_h)+\Delta t(c^{-1}\mathrm{div}\boldsymbol{\pi}^n,\,\mathrm{div}\boldsymbol{\psi}_h)\\&=-(\tilde{\boldsymbol{A}}^{-1}\boldsymbol{\epsilon}^n,\,\boldsymbol{\psi}_h)-\Delta t(c^{-1}\mathrm{div}\boldsymbol{\epsilon}^n,\,\mathrm{div}\boldsymbol{\psi}_h)+(\bar{u}^{n-1}-\bar{u}_h^{n-1},\,\mathrm{div}\boldsymbol{\psi}_h)\\&\quad+\Delta t(c^{-1}R_1^n,\,\mathrm{div}\boldsymbol{\psi}_h)+(\tilde{\boldsymbol{A}}^{-1}\boldsymbol{R}_2^n,\,\boldsymbol{\psi}_h)-\Delta t(\tilde{\boldsymbol{A}}^{-1}\boldsymbol{B}\sum_{i=1}^{n-1}\nabla(u^i-u_h^i),\,\boldsymbol{\psi}_h).\end{aligned}\tag{3.28}$$

Setting $\boldsymbol{\psi}_h=\boldsymbol{\pi}^n$ and using Cauchy's inequality and Lemma 3.2, we have

$$\begin{aligned}&\|\tilde{\boldsymbol{A}}^{-\frac{1}{2}}\boldsymbol{\pi}^n\|^2+\Delta t\|c^{-\frac{1}{2}}\mathrm{div}\boldsymbol{\pi}^n\|^2\\&\leq K(\|\boldsymbol{\epsilon}^n\|^2+\|\boldsymbol{R}_2^n\|^2)+K\Delta t(\|\mathrm{div}\boldsymbol{\epsilon}^n\|^2+\|R_1^n\|^2)\end{aligned}$$

$$+ K\Delta t^{-1}\|u^{n-1} - u_h^{n-1}\|^2 + K\Delta t \sum_{i=1}^{n-1} \|\nabla(u^i - u_h^i)\|^2$$
$$+ \delta(\|\tilde{\boldsymbol{A}}^{-\frac{1}{2}}\boldsymbol{\pi}^n\|^2 + \Delta t\|c^{-\frac{1}{2}}\mathrm{div}\boldsymbol{\pi}^n\|^2). \tag{3.29}$$

Therefore

$$\|\boldsymbol{\pi}^n\|^2 + \Delta t\|\mathrm{div}\boldsymbol{\pi}^n\|^2 \le K(\Delta t^2 + h_u^{2l} + \Delta t^{-1}h_u^{2(l+1)} + h_\sigma^{2(k+1)} + \Delta t h_\sigma^{2k_1}). \tag{3.30}$$

Hence, by using (3.1) and the triangle inequality, we are able to demonstrate (3.7). The proof of Theorem 3.1 is completed.

From Theorem 3.1, we see that the method leads to the approximate solutions with accuracy optimal in H^1 and L^2 norms for u and sub-optimal in $(L^2(\Omega))^d$ norm for $\boldsymbol{\sigma}$. The method is reasonable for such problem.

4 Numerical Example

In this section, we carry out the numerical example to demonstrate the theoretical results. Consider the model equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + \boldsymbol{d}(x)\cdot\nabla u - \nabla\cdot(a\nabla u + \displaystyle\int_0^t \nabla u(x,s)\mathrm{d}s) = f(x,t), & x\in\Omega,\ 0<t\le T, \\ u(x,t) = 0, & x\in\Gamma,\ 0<t\le T, \\ u(x,0) = \sin\pi x_1 \sin\pi x_2, & x\in\Omega, \end{cases} \tag{4.1}$$

where $\Omega = (0,1]\times(0,1]$, $a = 10^{-3}$. The exact solution is chosen to be $u = \mathrm{e}^t\sin\pi x_1\sin\pi x_2$, and the source term $f(x,t)$ is determined by the above data.

