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Existence of Solutions for a Four-point Boundary Value Problem with a $p(t)$-Laplacian

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Abstract: This paper deals with the existence of solutions to the $p(t)$-Laplacian equation with four-point boundary conditions. It is shown, by Leray-Schauder fixed point theorem and degree method, that under suitable conditions, solutions of the problem exist. The interesting point is that $p(t)$ is a general function.

Key words: $p(t)$-Laplacian, four-point boundary condition, fixed point theorem, Leray-Schauder degree method

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1 Introduction

In this paper, we investigate the existence of solutions to the following $p(t)$-Laplacian ordinary differential equations with four-point boundary conditions:

$$\begin{cases} (|u'(t)|^{p(t)-2}u'(t))' + a(t)f(t,\ u(t),\ u'(t)) = 0, & t \in (0,1), \\ u(0) - \alpha u'(\xi) = 0, \quad u(1) + \beta u'(\eta) = 0, \end{cases} \tag{1.1}$$

where the functions $f,\ p,\ a$, and the constants $\alpha,\ \beta,\ \xi,\ \eta$ satisfy:

(H1) $f \in C([0,1] \times \mathbf{R} \times \mathbf{R},\ \mathbf{R})$, $p \in C([0,1],\ \mathbf{R})$ and $p(t) > 1$, $a \in C((0,1),\ \mathbf{R})$ is probably singular at $t = 0$ or $t = 1$ and satisfies $0 < \int_0^1 |a(t)|\mathrm{d}t < +\infty$.

(H2) α, $\beta > 0$, and $0 < \xi < \eta < 1$.

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In the recent years, differential equations and variational problems with variable exponent have been studied extensively, for which the readers may refer to [1–6]. Such problems arise in the study of image processing, electrorheological fluids dynamics and elastic mechanics (see [7–10]).

In the case when p is a constant, the problem (1.1) becomes the classical p-Laplacian problem. Lian and Ge[11] discussed the following problem:

$$\begin{cases} (|u'(t)|^{p-2}u'(t))' + f(t,\ u(t)) = 0, & 0 < t < 1, \\ u(0) - \alpha u'(\xi) = 0, \quad u(1) + \beta u'(\eta) = 0, \end{cases}$$

and obtained the existence of multiple positive solutions. For more information about the existence of solutions for ordinary differential equations with p-Laplacian operator, the interested readers may refer to [12–17] and references therein.

Motivated by the results of the above papers, we study the existence of solutions to the problem (1.1). The main features of this paper are as follows. Firstly, $p(t)$ is a general function, which is more complicated than the case when p is constant. Secondly, the nonlinear term f may change sign and $a(t)$ is allowed to be singular at $t = 0$ or $t = 1$, which differ from those p-Laplacian problems.

The outline of this paper is as follows. In Section 2, we give some necessary preliminaries and important lemmas. Sections 3 is devoted to the proof of the existence of solutions to the problem (1.1).

2 Preliminaries

In this section, we give some preliminaries and lemmas.

Define $U = C^1[0, 1]$. It is well known that U is a Banach space with the norm $\|\cdot\|_1$ defined by

$$\|u\|_1 = \|u\| + \|u'\|,$$

where

$$\|u\| = \max_{t\in[0,1]} |u(t)|, \qquad \|u'\| = \max_{t\in[0,1]} |u'(t)|.$$

Set

$$p^- = \min_{t\in[0,1]} p(t), \qquad p^+ = \max_{t\in[0,1]} p(t).$$

Denote

$$\varphi(r, x) = |x|^{p(r)-2}x \qquad \text{for any fixed } r \in [0, 1],\ x \in \mathbf{R},$$

and denote $\varphi^{-1}(r,\ \cdot\ )$ as

$$\varphi^{-1}(r, x) = |x|^{\frac{2-p(r)}{p(r)-1}}x \qquad \text{for any fixed } r \in [0, 1],\ x \in \mathbf{R}\backslash\{0\},$$

where $\varphi^{-1}(r, 0) = 0$.

