Communications
in
Mathematical
Research
31(1)(2015), 31–39

Strong Convergence for a Countable Family of Total Quasi-ϕ-asymptotically Nonexpansive Nonself Mappings in Banach Space

Wang Xiong-rui and Quan Jing
(*Institute of Mathematics, Yibin University, Yibin, Sichuan, 644007*)

Communicated by Ji You-qing

Abstract: The purpose of this article is to introduce a class of total quasi-ϕ-asymptotically nonexpansive nonself mappings. Strong convergence theorems for common fixed points of a countable family of total quasi-ϕ-asymptotically nonexpansive mappings are established in the framework of Banach spaces based on modified Halpern and Mann-type iteration algorithm. The main results presented in this article extend and improve the corresponding results of many authors.
Key words: strong convergence, total quasi-ϕ-asymptotically nonexpansive nonself, generalized projection
2010 MR subject classification: 47H05, 47H09, 49M05
Document code: A
Article ID: 1674-5647(2015)01-0031-09
DOI: 10.13447/j.1674-5647.2015.01.04

1 Introduction and Preliminaries

Throughout this article we assume that E is a real Banach space with norm $\|\cdot\|$, E^* is the dual space of E, $\langle\cdot\,,\,\cdot\rangle$ is the duality pairing between E and E^*, C is a nonempty closed convex subset of E, $\mathbf{N}$ and $\mathbf{R}^+$ denote the set of natural numbers and the set of nonnegative real numbers, respectively. The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2;\ \|f^*\| = \|x\|,\ x \in E\}$$

is called the normalized duality mapping. Let $T : C \to C$ be a nonlinear mapping, and $F(T)$ denotes the set of fixed points of mapping T.

A subset C of E is said to be retract if there exists a continuous mapping $P : E \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach

Received date: Dec. 16, 2012.
Foundation item: Scientific Research Fund (2011JYZ010) of Science Technology Department of Sichuan Province and Scientific Research Fund (11ZA172 and 12ZB345) of Sichuan Provincial Education Department.
E-mail address: wxr888x@163.com (Wang X R).

space is a retraction. A mapping $P : E \to E$ is said to be a retraction if $P^2 = P$. Note that if a mapping P is a retraction, then $Pz = z$ for all $z \in R(P)$, the range of P. A mapping $P : E \to C$ is said to be a nonexpansive retraction, if it is nonexpansive and it is a retraction from E to C.

In this paper, we assume that E is a smooth, strictly convex and reflexive Banach space and C is a nonempty closed convex subset of E. We use $\phi : E \times E \to \mathbf{R}^+$ to denote the Lyapunov function, which is defined by

$$\phi(x,\, y) = \|x\|^2 - 2\langle x,\, Jy\rangle + \|y\|^2, \qquad x, y \in E.$$

It is obvious that

$$(\|x\| - \|y\|)^2 \le \phi(x,\, y) \le (\|x\| + \|y\|)^2, \qquad x, y \in E, \tag{1.1}$$

and

$$\begin{aligned}&\phi(x,\, J^{-1}(\lambda Jy + (1-\lambda)Jz)) \le \lambda\phi(x,\, y) + (1-\lambda)\phi(x,\, z),\\ &\phi(x,\, y) = \phi(x,\, z) + \phi(z,\, y) + 2\langle x - z,\, Jz - Jy\rangle, \qquad x, y, z \in E.\end{aligned} \tag{1.2}$$

Following Alber[1], the generalized projection $\Pi_C x : E \to C$ is defined by

$$\Pi_C x = \arg\inf_{y\in C} \phi(y,\, x), \qquad x \in E.$$

Lemma 1.1[1] *Let E be a smooth, strictly convex, and reflexive Banach space, and C be a nonempty closed convex subset of E. Then the following conclusions hold*:

(i) $\phi(x,\, \Pi_C y) + \phi(\Pi_C y,\, y) \le \phi(x,\, y)$ *for all* $x \in C$, $y \in E$;

(ii) *If* $x \in E$ *and* $z \in C$, *then* $z = \Pi_C x$ *if and only if* $\langle z - y,\, Jx - Jz\rangle \ge 0$ *for all* $y \in C$;

(iii) *For any* $x, y \in E$, $\phi(x,\, y) = 0$ *if and only if* $x = y$.