In computing the example, we use the C++ software package: AFEPack; it is available at http://dsec.pku.edu.cn/rli. Let S_{h_u} be piecewise linear spaces and H_{h_σ} be Raviart-Thomas element of the lowest order, that is, $l = 1$, $k = 0$, $k_1 = k+1 = 1$. We adopt the same mesh for T_{h_σ} and T_{h_u}, and Ω is triangulated into triangular elements with N nodal points. For simplicity, we also take $\Delta t = h$ in our test. The results of the scheme are as follows $\left(T = \dfrac{1}{2}\right)$. Table 4.1 lists the numerical results with different mesh sizes. Figs. 4.1–4.3 show the approximate solution at $\Delta t = h = \dfrac{1}{80}$. We see that our numerical results, as anticipated, are accordant with our theoretical prediction.

Table 4.1 The numerical results for Scheme I′

h	$\|u-u_h\|$	Rate	$\|u-u_h\|_1$	Rate	$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|$	Rate
$\frac{1}{10}$	1.05e-001	–	6.01e-001	–	2.08e-001	–
$\frac{1}{20}$	6.01e-002	0.80	3.36e-001	0.84	1.11e-001	0.90
$\frac{1}{40}$	3.28e-002	0.87	1.83e-001	0.88	5.79e-002	0.94
$\frac{1}{80}$	1.73e-002	0.93	9.75e-002	0.91	2.96e-002	0.97

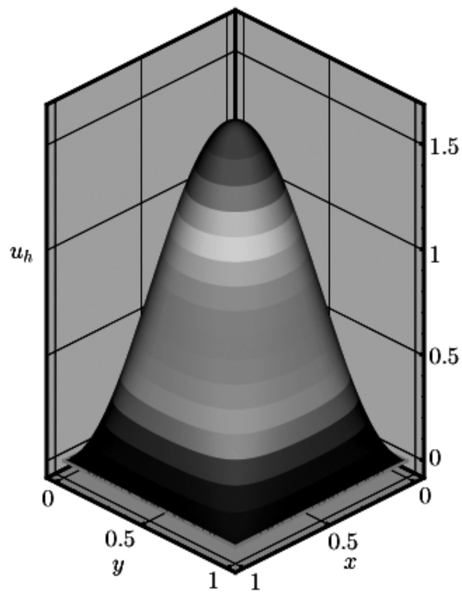

Fig. 4.1 The approximate solution u_h

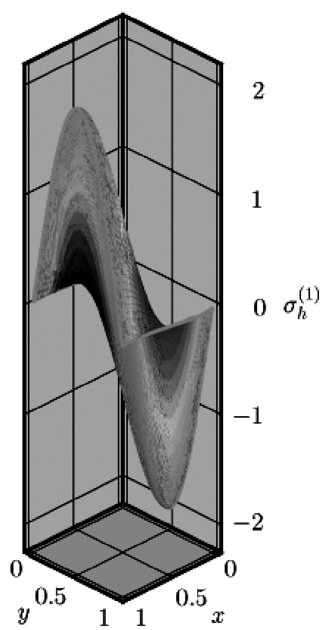

Fig. 4.2 The approximate solution $\boldsymbol{\sigma}_h^{(1)}$

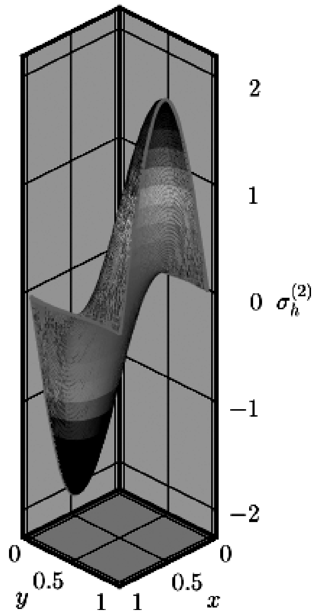

Fig. 4.3 The approximate solution $\boldsymbol{\sigma}_h^{(2)}$

5 Conclusion

We derive a split least-squares characteristic procedure for convection-dominated parabolic integro-differential equations in this paper. The resulting procedure can be split into two independent symmetric positive definite sub-schemes, and they are not subject to the LBB consistency condition. We have proved the method yields optimal estimates in the corresponding norms. Numerical experiments are given to confirm the efficiency of the scheme.

