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Weak Convergence Theorems for Nonself Mappings

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Abstract: Let E be a real uniformly convex and smooth Banach space, and K be a nonempty closed convex subset of E with P as a sunny nonexpansive retraction. Let T_1, $T_2 : K \to E$ be two weakly inward nonself asymptotically nonexpansive mappings with respect to P with a sequence $\{k_n^{(i)}\} \subset [1, \infty)$ $(i = 1, 2)$, and $F := F(T_1) \bigcap F(T_2) \neq \emptyset$. An iterative sequence for approximation common fixed points of the two nonself asymptotically nonexpansive mappings is discussed. If E has also a Fréchet differentiable norm or its dual E^* has Kadec-Klee property, then weak convergence theorems are obtained.

Key words: asymptotically nonexpansive nonself-mapping, weak convergence, uniformly convex Banach space, common fixed point, smooth Banach space

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1 Introduction and Preliminaries

Throughout this work, we assume that E is a real Banach space, E^* is the dual space of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \ \|f\| = \|x\|\}, \qquad x \in E,$$

where $\langle \cdot\, , \,\cdot \rangle$ denotes the duality pairing between E and E^*. A single-valued normalized duality mapping is denoted by j. It is well known that if E is a smooth Banach space, then J is single-valued.

A Banach space E is said to have a Fréchet differentiable norm (see [1]), if for all $x \in U = \{x \in E : \|x\| = 1\}$, the limit $\lim\limits_{t\to 0} \dfrac{\|x + ty\| - \|x\|}{t}$ exists and is attained uniformly in

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$y \in U$. In this case there exists an increasing function $b : [0, \infty) \to [0, \infty)$ with $\lim\limits_{t \to 0^+} \dfrac{b(t)}{t} = 0$ such that

$$\frac{1}{2}\|x\|^2 + \langle h, j(x) \rangle \le \frac{1}{2}\|x+h\|^2 \le \frac{1}{2}\|x\|^2 + \langle h, j(x) \rangle + b(\|h\|), \qquad x, h \in E. \tag{1.1}$$

A subset K of E is said to be retract of E if there exists a continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is retract. A mapping $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. Let C and K be subsets of a Banach space E. A mapping P from C into K is called sunny if $P(Px + t(x - Px)) = Px$ for $x \in C$ with $Px + t(x - Px) \in C$ and $t \ge 0$.

For any $x \in K$, the inward set $I_K(x)$ is defined as follows:

$$I_K(x) = \{y \in E : y = x + \lambda(z - x),\ z \in K,\ \lambda \ge 0\}.$$

A mapping $T : K \to E$ is said to satisfy the inward condition if $Tx \in I_K(x)$ for all $x \in K$. T is said to be weakly inward if $Tx \in \mathrm{cl}I_K(x)$ for each $x \in K$, where $\mathrm{cl}I_K(x)$ is the closure of $I_K(x)$.

A Banach space E is said to have the Kadec-Klee property (see [2]) if for every sequence $\{x_n\}$ in E, with $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, it follows that $x_n \to x$ strongly.

We denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in K : Tx = x\}$, and by $F := F(T_1) \bigcap F(T_2)$ the set of common fixed points of two mappings T_1 and T_2.

Definition 1.1[3] *Let E be a real normed linear space, and K be a nonempty subset of E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that for any $x, y \in K$, $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \le k_n\|x - y\|$, $n \ge 1$. T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$, $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \le L\|x - y\|$, $n \ge 1$.*

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Nonself asymptotically nonexpansive mappings have been studied by many authors (see [3–8]). Chidume *et al.*[3] studied the following iteration scheme:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \qquad n \ge 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in (0, 1), and proved some strong and weak convergence theorems of the iteration scheme (1.2).

Wang[4] studied the following iteration scheme:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}y_n), \\ y_n = P((1 - \beta_n)x_n + \beta_n T(PT)^{n-1}x_n), \qquad n \ge 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0, 1), $T_1, T_2 : K \to E$ are two asymptotically nonexpansive nonself mappings, and proved strong and weak convergence theorems of the iteration scheme (1.3). Guo and Guo[5] completed the weak convergence theorems of the iteration scheme (1.3).