Evidently, $\varphi^{-1}(r,\ \cdot\ )$ is continuous and sends a bounded sets into a bounded sets.

To obtain the existence of solutions of the problem (1.1), we need the following lemmas. The proofs are standard, and we omit the details.

Lemma 2.1 *Let U be a Banach space. Suppose that the operator $T(u,\lambda): U\times[0,1]\to U$ is a map satisfying the following conditions*:

(S1) *T is a compact map*;

(S2) *$T(u,0)=0$ for any $u\in U$*;

(S3) *If one has $u=T(u,\lambda)$ for some $\lambda\in[0,1]$, then there exists an $M>0$ such that $\|u\|_U\le M$ for any $u\in U$.*

Then, $T(u,1)$ has a fixed point in U.

Lemma 2.2 *Assume that $g\in L^1[0,1]$ and $g(t)\not\equiv 0$ on any subinterval of $[0,1]$. Then the boundary value problem*

$$\begin{cases} (\varphi(t,u'))'+g(t)=0, & 0<t<1,\\ u(0)-\alpha u'(\xi)=0, & u(1)+\beta u'(\eta)=0\end{cases}\tag{2.1}$$

has a unique solution $u(t)$ which is

$$u(t)=\alpha\varphi^{-1}\Big(\rho-\int_0^\xi g(s)\mathrm{d}s\Big)+\int_0^t\varphi^{-1}\Big(\rho-\int_0^s g(r)\mathrm{d}r\Big)\mathrm{d}s$$

or

$$u(t)=-\beta\varphi^{-1}\Big(\rho-\int_0^\eta g(s)\mathrm{d}s\Big)-\int_t^1\varphi^{-1}\Big(\rho-\int_0^s g(r)\mathrm{d}r\Big)\mathrm{d}s,$$

where $\rho=\varphi(0,\ u'(0))$ and ρ is dependent of g.

Now, for any $h\in C[0,1]$, we define

$$\Lambda_h(\rho)=\alpha\varphi^{-1}\Big(\rho-\int_0^\xi h(s)\mathrm{d}s\Big)+\beta\varphi^{-1}\Big(\rho-\int_0^\eta h(s)\mathrm{d}s\Big)+\int_0^1\varphi^{-1}\Big(\rho-\int_0^s h(r)\mathrm{d}r\Big)\mathrm{d}s.$$

The properties of the operator Λ_h is stated in the following lemma.

Lemma 2.3 *For any $h\in C[0,1]$, the equation*

$$\Lambda_h(\rho)=0$$

has a unique solution $\bar\rho(h)\in\mathbf{R}$.

Proof. The proof is similar to the proof of Lemma 2.1 in [5], and we omit the details here.

Using Lemma 2.2 and Lemma 2.3, we also have the following lemma.

Lemma 2.4 *If u is the solution of the problem* (2.1), *then it can also be rewritten in the form as*

$$u(t)=\begin{cases}\alpha\varphi^{-1}\Big(\int_\xi^\sigma g(s)\mathrm{d}s\Big)+\int_0^t\varphi^{-1}\Big(\int_s^\sigma g(r)\mathrm{d}r\Big)\mathrm{d}s, & 0\le t\le\sigma;\\ \beta\varphi^{-1}\Big(\int_\sigma^\eta g(s)\mathrm{d}s\Big)+\int_t^1\varphi^{-1}\Big(\int_\sigma^s g(r)\mathrm{d}r\Big)\mathrm{d}s, & \sigma\le t\le 1,\end{cases}\tag{2.2}$$

where $\sigma\in(0,1)$.

Proof. Assume that $u(t)$ is the solution of the problem (2.1). Then there exists a $\sigma\in(0,1)$ such that $u'(\sigma)=0$. On the contrary, suppose that $u'(t)<0$ for any $t\in(0,1)$. We know that $u(t)$ is nonincreasing. From the boundary value conditions it follows that

$$u(0)=\alpha u'(\xi)<0,$$

but

$$u(1) = -\beta u'(\beta) > 0,$$

which is a contradiction. Similarly, if $u'(t) > 0$ for any $t \in (0, 1)$, we find that $u(t)$ is nondecreasing, which together with boundary conditions yields a contradiction. By direct computations, we see that (2.2) holds.