Lemma 1.2[2] *Let E be a uniformly convex and smooth Banach space, and $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n,\, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Recently, many researchers have focused on studying the convergence of iterative scheme for quasi-ϕ-asymptotically nonexspansive mappings and total quasi-ϕ-asymptotically nonexspansive mappings. Related works can be found in [3–10]. The quasi-ϕ-nonexspansive, quasi-ϕ-asymptotically nonexspansive and total quasi-ϕ-asymptotically nonexspansive mappings are defined as:

Definition 1.1 *A mapping $T : C \to C$ is said to be quasi-ϕ-nonexpansive, if $F(T) \ne \emptyset$ and $\phi(u, Tx) \le \phi(u, x)$ holds for all $x \in C$, $u \in F(T)$.*

A mapping $T : C \to C$ is said to be quasi-ϕ-asymptotically nonexpansive, if $F(T) \ne \emptyset$, and there exists a sequence $\{k_n\} \subset [1, +\infty]$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p,\, T^n x) \le k_n\phi(p,\, x)$ holds for all $x \in C$, $p \in F(T)$ and all $n \in \mathbf{N}$.

A mapping $T : C \to C$ is said to be total quasi-ϕ-asymptotically nonexpansive, if $F(T) \ne \emptyset$, and there exist sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n, \nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$ such that

$$\phi(p,\, T^n x) \le \phi(p,\, x) + \mu_n\psi(\phi(p,\, x)) + \nu_n$$

holds for all $x \in C$, $p \in F(T)$ and all $n \in \mathbf{N}$.

Recently, the strong and weak convergence of nonself mappings has been considered extensively by several authors in the setting of Hilbert or Banach spaces (see, for example, [2, 11–17]). Especially, Chang *et al.*[3] studied the convergence theorems for a countable family of quasi-ϕ-asymptotically nonexpansive nonself mappings in the framework of Banach spaces based on modified Halpern and Mann-type iteration algorithm. Now we recall the following nonself mappings.

Definition 1.2 *Let $P : E \to C$ be the nonexpansive retraction.*

A mapping $T : C \to E$ is said to be quasi-ϕ-nonexpansive nonself mapping, if $F(T) \neq \emptyset$ and $\phi(u,\, T(PT)^{n-1}x) \leq \phi(u,\, x)$ holds for all $x \in C$, $u \in F(T)$ and all $n \in \mathbf{N}$.

A mapping $T : C \to E$ is said to be quasi-ϕ-asymptotically nonexpansive nonself mapping, if $F(T) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [1, +\infty]$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(u,\, T(PT)^{n-1}x) \leq k_n\phi(u,\, x)$ holds for all $x \in C$, $u \in F(T)$ and all $n \in \mathbf{N}$.

A mapping $T : C \to E$ is said to be total quasi-ϕ-asymptotically nonexpansive nonself mapping, if $F(T) \neq \emptyset$, and there exist sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n, \nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$ such that

$$\phi(u,\, T(PT)^{n-1}x) \leq \phi(u,\, x) + \mu_n\psi(\phi(u,\, x)) + \nu_n$$

holds for all $x \in C$, $u \in F(T)$ and all $n \in \mathbf{N}$.

Lemma 1.3 *Let E be a real uniformly smooth, strictly convex and reflexive Banach space, and C be a nonempty closed convex subset of E. Let $T : C \to E$ be a total quasi-ϕ-asymptotically nonexpansive nonself mapping with respect to P defined by Definition 1.2. If $\nu_1 = 0$, then the fixed point set $F(T)$ is a closed and convex set of C.*

Proof. Let u_n be any sequence in $F(T)$ such that $u_n \to u$. Now we prove that $u \in F(T)$. In fact, since $T : C \to E$ is a total quasi-ϕ-asymptotically nonexpansive nonself mapping, we have

$$\phi(u,\, Tu) = \lim_{n\to\infty} \phi(u_n,\, Tu) \leq \lim_{n\to\infty} [\phi(u_n,\, u) + \mu_1\psi(\phi(u_n,\, u)) + \nu_1] = 0.$$

By Lemma 1.1(iii), we have $u = Tu$.