Remark 1.1 If $T : K \to E$ is an asymptotically nonexpansive mapping and $P : E \to K$ is a nonexpansive retraction, then $PT : K \to K$ is asymptotically nonexpansive. Indeed, for all $x, y \in K$ and $n \geq 1$, it follows that

$$\begin{aligned}\|(PT)^n x - (PT)^n y\| &= \|PT(PT)^{n-1}x - PT(PT)^{n-1}y\| \\ &\leq \|T(PT)^{n-1}x - T(PT)^{n-1}y\| \\ &\leq k_n\|x - y\|.\end{aligned}$$

Therefore, Zhou *et al.*[7] introduced the following generalized definition:

Definition 1.2[7] *Let E be a real normed linear space, and K be a nonempty subset of E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is said to be asymptotically nonexpansive with respect to P if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for any $x, y \in K$,*

$$\|(PT)^n x - (PT)^n y\| \leq k_n\|x - y\|, \qquad n \geq 1. \tag{1.4}$$

T is said to be uniformly L-Lipschitzian with respect to P if there exists a constant $L > 0$ such that for all $x, y \in K$,

$$\|(PT)^n x - (PT)^n y\| \leq L\|x - y\|, \qquad n \geq 1.$$

Furthermore, by studying the following iterative scheme:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \alpha_n x_n + \beta_n (PT)^n x_n + \gamma_n (PT)^n x_n, \qquad n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[a, 1-a]$ for some $a \in (0, 1)$, satisfying $\alpha_n + \beta_n + \gamma_n = 1$, Zhou *et al.*[7] obtained some strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings with respect to P in uniformly convex Banach spaces. As a consequence, the main results of Chidume *et al.*[3] can be deduced.

Recently, Turkmen *et al.*[8] generalized the iteration process (1.5) as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)(PT_1)^n y_n + \alpha_n (PT_2)^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n (PT_1)^n x_n, \qquad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0, 1)$, $T_1, T_2 : K \to E$ are two asymptotically nonexpansive nonself mappings, and P is as in Definition 1.2, and obtained the following weak convergence theorem:

Theorem 1.1[8] *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E satisfying Opial's condition with P as a sunny nonexpansive retraction. Let T_1, $T_2 : K \to E$ be two weakly inward and nonself asymptotically nonexpansive mappings with respect to P with a sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by* (1.6)*, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[a, 1 - a]$ for some $a \in (0, 1)$. If $F \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.*

Only Theorem 1.1 has been obtained from the weak convergence problem for the sequence defined by (1.6). The purpose of this paper is to prove some new weak convergence theorems of the iteration scheme (1.6) for two asymptotically nonexpansive nonself-mappings in uniformly convex and smooth Banach spaces.

2 Some Lemmas

Let T_1, $T_2 : K \to E$ be two nonself asymptotically nonexpansive mappings with respect to P with sequences $\{k_n^{(i)}\} \subset [1, \infty)$ satisfying $\sum\limits_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty$, $i = 1, 2$, respectively. Put $k_n = \max\{k_n^{(1)}, k_n^{(2)}\}$. Then obviously $\sum\limits_{n=1}^{\infty} (k_n - 1) < \infty$. From now on, we take this sequence $\{k_n\}$ for both T_1 and T_2.

In order to prove the main results, we need the following lemmas:

Lemma 2.1[8] *Let E be a real normed linear space, K be a nonempty closed convex subset of E, and T_1, $T_2 : K \to E$ be two asymptotically nonexpansive mappings with respect to P with a sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum\limits_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by* (1.6) *and $F \neq \emptyset$. Then $\lim\limits_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.*

Lemma 2.2[8] *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E, and $T_1, T_2 : K \to E$ be two asymptotically nonexpansive mappings with respect to P with a sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum\limits_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by* (1.6), *where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[a, 1-a]$ for some $a \in (0, 1)$. If $F \neq \emptyset$, then*

$$\lim_{n\to\infty} \|x_n - (PT_1)x_n\| = \lim_{n\to\infty} \|x_n - (PT_2)x_n\| = 0.$$

Lemma 2.3[9] *Let X be a uniformly convex Banach space and C be a convex subset of X. Then there exists a strictly increasing continuous convex function $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ such that for each $S : C \to C$ with Lipschitz constant L,*

$$\|\alpha Sx+(1-\alpha)Sy-S[\alpha x+(1-\alpha)y]\| \leq L\gamma^{-1}\Big(\|x-y\|-\frac{1}{L}\|Sx-Sy\|\Big), \qquad x, y \in C,\ 0 < \alpha < 1.$$