3 Existence of Solutions

In this section, we show the existence of solutions to the problem (1.1).

Theorem 3.1 *Assume that* (H1), (H2) *hold, and* f *satisfies*

$$\lim_{|u|+|v|\to\infty} \frac{f(t,u,v)}{(|u|+|v|)^{q(t)-1}} = 0, \qquad 1 < q^- \le q^+ < p^-.$$

Then the problem (1.1) *has at least one solution.*

Proof. In order to obtain the existence of solutions of the problem (1.1), we consider the boundary value problem

$$\begin{cases} (|u'(t)|^{p(t)-2}u'(t))' + \lambda a(t) f(t,\ u(t),\ u'(t)) = 0, & t \in (0,1),\ \lambda \in [0,1], \\ u(0) - \alpha u'(\xi) = 0, \quad u(1) + \beta u'(\eta) = 0, \end{cases}$$

and define the integral operator $T:\ U \times [0, 1] \to U$ by

$$T(u,\lambda)(t) = \begin{cases} \alpha\varphi^{-1}\left(\displaystyle\int_\xi^\sigma \lambda a(s) f(s,\ u(s),\ u'(s))\mathrm{d}s\right) \\ \qquad + \displaystyle\int_0^t \varphi^{-1}\Big(\int_s^\sigma \lambda a(r) f(r,\ u(r),\ u'(r))\mathrm{d}r\Big)\mathrm{d}s, & 0 \le t \le \sigma; \\ \beta\varphi^{-1}\left(\displaystyle\int_\sigma^\eta \lambda a(s) f(s,\ u(s),\ u'(s))\mathrm{d}s\right) \\ \qquad + \displaystyle\int_t^1 \varphi^{-1}\left(\int_\sigma^s \lambda a(r) f(r,\ u(r),\ u'(r))\mathrm{d}r\right)\mathrm{d}s, & \sigma \le t \le 1, \end{cases}$$

where $\sigma \in (0, 1)$. From the continuity of f, φ^{-1} and the definition of a, it is easy to see that u is a fixed point of the integral operator T if and only if u is a solution of the problem (1.1) when $\lambda = 1$. In order to apply Lemma 2.1, the proof is divided into three steps.

(1) T is compact.

Let $D \subset U \times [0, 1]$ be an arbitrary bounded subset. Then there exists an M such that

$$\|u\|_1 \le M.$$

Let $\{(u_n, \lambda_n)\}$ be a sequence in D. First, we prove that $\{T(u_n,\ \lambda_n)\}$ has a convergent subsequence in $C[0, 1]$. By the conditions (H1), we know that there exists an $N > 1$ such that

$$|f(t,\ u_n(t),\ u_n'(t))| \le N, \qquad t \in [0, 1],\ \|u_n\|_1 \le M.$$

Then for any $(u_n, \lambda_n) \in D$, if $0 \le t \le \sigma$, one has

$$\begin{aligned}|T(u_n,\lambda_n)(t)| &= \Big|\alpha\varphi^{-1}\Big(\int_\xi^\sigma \lambda_n a(s)f(s,\ u_n(s),\ u_n'(s))\mathrm{d}s\Big)\\ &\quad+\int_0^t \varphi^{-1}\Big(\int_s^\sigma \lambda_n a(r)f(r,\ u_n(r),\ u_n'(r))\mathrm{d}r\Big)\mathrm{d}s\Big|\\ &\le \alpha\varphi^{-1}\Big(\int_\xi^\sigma \lambda_n|a(s)f(s,u_n(s),u_n'(s))|\mathrm{d}s\Big)\\ &\quad+\int_0^1 \varphi^{-1}\Big(\int_s^\sigma \lambda_n|a(r)f(r,\ u_n(r),\ u_n'(r))|\mathrm{d}r\Big)\mathrm{d}s\\ &\le (\alpha+1)N^{\frac{1}{p^--1}}\max\Big\{\Big(\int_0^1|a(s)|\mathrm{d}s\Big)^{\frac{1}{p^--1}},\ \Big(\int_0^1|a(s)|\mathrm{d}s\Big)^{\frac{1}{p^+-1}}\Big\}.\end{aligned}$$