We now prove that $F(T)$ is convex. Let $u_1, u_2 \in F(T)$ and $u = tu_1 + (1-t)u_2$, where $t \in (0, 1)$. By the definition of T, we have

$$\phi(u_1,\, T(PT)^{n-1}u) \leq \phi(u_1,\, u) + \mu_n\psi(\phi(u_1,\, u)) + \nu_n$$

and

$$\phi(u_2,\, T(PT)^{n-1}u) \leq \phi(u_2,\, u) + \mu_n\psi(\phi(u_2,\, u)) + \nu_n.$$

In view of (1.2), we obtain

$$\phi(u_1,\, T(PT)^{n-1}u) = \phi(u_1,\, u) + \phi(u,\, T(PT)^{n-1}u) + 2\langle u_1 - u,\, Ju - JT(PT)^{n-1}u\rangle,$$
$$\phi(u_2,\, T(PT)^{n-1}u) = \phi(u_2,\, u) + \phi(u,\, T(PT)^{n-1}u) + 2\langle u_2 - u,\, Ju - JT(PT)^{n-1}u\rangle.$$

So we have

$$\phi(u,\, T(PT)^{n-1}u) \leq 2\langle u - u_1,\, Ju - JT(PT)^{n-1}u\rangle + \mu_n\psi(\phi(u_1,\, u)) + \nu_n,$$
$$\phi(u,\, T(PT)^{n-1}u) \leq 2\langle u - u_2,\, Ju - JT(PT)^{n-1}u\rangle + \mu_n\psi(\phi(u_2,\, u)) + \nu_n.$$

Multiply both sides of the above two inequalities by t and $1-t$, respectively, and yield that

$$\phi(u,\, T(PT)^{n-1}u) \le \mu_n[t\psi(\phi(u_1,\, u)) + (1-t)\psi(\phi(u_2,\, u))] + \nu_n.$$

It follows that

$$\lim_{n\to\infty} \phi(u,\, T(PT)^{n-1}u) = 0.$$

In light of (1.1), we arrive at

$$\lim_{n\to\infty} \|T(PT)^{n-1}u\| = \|u\| \quad \text{and} \quad \lim_{n\to\infty} \|J(T(PT)^{n-1}u)\| = \|Ju\|.$$

Since E^* is reflexive, without loss of generality, we assume that $J(T(PT)^{n-1}u) \rightharpoonup e^* \in E^*$. In view of the reflexivity of E, we have $JE = E^*$. So there exists an element $e \in E$ such that $Je = e^*$. It follows that

$$\begin{aligned}\phi(u,\, T(PT)^{n-1}u) &= \|u\|^2 - 2\langle u,\, J(T(PT)^{n-1}u)\rangle + \|T(PT)^{n-1}u\|^2 \\ &= \|u\|^2 - 2\langle u,\, J(T(PT)^{n-1}u)\rangle + \|J(T(PT)^{n-1}u)\|^2.\end{aligned}$$

Taking $\liminf\limits_{n\to\infty}$ on the both sides of the equality above, we obtain that

$$\begin{aligned}0 &\ge \|u\|^2 - 2\langle u,\, e^*\rangle + \|e^*\|^2 \\ &= \|u\|^2 - 2\langle u,\, Je\rangle + \|Je\|^2 \\ &= \|u\|^2 - 2\langle u,\, Je\rangle + \|e\|^2 \\ &= \phi(u,\, e).\end{aligned}$$

This implies that $u = e$, that is, $Ju = e^*$. So $J(T(PT)^{n-1}u) \rightharpoonup Ju \in E^*$. By Kadec-Klee property of E^*, from

$$\lim_{n\to\infty} \|J(T(PT)^{n-1}u)\| = \|Ju\|,$$

we obtain that

$$\lim_{n\to\infty} \|J(T(PT)^{n-1}u) - Ju\| = 0.$$

Since $J^{-1} : E^* \to E$ is demicontinuous, we see that $T(PT)^{n-1}u \rightharpoonup u$. By virtue of Kadec-Klee property of E, from

$$\lim_{n\to\infty} \|T(PT)^{n-1}u\| = \|u\|,$$

we see that

$$T(PT)^{n-1}u \to u \quad \text{as} \quad n \to \infty.$$

Hence

$$T(PT)^{n}u \to u \quad \text{as} \quad n \to \infty,$$

i.e.,

$$TP[T(PT)^{n-1}u] \to u \quad \text{as} \quad n \to \infty.$$

In view of the closedness of T, we can obtain that $TPu = u$. Since $u \in C$, $Pu = u$, it shows that $Tu = u$. This proves that $F(T)$ is convex. The conclusion of Lemma 1.3 is proved.