Lemma 2.4[9] *Let X be a uniformly convex Banach space such that its dual X^* has the Kadec-Klee property. If $\{x_n\}$ is a bounded sequence and $f_1, f_2 \in W_w(\{x_n\})$, where $W_w(\{x_n\})$ denotes the set of all weak subsequential limits of a bounded sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} \|\alpha x_n + (1-\alpha)f_1 - f_2\|$ exists for all $\alpha \in [0, 1]$, then $f_1 = f_2$.*

Lemma 2.5[10] *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E, and $T : K \to K$ be an asymptotically nonexpansive mapping with $F \neq \emptyset$. Then $I - T$ is demiclosed at zero, i.e., for each sequence $\{x_n\}$ in K, if $\{x_n\}$ converges weakly to $q \in K$ and $\{(I - T)x_n\}$ converges strongly to* 0, *then $(I - T)q = 0$.*

Lemma 2.6[8] *Let E be a real smooth Banach space, K be a nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and $T : K \to E$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.*

3 Main Results

In this section, we prove weak convergence theorems for the iterative scheme (1.6) for two asymptotically nonexpansive nonself-mappings in uniformly convex and smooth Banach spaces.

Lemma 3.1 *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space, and T_1, $T_2 : K \to E$ be two nonself asymptotically nonexpansive mappings with respect to P with a sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F \neq \emptyset$. Let $\{x_n\}$ be defined by (1.6), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1)$. Then for all $q_1, q_2 \in F$, the limit $\lim_{n\to\infty} \|tx_n + (1-t)q_1 - q_2\|$ exists for all $t \in [0, 1]$.*

Proof. Set $a_n(t) = \|tx_n + (1-t)q_1 - q_2\|$. Then $\lim_{n\to\infty} a_n(0) = \|q_1 - q_2\|$, and from Lemma 2.1, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - q_2\|$ exists. It remains to prove that the Lemma 3.1 holds for all $t \in (0, 1)$.

Define the mapping $H_n : K \to K$ by

$$H_nx = (1-\alpha_n)(PT_1)^n[(1-\beta_n)x+\beta_n(PT_1)^nx]+\alpha_n(PT_2)^n[(1-\beta_n)x+\beta_n(PT_1)^nx], \quad x \in K.$$

Then, for all $x, y \in K$, by (1.4) we have

$$\begin{aligned}\|H_nx - H_ny\| &\leq (1-\alpha_n)k_n\|(1-\beta_n)(x-y) + \beta_n[(PT_1)^nx - (PT_1)^ny]\| \\ &\quad + \alpha_nk_n\|(1-\beta_n)(x-y) + \beta_n[(PT_1)^nx - (PT_1)^ny]\| \\ &\leq k_n(1-\beta_n)\|x-y\| + k_n^2\beta_n\|x-y\| \\ &\leq k_n^2\|x-y\|. \end{aligned} \tag{3.1}$$

Set

$$R_{n,m} = H_{n+m-1}H_{n+m-2}\cdots H_n, \qquad m \geq 1. \tag{3.2}$$

From (3.1) and (3.2), we can obtain that

$$\|R_{n,m}x - R_{n,m}y\| \leq \Big(\prod_{j=n}^{n+m-1} k_j^2\Big)\|x-y\|, \qquad x, y \in K, \tag{3.3}$$

and $R_{n,m}x_n = x_{n+m}$, $R_{n,m}q = q$ for each $q \in F$. Let

$$b_{n,m} = \|tR_{n,m}x_n + (1-t)R_{n,m}q_1 - R_{n,m}(tx_n + (1-t)q_1)\|. \tag{3.4}$$

Using (3.3), (3.4) and Lemma 2.3, we have

$$\begin{aligned} b_{n,m} &\leq \Big(\prod_{j=n}^{n+m-1} k_j^2\Big)\gamma^{-1}\Big(\|x_n - q_1\| - \Big(\prod_{j=n}^{n+m-1} k_j^2\Big)^{-1}\|R_{n,m}x_n - R_{n,m}q_1\|\Big) \\ &\leq \Big(\prod_{j=n}^{\infty} k_j^2\Big)\gamma^{-1}\Big(\|x_n - q_1\| - \Big(\prod_{j=n}^{\infty} k_j^2\Big)^{-1}\|R_{n,m}x_n - R_{n,m}q_1\|\Big). \end{aligned} \tag{3.5}$$