Similarly, if $\sigma\le t\le 1$, we conclude that

$$|T(u_n,\ \lambda_n)(t)|\le(\beta+1)N^{\frac{1}{p^--1}}\max\Big\{\Big(\int_0^1|a(s)|\mathrm{d}s\Big)^{\frac{1}{p^--1}},\ \Big(\int_0^1|a(s)|\mathrm{d}s\Big)^{\frac{1}{p^+-1}}\Big\}.$$

On the other hand,

$$\begin{aligned}|T'(u_n,\ \lambda_n)(t)| &= \Big|\varphi^{-1}\Big(\int_\sigma^t \lambda_n a(s)f(s,\ u_n(s),\ u_n'(s))\mathrm{d}s\Big)\Big|\\ &\le N^{\frac{1}{p^--1}}\max\Big\{\Big(\int_0^1|a(s)|\mathrm{d}s\Big)^{\frac{1}{p^--1}},\ \Big(\int_0^1|a(s)|\mathrm{d}s\Big)^{\frac{1}{p^+-1}}\Big\}.\end{aligned}$$

So

$$\begin{aligned}&\|T(u_n,\ \lambda_n)(t)\|\\ &\le\ \max\{(\alpha+1),\ (\beta+1)\}N^{\frac{1}{p^--1}}\max\Big\{\Big(\int_0^1|a(s)|\mathrm{d}s\Big)^{\frac{1}{p^--1}},\ \Big(\int_0^1|a(s)|\mathrm{d}s\Big)^{\frac{1}{p^+-1}}\Big\},\\ &\|T'(u_n,\ \lambda_n)(t)\|\le N^{\frac{1}{p^--1}}\max\Big\{\Big(\int_0^1|a(s)|\mathrm{d}s\Big)^{\frac{1}{p^--1}},\ \Big(\int_0^1|a(s)|\mathrm{d}s\Big)^{\frac{1}{p^+-1}}\Big\}.\end{aligned}$$

Meanwhile, we find that

$$\begin{aligned}&|T(u_n,\ \lambda_n)(t_2)-T(u_n,\ \lambda_n)(t_1)|\\ &=\Big|\int_{t_1}^{t_2}T'(u_n,\ \lambda_n)(t)\mathrm{d}t\Big|\\ &\le N^{\frac{1}{p^--1}}\max\Big\{\Big(\int_0^1|a(s)|\mathrm{d}s\Big)^{\frac{1}{p^--1}},\ \Big(\int_0^1|a(s)|\mathrm{d}s\Big)^{\frac{1}{p^+-1}}\Big\}|t_1-t_2|\end{aligned}$$

for any $0\le t_1\le t_2\le 1$. Hence, $\{T(u_n,\ \lambda_n)\}$ is uniformly bounded and equi-continuous. Applying the Ascoli-Arzelà theorem, there exists a convergent subsequence of $\{T(u_n,\ \lambda_n)\}$ in $C[0,1]$, and without loss of generality, we denote again by $\{T(u_n,\ \lambda_n)\}$.

Next, we show that $\{T'(u_n,\ \lambda_n)\}$ also has a convergent subsequence in $C[0,1]$.

Denote

$$F_n(t)=\int_\sigma^t \lambda_n a(s)f(s,\ u_n(s),\ u_n'(s))\mathrm{d}s.$$

Similarly to the proof above, we know that $\{F_n(t)\}$ has a convergent subsequence in $C[0,1]$, which we also denote by $\{F_n(t)\}$. According to the continuity of φ^{-1}, we see that

$\{T'(u_n,\ \lambda_n)\}$ is convergent in $C[0,1]$. Consequently, the condition (S1) in Lemma 2.1 is satisfied.

(2) Obviously, $T(u,0)=0$ for $u\in U$, and so the condition (S2) in Lemma 2.1 holds.