Definition 1.3 *A countable family of nonself mappings $\{T_i\} : C \to E$ is said to be uniformly total quasi-ϕ-asymptotically nonexpansive nonself mapping if*

$$\bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset,$$

there exist sequences $\{\mu_n\}$, $\{\nu_n\}$ *with* $\mu_n, \nu_n \to 0$ *as* $n \to \infty$ *and a strictly increasing continuous function* $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ *with* $\psi(0) = 0$ *such that*

$$\phi(u, T_i(PT_i)^{n-1}x) \leq \phi(u, x) + \mu_n \psi(\phi(u, x)) + \nu_n$$

holds for all $x \in C$, $u \in \bigcap_{i=1}^{\infty} F(T_i)$ *and all* $n \in \mathbf{N}$.

A nonself mapping $T : C \to E$ *is said to be uniformly L-Lipschitz continuous if there exists a constant* $L > 0$ *such that*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|$$

holds for all $x, y \in C$, $n \in \mathbf{N}$.

Next, we prove the strong convergence theorems for common fixed points of a countable family of total quasi-ϕ-asymptotically nonexpansive mappings in the framework of Banach spaces based on modified Halpern and Mann-type iteration algorithm. The results improve and extend the corresponding results of many others.

2 Main Results

Theorem 2.1 *Let* E *be a a real uniformly convex and uniformly smooth Banach space, and* C *be a nonempty closed convex subset of* E. *Let* $T_i : C \to E$, $i \in \mathbf{N}$ *be a family of uniformly total quasi-*ϕ*-asymptotically nonexpansive nonself mappings defined by Definition* 1.3. *Suppose that* T_i *is uniformly* L_i*-Lipschitz and*

$$F(T) := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset.$$

Suppose that there exists an $M^* > 0$ *such that* $\psi(\eta_n) \leq M^* \eta_n$. *Let* α_n *be a sequence in* $[0, 1]$, *and* β_n *be a sequence in* $(0, 1)$ *satisfying the following conditions*:

$$\lim_{n\to\infty} \alpha_n = 0, \qquad 0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1.$$

Let x_n *be a sequence generated by*

$$\begin{cases} x_1 \in E, \text{chosen arbitrarily; } C_1 = C, \\ l_{n,i} = \beta_n J x_n + (1 - \beta_n) J T_i (PT_i)^{n-1} x_n, \qquad i \geq 1, \\ y_{n,i} = J^{-1}[\alpha_n J x_1 + (1 - \alpha_n) l_{n,t}], \qquad i \geq 1, \\ C_{n+1} = \{z \in C_n : \sup_{i\geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \qquad n \geq 1, \end{cases} \tag{2.1}$$

where

$$\xi_n = \mu_n M^* \sup_{p\in F(T)} \phi(p, x_n) + \nu_n.$$

If $\nu_1 = 0$ *and* $F(T)$ *is bounded in* C, *then the iterative sequence* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)} x_1$ *in* C.

Proof. (I) We prove that $F(T)$ and C_n $(n \in \mathbf{N})$ are all closed and convex subsets in C.

It follows from Lemma 1.3 that for each i, $F(T_i)$ is a closed and convex subset of C. So $F(T)$ is closed and convex in C. By the assumption we know that $C_1 = C$ is closed and convex. We suppose that C_n is closed and convex for some $n \geq 2$. By the definition of ϕ, we have

$$\begin{aligned} C_{n+1} &= \{z \in C_n : \sup_{i\geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\} \\ &= \bigcap_{i\geq 1} \{z \in C : \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\} \bigcap C_n \\ &= \bigcap_{i\geq 1} \{z \in C : 2\alpha_n \langle z, Jx_1\rangle + 2(1-\alpha_n)\langle z, Jx_n\rangle - 2\langle z, Jy_{n,i}\rangle \leq \alpha_n \|x_1\|^2 \\ &\qquad + (1-\alpha_n)\|x_n\|^2 - \|y_{n,i}\|^2\} \bigcap C_n. \end{aligned}$$

This shows that C_{n+1} is closed and convex.

(II) We prove that $F(T) \subset C_n$ for all $n \in \mathbf{N}$.