It follows from Lemma 2.1, (3.3), (3.5) and $\lim_{n\to\infty} \prod_{j=n}^{\infty} k_j^2 = 1$ that $\lim_{n\to\infty} b_{n,m} = 0$ uniformly for all m. Observe that

$$\begin{aligned} a_{n+m}(t) &\leq \|tx_{n+m} + (1-t)q_1 - q_2 + R_{n,m}(tx_n + (1-t)q_1) - tR_{n,m}x_n - (1-t)R_{n,m}q_1\| \\ &\quad + \| - R_{n,m}(tx_n + (1-t)q_1) + tR_{n,m}x_n + (1-t)R_{n,m}q_1\| \end{aligned}$$

$$
\begin{aligned}
&= \|R_{n,m}(tx_n + (1-t)q_1) - q_2\| + b_{n,m} \\
&= \|R_{n,m}(tx_n + (1-t)q_1) - R_{n,m}q_2\| + b_{n,m} \\
&\leq \Big(\prod_{j=n}^{n+m-1} k_j^2\Big)\|tx_n + (1-t)q_1 - q_2\| + b_{n,m}.
\end{aligned}
\tag{3.6}
$$

By $\lim\limits_{n\to\infty}\prod\limits_{j=n}^{\infty} k_j^2 = 1$, $\lim\limits_{n\to\infty} b_{n,m} = 0$ and (3.6), we have

$$\limsup_{n\to\infty} a_n \leq \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} a_n(t).$$

That is, $\lim\limits_{n\to\infty}\|tx_n + (1-t)q_1 - q_2\|$ exists for all $t \in (0,1)$. This completes the proof.

Lemma 3.2 *Let E be a uniformly convex Banach space which has a Fréchet differentiable norm, K be a nonempty closed convex subset of E, and T_1, $T_2 : K \to E$ be two nonself asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1,\infty)$ such that $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$ and $F \neq \emptyset$. Let $\{x_n\}$ be defined by (1.6), where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1)$. Then for all $q_1, q_2 \in F$, the limit $\lim\limits_{n\to\infty}\langle x_n,\ j(p-q)\rangle$ exists. Furthermore, if $W_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$, then $\langle x^* - y^*,\ j(q_1 - q_2)\rangle = 0$ for all q_1, $q_2 \in F$ and x^*, $y^* \in W_w(x_n)$.*

Proof. This follows basically as in the proof of Lemma 4 in [1]. For completeness, we sketch the details. Set $x = q_1 - q_2$ and $h = t(x_n - q_1)$, $0 \leq t \leq 1$ in (1.1). Since b is increasing, and $\|x_n - q_1\| \leq M$ for some $M > 0$ and all $n \geq 1$, by Lemma 2.1, we have

$$
\begin{aligned}
&\frac{1}{2}\|q_1 - q_2\|^2 + t\langle x_n - q_1,\ j(q_1 - q_2)\rangle \\
\leq\ &\frac{1}{2}\|tx_n + (1-t)q_1 - q_2\|^2 \\
\leq\ &\frac{1}{2}\|q_1 - q_2\|^2 + t\langle x_n - q_1,\ j(q_1 - q_2)\rangle + b(tM).
\end{aligned}
\tag{3.7}
$$

It follows from (3.7) and Lemma 3.1 that

$$
\begin{aligned}
&\frac{1}{2}\|q_1 - q_2\|^2 + t\limsup_{n\to\infty}\langle x_n - q_1,\ j(q_1 - q_2)\rangle \\
\leq\ &\frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)q_1 - q_2\|^2 \\
\leq\ &\frac{1}{2}\|q_1 - q_2\|^2 + t\liminf_{n\to\infty}\langle x_n - q_1,\ j(q_1 - q_2)\rangle + b(tM).
\end{aligned}
\tag{3.8}
$$

So by (3.8) we have

$$\limsup_{n\to\infty}\langle x_n - q_1,\ j(q_1 - q_2)\rangle \leq \liminf_{n\to\infty}\langle x_n - q_1,\ j(q_1 - q_2)\rangle + \frac{b(tM)}{t}, \qquad 0 < t \leq 1. \tag{3.9}$$