6 Appendix – Proof of Lemma 3.1

We first prove the first inequality in (3.3). The proof of the second one is similar. It is clear that when $n = 0$, $(\boldsymbol{A}\nabla(Ru^0 - u^0), \nabla v_h) = 0$. It is clear that the projection operator Ru^0 is the orthogonal elliptic projection operator, then the following inequality holds (see [15]):

$$\|u^0 - Ru^0\|_s \le K h_u^{l+1-s} \|u^0\|_{l+1}. \tag{6.1}$$

Let $\pi_h u^n \in S_{h_u}$ be the interpolant of u^n. From the definition we have

$$\begin{aligned}
&(\boldsymbol{A}\nabla(Ru^n - \pi_h u^n),\ \nabla v_h) \\
&= (\boldsymbol{A}\nabla(u^n - \pi_h u^n),\ \nabla v_h) + \sum_{i=1}^{n} \Delta t(\boldsymbol{B}\nabla(\pi_h u^i - Ru^i),\ \nabla v_h) \\
&\quad + \sum_{i=1}^{n} \Delta t(\boldsymbol{B}\nabla(u^i - \pi_h u^i),\ \nabla v_h), \qquad v_h \in S_{h_u}.
\end{aligned} \tag{6.2}$$

Letting $v_h = Ru^n - \pi_h u^n$, we have that

$$\begin{aligned}
&\|\boldsymbol{A}^{\frac{1}{2}}\nabla(Ru^n - \pi_h u^n)\|^2 \\
&\le K\|\boldsymbol{A}^{\frac{1}{2}}\nabla(Ru^n - \pi_h u^n)\|[\|\nabla(u^n - \pi_h u^n)\| \\
&\quad + \sum_{i=1}^{n}\Delta t\|\boldsymbol{A}^{\frac{1}{2}}\nabla(Ru^i - \pi_h u^i)\| + \sum_{i=1}^{n}\Delta t\|\nabla(u^i - \pi_h u^i)\|]. \qquad (6.3)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\boldsymbol{A}^{\frac{1}{2}}\nabla(Ru^n - \pi_h u^n)\| \\
&\le K\sum_{i=1}^{n}\Delta t\|\boldsymbol{A}^{\frac{1}{2}}\nabla(Ru^i - \pi_h u^i)\| + Kh_u^l\|u\|_{L^\infty(H^{l+1})}. \qquad (6.4)
\end{aligned}$$

Gronwall's inequality shows that

$$\|\nabla(Ru^n - \pi_h u^n)\| \le Kh_u^l\|u\|_{L^\infty(H^{l+1})}. \qquad (6.5)$$

Combining with the estimate of $\|\pi_h u^n - u^n\|_1 = O(h_u^l)$ completes the proof of the first inequality with $s = 1$.

Now we estimate $\|Ru^n - u^n\|$. For this purpose we define w satisfying

$$\begin{cases} -\nabla\cdot(\boldsymbol{A}\nabla w) = Ru^n - u^n & \text{in } \Omega, \\ w|_\Gamma = 0. \end{cases}$$

By using the H^2-regular assumption we have that

$$\|w\|_2 \le \|Ru^n - u^n\|. \qquad (6.6)$$

Now we consider $\|Ru^n - u^n\|$. Direct calculation shows that

$$\begin{aligned}
&\|Ru^n - u^n\|^2 \\
&= (Ru^n - u^n,\ Ru^n - u^n) \\
&= (\boldsymbol{A}\nabla(Ru^n - u^n),\ \nabla w) \\
&= (\boldsymbol{A}\nabla(Ru^n - u^n),\ \nabla(w - \pi_h w)) + (\boldsymbol{A}\nabla(Ru^n - u^n),\ \nabla\pi_h w) \\
&= (\boldsymbol{A}\nabla(Ru^n - u^n),\ \nabla(w - \pi_h w)) - \sum_{i=1}^{n}\Delta t(\boldsymbol{B}\nabla(Ru^i - u^i),\ \nabla\pi_h w) \\
&= (\boldsymbol{A}\nabla(Ru^n - u^n),\ \nabla(w - \pi_h w)) + \sum_{i=1}^{n}\Delta t(\boldsymbol{B}\nabla(Ru^i - u^i),\ \nabla(w - \pi_h w)) \\
&\quad + \sum_{i=1}^{n}\Delta t(Ru^i - u^i,\ \nabla\cdot(\boldsymbol{B}\nabla w)). \qquad (6.7)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|Ru^n - u^n\|^2 \\
&\le K\|\nabla(R^n u^n - u^n)\|\|\nabla(w - \pi_h w)\| + K\sum_{i=1}^{n}\Delta t\|\nabla(Ru^i - u^i)\|\|\nabla(w - \pi_h w)\| \\
&\quad + K\sum_{i=1}^{n}\Delta t\|Ru^i - u^i\|\|\nabla\cdot(\boldsymbol{B}\nabla w)\| \\
&\le Kh_u^{l+1}\|w\|_2 + K\sum_{i=1}^{n}\Delta t\|Ru^i - u^i\|\|w\|_2. \qquad (6.8)
\end{aligned}$$