In fact, $F(T) \subset C_1 = C$. Suppose that $F(T) \subset C_n$, $n \geq 2$. Let

$$\omega_{n,t} = J^{-1}(\beta_n Jx_n + (1-\beta_n)JT_i(PT_i)^{n-1}x_n).$$

It follows from (1.2) that for any $u \in F(T) \subset C_n$, we have

$$\begin{aligned} \phi(u,\, y_{n,i}) &= \phi(u,\, J^{-1}(\alpha_n Jx_1 + (1-\alpha_n)J\omega_{n,i}) \\ &\leq \alpha_n \phi(u,\, x_1) + (1-\alpha_n)\phi(u,\, \omega_{n,i}) \end{aligned}$$

and

$$\begin{aligned} \phi(u,\, \omega_{n,i}) &= \phi(u,\, J^{-1}(\beta_n Jx_n + (1-\beta_n)JT_i(PT_i)^{n-1}x_n) \\ &\leq \beta_n \phi(u,\, x_n) + (1-\beta_n)\phi(u,\, T_i(PT_i)^{n-1}x_n) \\ &\leq \beta_n \phi(u,\, x_n) + (1-\beta_n)[\phi(u,\, x_n) + \mu_n \psi(\phi(u,\, x_n)) + \nu_n] \\ &\leq \phi(u, x_n) + (1-\beta_n)(\mu_n M^* \phi(u,\, x_n) + \nu_n). \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{i\geq 1} \phi(u,\, y_{n,i}) &\leq \alpha_n \phi(u,\, x_1) + (1-\alpha_n)[\phi(u,\, x_n) + (1-\beta_n)(\mu_n M^* \phi(u,\, x_n) + \nu_n)] \\ &\leq \alpha_n \phi(u,\, x_1) + (1-\alpha_n)[\phi(u,\, x_n) + \mu_n M^* \sup_{p\in F(T)} \phi(p,\, x_n) + \nu_n] \\ &= \alpha_n \phi(u,\, x_1) + (1-\alpha_n)\phi(u,\, x_n) + \xi_n, \end{aligned}$$

where

$$\xi_n = \mu_n M^* \sup_{p\in F(T)} \phi(p, x_n) + \nu_n.$$

This shows that

$$u \in C_{n+1}.$$

So

$$F(T) \subset C_{n+1}.$$

(III) We prove that $\{x_n\}$ is a Cauchy sequence in C.

Since $x_n = \Pi_{C_n} x_1$, from Lemma 1.1(ii) we have

$$\langle x_n - y,\, Jx_1 - Jx_n\rangle \geq 0, \qquad y \in C_n.$$

Again, since $F(T) \subset C_n$, $n \geq 1$, we have

$$\langle x_n - u,\, Jx_1 - Jx_n\rangle \geq 0, \qquad u \in F(T).$$

It follows from Lemma 1.1(i) that for each $u \in F(T)$, $n \geq 1$,

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(u, x_1) - \phi(u, x_n) \leq \phi(u, x_1).$$

Therefore, $\{\phi(x_n, x_1)\}$ is bounded. By virtue of (1.1), x_n is also bounded. Since

$$x_n = \Pi_{C_n} x_1, \qquad x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n,$$

we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing. Hence, $\lim\limits_{n\to\infty} \phi(x_n, x_1)$ exists. By the construction of C_n, for any positive integer $m \geq n$, we have

$$C_m \subset C_n \quad \text{and} \quad x_m = \Pi_{C_m} x_1 \in C_n.$$

This shows that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_1) \leq \phi(x_m, x_1) - \phi(x_n, x_1) \to 0, \qquad m, n \to \infty.$$

It follows from Lemma 1.2 that

$$\lim_{n,m\to\infty} \|x_m - x_n\| = 0.$$

Hence x_n is a Cauchy sequence in C. Since C is complete, there is $p^* \in C$ such that $x_n \to p^*$. By the assumption, we have that

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} [\mu_n M^* \sup_{p\in F(T)} \phi(p, x_n) + \nu_n] = 0. \tag{2.2}$$

(IV) Now we prove that $p^* \in F(T)$.

Since $x_{n+1} \in C_{n+1}$ and $\alpha_n \to 0$, it follows from (2.1) and (2.2) that

$$\sup_{i\geq 1} \phi(x_{n+1}, y_{n,i}) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n)\phi(x_{n+1}, x_n) + \xi_n \to 0, \qquad n \to \infty.$$

Since $x_n \to p^*$, by Lemma 1.2, for each $i \geq 1$ we have

$$\lim_{n\to\infty} y_{n,i} = p^*. \tag{2.3}$$

Since x_n is bounded, and $\{T_i\}_{i=1}^{\infty}$ are total quasi-ϕ-asymptotically nonexpansive nonself mappings with sequences μ_n, ν_n, $p \in F(T)$, we have

$$\phi(p, T_i(PT_i)^{n-1}x) \leq \phi(p, x) + \mu_n \psi(\phi(p, x)) + \nu_n \leq \phi(p, x) + \mu_n M^* \phi(p, x) + \nu_n.$$