From (3.9) and

$$\lim_{t\to 0^+}\frac{b(tM)}{t} = \lim_{t\to 0^+}\frac{Mb(tM)}{tM} = 0,$$

we know that the limit $\lim\limits_{n\to\infty}\langle x_n - q_1,\ j(q_1 - q_2)\rangle$ exists. Furthermore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim\limits_{k\to\infty}\langle x_{n_k} - q_1,\ j(q_1 - q_2)\rangle = \langle x^* - q_1,\ j(q_1 - q_2)\rangle$, and so

$$\lim_{n\to\infty}\langle x_n - q_1,\ j(q_1 - q_2)\rangle = \langle x^* - q_1,\ j(q_1 - q_2)\rangle, \qquad x^* \in W_w(x_n).$$

This shows that $\langle x^* - y^*,\ j(q_1 - q_2)\rangle = 0$ for all $q_1, q_2 \in F$ and x^*, $y^* \in W_w(x_n)$. This completes the proof.

Theorem 3.1 *Let E be a real uniformly convex and smooth Banach space which has a Fréchet differentiable norm, K be a nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and T_1, $T_2 : K \to E$ be two weakly inward nonself asymptotically nonexpansive mappings with respect to P with a sequence $\{k_n\} \subset [1, \infty)$ such that $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$ and $F \neq \emptyset$. Let $\{x_n\}$ be defined by (1.6), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[a, 1-a]$ for some $a > 0$. Then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.*

Proof. By Lemma 2.1, $\{x_n\}$ is bounded. Since E is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \in K$. By Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_{n_k} - (PT_1)x_{n_k}\| = \lim_{n\to\infty} \|x_{n_k} - (PT_2)x_{n_k}\| = 0.$$

It follows from Remark 1.1 and Lemma 2.5 that $q \in F(PT_1) \bigcap F(PT_2)$, where $F(PT_i)$ is the set of fixed points of the asymptotically nonexpansive mapping PT_i, $i = 1, 2$. By Lemma 2.6, we know that $F(PT_i) = F(T_i)$. So, we have $q \in F$.

Now, we prove that $\{x_n\}$ converges weakly to q. Suppose that there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some $q_1 \in K$. Then by the same method as given above, we can also prove that $q_1 \in F$. So $q, q_1 \in F \bigcap W_w(x_n)$. It follows from Lemma 3.2 that

$$\|q - q_1\|^2 = \langle q - q_1,\ j(q - q_1)\rangle = 0.$$

Therefore, $q = q_1$ and so $\{x_n\}$ converges weakly to q. This completes the proof.

Theorem 3.2 *Let E be a real uniformly convex and smooth Banach space E such that its dual E^* has the Kadec-Klee property, and K be a nonempty closed convex subset of E with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward nonself asymptotically nonexpansive mappings with respect to P with a sequence $\{k_n\} \subset [1, \infty)$ such that $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$ and $F \neq \emptyset$. Let $\{x_n\}$ be defined by (1.6), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[a, 1-a]$ for some $a > 0$. Then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.*

Proof. Using the same method as given in Theorem 3.1, we can prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \in F$.

Now, we prove that $\{x_n\}$ converges weakly to q. Suppose that there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $p \in K$. Then, for $q, p \in F$, it follows from Lemma 3.1 that the limit $\lim\limits_{n\to\infty} \|tx_n + (1-t)q - p\|$ exists for all $t \in [0, 1]$. Again, since $q, p \in W_w(x_n)$, and so $p = q$ by Lemma 2.4, we know that $\{x_n\}$ converges weakly to a common fixed point q. This completes the proof.

Remark 3.1 (1) It is well known (see, for example, [11]) that some Banach spaces, such as L_p space with $p \neq 2$, do not satisfy Opial's condition. Also, it is well known that every Banach space, which is both uniformly convex and uniformly smooth, has a Fréchet differentiable norm. In particular, L_p space, $1 < p < \infty$, has a Fréchet differentiable norm. That shows that a Banach space which has a Fréchet differentiable norm is different from the Banach space satisfying Opial's condition.

(2) It is also needed to point out that even if a Banach space neither has a Fréchet differentiable norm nor satisfies Opial's condition, its dual still may have the Kadec-Klee property. For example, see [9], [12]. Take $X_1 = \mathbf{R}^2$ with the norm defined by $|x| = \sqrt{\|x\|_2^2 + \|x\|_1^2}$ and $X_2 = L^p[0,1]$ with $1 < p < \infty$ and $p \neq 2$. The Cartesian product of X_1 and X_2 furnished with l^2-norm is uniformly convex, it does not satisfy Opial's condition, and its norm is not Fréchet differentiable, but its dual does have the Kadec-Klee property.

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