Noticing (6.6), we have

$$\|Ru^n - u^n\| \le K h_u^{l+1} \|u\|_{L^\infty(H^{l+1})} + K \sum_{i=1}^{n} \Delta t \|Ru^i - u^i\|. \tag{6.9}$$

By Gronwall's inequality, we have

$$\|Ru^n - u^n\| \le C h_u^{l+1} \|u\|_{L^\infty(H^{l+1})}. \tag{6.10}$$

The proof of the second inequality is similar.

References

[1] Allegretto W, Lin Y, Zhou A. A box scheme for coupled systems resulting from microsensor thermistor problems. *Dyn. Contin. Discrete Impuls. Syst.*, 1999, **5**: 209–223.

[2] Cushman J H, Hu X, Deng F. Nonlocal reactive transfer with physical and chemical heterogeneity: Localization error. *Water Resource Res.,* 1995, **31**: 2219–2237.

[3] Douglas J, Russell T F. Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures. *SIAM J. Numer. Anal.*, 1982, **19**(5): 871–885.

[4] Ewing R E, Russell T F, Wheeler M F. Convergence analysis of an approximation of miscible displacement in porous media by mixed finite elements and a modified method of characteristics. *Comput. Methods Appl. Mech. Engrg.,* 1984, **47**: 73–92.

[5] Guo H. Least-squares Galerkin procedures with the method of characteristics for convection-dominated parabolic integro-differential equations (in Chinese). *Chinese J. Engrg. Math.*, 2004, **21**(6): 979–983.

[6] Ladyzhenskaya O A. The Mathematical Theory of Viscous Incompressible Flows. London: Gordon and Breach, 1969.

[7] Babuška I. The finite element method with lagrangian multipliers. *Numer. Math.*, 1973, **20**: 179–192.

[8] Brezzi F. On the existence uniqueness and approximation of saddle point problems arising from Lagrangian multipliers. *RAIRO Modél. Math. Anal. Numér.*, 1974, **8**: 129–151.

[9] Rui H, Kim S, Kim S D. A remark on least-squares mixed element methods for reaction-diffusion problems. *J. Comput. Appl. Math.,* 2007, **202**: 230–236.

[10] Rui H, Kim S D, Kim S. Split least-squares finite element methods for linear and nonlinear parabolic problems. *J. Comput. Appl. Math.,* 2009, **223**: 938–952.

[11] Duran R E. On the approximation of miscible displacement in porous media by a method of characteristic combined with a mixed method. *SIAM J. Numer. Anal.*, 1088, **25**: 989–1001.

[12] Adams R A. Sobolev Spaces. New York: Academic Press, 1975.

[13] Raviart P A, Thomas J M. A Mixed Finite Element Method for 2nd Order Elliptic Problems, Mathematical Aspects of Finite Element Methods: Lecture Notes Math. vol. 606. Berlin: Springer, 1977: 292–315.

[14] Nedelec J C. Mixed finite element in $\mathbf{R}^3$. *Numer. Math.,* 1980, **35**: 315–341.

[15] Ciarlet P G. Finite Element Methods for Elliptic Problems. New York: North-Holland, 1978.

[16] Wheeler M F. A priori L^2 error estimates for Galerkin approximation to parabolic partial differential equations. *SIAM Numer. Anal.*, 1973, **10**: 723–759.