This implies that $\{T_i(PT_i)^{n-1}x_n\}$ is uniformly bounded. For each $i \geq 1$, we have

$$\begin{aligned}
\|\omega_{n,i}\| &= \|J^{-1}(\beta_n Jx_n + (1 - \beta_n)JT_i(PT_i)^{n-1}x_n)\| \\
&\leq \beta_n \|x_n\| + (1 - \beta_n)\|T_i(PT_i)^{n-1}x_n\| \\
&\leq \max\{\|x_n\|, \|T_i(PT_i)^{n-1}x_n\|\}.
\end{aligned}$$

This implies that $\{\omega_{n,i}\}$, $t \geq 0$ is also uniformly bounded. Since $\alpha_n \to 0$, from (2.1) we have

$$\lim_{n\to\infty} \|Jy_{n,i} - J\omega_{n,i}\| = \lim_{n\to\infty} \alpha_n \|Jx_1 - J\omega_{n,i}\| = 0, \qquad i \geq 1. \tag{2.4}$$

Since E is uniformly smooth and J^{-1} is uniformly continuous on each bounded subset of E^*, it follows from (2.3) and (2.4) that

$$\lim_{n\to\infty} \omega_{n,i} = p^*, \qquad i \geq 1.$$

Since $x_n \to p^*$ and J is uniformly continuous on each bounded subset of E, we have that $Jx_n \to Jp^*$, and for each $i \geq 1$,

$$\begin{aligned}
0 &= \lim_{n\to\infty} \|J\omega_{n,i} - Jp^*\| \\
&= \lim_{n\to\infty} \|\beta_n Jx_n + (1 - \beta_n)JT_i(PT_i)^{n-1}x_n - Jp^*\| \\
&= \lim_{n\to\infty} \|\beta_n(Jx_n - Jp^*) + (1 - \beta_n)(JT_i(PT_i)^{n-1}x_n - Jp^*)\| \\
&= \lim_{n\to\infty} (1 - \beta_n)\|JT_i(PT_i)^{n-1}x_n - Jp^*\|.
\end{aligned}$$

By the condition

$$0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1,$$

we have

$$\lim_{n\to\infty} \|JT_i(PT_i)^{n-1}x_n - Jp^*\| = 0.$$

Since J is uniformly continuous, this shows that

$$\lim_{n\to\infty} T_i(PT_i)^{n-1}x_n = p^*.$$

By the assumptions that $T_i : i \geq 1$ is closed and uniformly L_i-Lipschitz, we have

$$\begin{aligned} & \|T_i(PT_i)^n x_n - T_i(PT_i)^{n-1}x_n\| \\ \leq\ & \|T_i(PT_i)^n x_n - T_i(PT_i)^n x_{n+1}\| + \|T_i(PT_i)^n x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ & + \|x_n - T_i(PT_i)^{n-1}x_n\| \\ \leq\ & (L_i + 1)\|x_{n+1} - x_n\| + \|T_i(PT_i)^n x_{n+1} - x_{n+1}\| + \|x_n - T_i(PT_i)^{n-1}x_n\|. \end{aligned} \tag{2.5}$$

By

$$\lim_{n\to\infty} T_i(PT_i)^{n-1}x_n = p^*, \quad i \geq 1, \qquad x_n \to p^*$$

and (2.5), we have

$$\lim_{n\to\infty} \|T_i(PT_i)^n x_n - T_i(PT_i)^{n-1}x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} T_i(PT_i)^n x_n = p^*.$$

So we get

$$\lim_{n\to\infty} T_iP(T_i(PT_i)^{n-1}x_n) = p^*.$$

By virtue of the continuity of T_iP, we have $T_iPp^* = p^*$. Since $p^* \in C$ and $Pp^* = p^*$, we get $T_ip^* = p^*$. By the arbitrariness of $i \geq 1$, we have $p^* \in F(T)$.

(V) Finally, we prove that $x_n \to p^* = \Pi_{F(T)}x_1$.