(3) We verify the condition (S3) in Lemma 2.1.

If it were false, we would find that there exists a subsequence $\{(u_n,\ \lambda_n)\}$ such that $\|u_n\|_1\to\infty$ as $n\to\infty$ and $\|u_n\|_1>1$. Then by Lemma 2.4 we have

$$\begin{aligned}|u_n'(t)|^{p(t)-2}u_n'(t) &= \int_\sigma^t (|u_n'(s)|^{p(s)-2}u_n'(s))'\mathrm{d}s\\ &= -\lambda_n\int_\sigma^t a(s)f(s,\ u_n(s),\ u_n'(s))\mathrm{d}s.\end{aligned}$$

Noting that

$$\lim_{|u|+|u'|\to\infty}\frac{f(t,\ u,\ u')}{(|u|+|u'|)^{q(t)-1}}=0,$$

we get that there exist $M_1>0$ and $c_1>0$ such that

$$|f(t,\ u,\ u')|\le c_1(|u|+|u'|)^{q(t)-1},\qquad t\in[0,1],\ |u|+|u'|\in[M_1,+\infty).$$

Thus, for $|u_n|+|u_n'|\ge M_1$ and $t\in[0,1]$, we have

$$\begin{aligned}||u_n'(t)|^{p(t)-2}u_n'(t)| &\le \lambda_n\int_\sigma^t |a(s)f(s,\ u_n(s),\ u_n'(s))|\mathrm{d}s\\ &\le c_1\int_0^1 |a(s)|(|u_n(s)|+|u_n'(s)|)^{q(s)-1}\mathrm{d}s\\ &\le c_1\|u_n\|_1^{q^+-1}\int_0^1|a(s)|\mathrm{d}s.\end{aligned}$$

So

$$|u_n'(t)|\le C\|u_n\|_1^{\frac{q^+-1}{p^--1}}\max\left\{\left(\int_0^1|a(s)|\mathrm{d}s\right)^{\frac{1}{p^--1}},\ \left(\int_0^1|a(s)|\mathrm{d}s\right)^{\frac{1}{p^+-1}}\right\},$$

and

$$|u_n(t)|=\left|\int_\sigma^t u_n'(s)\mathrm{d}s\right|\le C\|u_n\|_1^{\frac{q^+-1}{p^--1}}\max\left\{\left(\int_0^1|a(s)|\mathrm{d}s\right)^{\frac{1}{p^--1}},\ \left(\int_0^1|a(s)|\mathrm{d}s\right)^{\frac{1}{p^+-1}}\right\},$$

where C is a constant.

Consequently, we can conclude that $\{(u_n,\ \lambda_n)\}$ is bounded, which is a contradiction. Then, the condition (S3) in Lemma 2.1 holds.

Applying Lemma 2.1, we obtain that $T(u,1)$ has a fixed point in U. Therefore, the problem (1.1) has at least one solution. This completes the proof.

Now, we prove the existence of solutions to the problem (1.1) when f satisfies some other conditions.

Theorem 3.2 *Assume that $\Omega_r=\{u\in C^1[0,1],\ \|u\|_1<r\}$ is a bounded open set in U and* (H1), (H2) *hold. Then the problem* (1.1) *has at least one solution $u(t)$ if there exists an $r>0$ such that*

$$|f(t,\ u,\ u')|\le\min\left\{\left(\frac{r}{3}\right)^{p^--1},\ \left(\frac{r}{3}\right)^{p^+-1}\right\}\frac{1}{\displaystyle\int_0^1|a(t)|\mathrm{d}t},$$

where $u\in\bar{\Omega}_r$, $t\in[0,1]$.

Proof. Consider the boundary value problem

$$\begin{cases} (|u'(t)|^{p(t)-2}u'(t))' + \lambda a(t)f(t,\ u(t),\ u'(t)) = 0, & t \in (0,1),\ \lambda \in [0,1], \\ u(0) - \alpha u'(\xi) = 0, \quad u(1) + \beta u'(\eta) = 0, \end{cases} \tag{3.1}$$

and define an operator $T:\ U \times [0,1] \to U$ by

$$T(u,\ \lambda)(t) = \begin{cases} \alpha\varphi^{-1}\Big(\displaystyle\int_\xi^\sigma \lambda a(s)f(s,\ u(s),\ u'(s))\mathrm{d}s\Big) \\ \quad + \displaystyle\int_0^t \varphi^{-1}\Big(\int_s^\sigma \lambda a(r)f(r,\ u(r),\ u'(r))\mathrm{d}r\Big)\mathrm{d}s, & 0 \le t \le \sigma; \\ \beta\varphi^{-1}\Big(\displaystyle\int_\sigma^\eta \lambda a(s)f(s,\ u(s),\ u'(s))\mathrm{d}s\Big) \\ \quad + \displaystyle\int_t^1 \varphi^{-1}\Big(\int_\sigma^s \lambda a(r)f(r,\ u(r),\ u'(r))\mathrm{d}r\Big)\mathrm{d}s, & \sigma \le t \le 1, \end{cases}$$

where $\sigma \in (0,1)$. Then T is a compact operator. Moreover, u is a solution of the problem (1.1) if and only if u is a fixed point of $u = T(u,1)$. In order to obtain the existence of solutions to the problem (1.1) by Leray-Schauder degree theory, we only need to prove that

(i) $u = T(u,\ \lambda)$ has no solution on $\partial\Omega_r$ for any $\lambda \in [0,1)$;

(ii) $\deg(I - T(u,0),\ \Omega_r,\ 0) \neq 0$.

First, we prove that (i) is satisfied. Without loss of generality, if there exists a $\lambda \in [0,1)$ and $u \in \partial\Omega_r$ such that $u = T(u,\ \lambda)$, then

$$|u'(t)|^{p(t)-2}u'(t) = -\lambda \int_\sigma^t a(s)f(s,\ u(s),\ u'(s))\mathrm{d}s, \qquad t \in (0,1).$$

Since $u \in \partial\Omega_r$, one has

$$\|u\| + \|u'\| = r.$$

If $\|u\| \ge \dfrac{2r}{3}$, then

$$\|u'\| \le \frac{r}{3},$$

but

$$|u(t)| = \Big|\int_\sigma^t u'(s)\mathrm{d}s\Big| \le \int_0^1 |u'(s)|\mathrm{d}s \le \frac{r}{3},$$

a contradiction.

Similarly, if $\|u\| < \dfrac{2r}{3}$, we have

$$\|u'\| > \frac{r}{3}.$$

Then there exists some $t_0 \in [0,1]$ such that

$$|u'(t_0)|^{p(t_0)-1} > \Big(\frac{r}{3}\Big)^{p(t_0)-1}.$$

From the condition

$$|f(t,\ u(t),\ u'(t))| \le \min\Big\{\Big(\frac{r}{3}\Big)^{p^- -1},\ \Big(\frac{r}{3}\Big)^{p^+ -1}\Big\}\frac{1}{\displaystyle\int_0^1 |a(t)|\mathrm{d}t},$$

we obtain that

$$|u'(t_0)|^{p(t_0)-1} = \Big|\int_\sigma^{t_0} \lambda a(s)f(s,\ u(s),\ u'(s))\mathrm{d}s\Big|$$

$$\leq \int_0^1 |a(s)f(s,\ u(s),\ u'(s))|\mathrm{d}s$$

$$\leq \min\left\{\left(\frac{r}{3}\right)^{p^- -1},\ \left(\frac{r}{3}\right)^{p^+ -1}\right\},$$

which is again a contradiction. Hence, the problem (3.1) has no solution on $\partial\Omega_r$.

Next, when $\lambda = 0$, it is easy to see that the problem (3.1) has a solution on Ω_r. Obviously,

$$\deg(I - T(u, 0),\ \Omega_r,\ 0) \neq 0.$$

So the condition (ii) holds.

Thus, upon an application of Leray-Schauder degree method, we obtain that the problem (1.1) has at least one solution.

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