Let $\omega = \Pi_{F(T)}x_1$. Since $\omega \in F(T) \subset C_n$ and $x_n = \Pi_{C_n}x_1$, we get

$$\phi(x_n, x_1) \leq \phi(\omega, x_1), \qquad n \geq 1.$$

This implies that

$$\phi(p^*, x_1) = \lim_{n\to\infty} \phi(x_n, x_1) \leq \phi(\omega, x_1). \tag{2.6}$$

By the definition of $\Pi_{F(T)}x_1$ and from (2.6) we have $p^* = \omega$. Therefore,

$$x_n \to p^* = \Pi_{F(T)}x_1.$$

This completes the proof of Theorem 2.1.

References

[1] Alber Y I. Metric and Generalized Projection Operators in Banach Spaces: Properties and applications. in: Kartosator A G. Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. New York: Marcel Dekker, 1996: 15–50.

[2] Kamimura S, Takahashi W. Strong convergence of a proximal-type algorithm in a Banach space. *SIAM J. Optim.*, 2002, **13**: 938–945.

[3] Chang S S, Chan C K, Joseph Lee H W. Modified block iterative algorithm for quasi-ϕ-asymptotically nonexpansive mappings and equilibrium problem in Banach spaces. *Appl. Math. Comput.*, 2011, **217**: 7520–7530.

[4] Qin X L, Cho S Y, Kang S M. On hybrid projection methods for asymptotically quasi-ϕ-nonexpansive mappings. *Appl. Math. Comput.*, 2010, **215**(11): 3874–3883.

[5] Saewan S, Kumam P. Modified hybrid block iterative algorithm for convex feasibility problems and generalized equilibrium problems for uniformly quasi-ϕ-asymptotically nonexpansive mappings. *Abstr. Appl. Anal.*, 2010: Article ID 357120, 22pp.

[6] Qin X L, Huang S C, Wang T Z. On the convergence of hybrid projection algorithms for asymptotically quasi-ϕ-nonexpansive mappings. *Comput. Math. Appl.*, 2011, **61**(4): 851–859.

[7] Chang S S, Joseph Lee H W, Chan C K, Yang L. Approximation theorems for total quasi-ϕ-asymptotically nonexpansive mappings with applications. *Appl. Math. Comput.*, 2011, **218**: 2921–2931.

[8] Chang S S, Joseph Lee H W, Chan C K, Zhang W B. A modified Halpearn type iterative algorithm for total quasi-ϕ-asymptotically nonexpansive mappings with applications. *Appl. Math. Comput.*, 2012, **218**(11): 6489–6497.

[9] Wang X R, Chang S S, Wang L, Tang Y K, Xu Y G. Strong convergence theorems for nonlinear operator equations with total quasi-ϕ-asymptotically nonexpansive mappings and applications. *Fixed Point Theory Appl.*, 2012, doi:10.1186/1687-1812-2012-34.

[10] Zuo P, Chang S S, Liu M. On a hybrid algorithm for a family of total quasi-ϕ-asymptotically nonexpansive mappings in Banach spaces. *Fixed Point Theory Appl.*, 2012, doi:10.1186/1687-1812-2012-70.

[11] Wang L. Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings. *J. Math. Anal. Appl.*, 2006, **323**(1): 550–557.

[12] Wang L. Explicit iteration method for common fixed points of a finite family of nonself asymptotically nonexpansive mappings. *Comput. Math. Appl.*, 2007, **53**(7): 1012–1019.

[13] Yildirim I, Ozdemir M. A new iterative process for common fixed points of finite families of non-self-asymptotically non-expansive mappings. *Nonlinear Anal.*, 2009, **71**: 991–999.

[14] Hao Y, Cho S Y, Qin X. Some weak convergence theorems for a family of asymptotically nonexpansive nonself mappings. *Fixed Point Theory Appl.*, 2010, Article ID 218573, 11pp.

[15] Qin X L, Cho S Y, Wang T Z, Kang S M. Convergence of an implicit iterative process for asymptotically pseudocontractive nonself mappings. *Nonlinear Anal.*, 2011, doi:10.1016/j.na.2011.04.031.

[16] Guo W P, Guo W. Weak convergence theorems for asymptotically nonexpansive nonself-mappings. *Appl. Math. Lett.*, 2011, **24**: 2181–2185.

[17] Kiziltunc H, Temir S. Convergence theorems by a new iteration process for a finite family of nonself asymptotically nonexpansive mappings with errors in Banach spaces. *Comput. Math. Appl.*, 2011, **61**(9): 2480–2489.