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Equivalent Conditions of Complete Convergence for Weighted Sums of Sequences of Extended Negatively Dependent Random Variables

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Abstract: By using Rosenthal type moment inequality for extended negatively dependent random variables, we establish the equivalent conditions of complete convergence for weighted sums of sequences of extended negatively dependent random variables under more general conditions. These results complement and improve the corresponding results obtained by Li *et al.* (Li D L, RAO M B, Jiang T F, Wang X C. Complete convergence and almost sure convergence of weighted sums of random variables. *J. Theoret. Probab.*, 1995, **8**: 49–76) and Liang (Liang H Y. Complete convergence for weighted sums of negatively associated random variables. *Statist. Probab. Lett.*, 2000, **48**: 317–325).

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1 Introduction

In many stochastic model, the assumption that random variables are independent is not plausible. Many researchers focus on weakening the restriction of independence in recent years. The concept of extended negatively dependent random variables was firstly introduced

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by Liu[1] as follows.

Definition 1.1[1] *Random variables $\{X_i,\ i \geq 1\}$ are said to be extended negatively dependent if there exists a constant $M > 0$ such that both*

$$P\Big(\bigcap_{i=1}^{n}(X_i \leq x_i)\Big) \leq M \prod_{i=1}^{n} P(X_i \leq x_i) \tag{1.1}$$

and

$$P\Big(\bigcap_{i=1}^{n}(X_i > x_i)\Big) \leq M \prod_{i=1}^{n} P(X_i > x_i) \tag{1.2}$$

hold for each $n \geq 1$ and all real numbers $x_1,\ x_2,\ \cdots,\ x_n$.

In the case $M = 1$ the notion of extended negatively dependent random variables reduces to the well-known notion of so-called negatively dependent random variables which was introduced by Lehmann[2]. Recall that random variables $\{X_i,\, i \geq 1\}$ are said to be positively dependent if the inequalities (1.1) and (1.2) hold both in the reverse direction when $M = 1$. Not looking that the notion of extended negatively dependent random variables seems to be a straightforward generalization of the notion of negative dependence, the extended negative dependent structure is substantially more comprehensive. As it is mentioned in [1], the extended negatively dependent structure can reflect not only a negative dependent structure but also a positive one, to some extend. Joag-Dev and Proschan[3] also pointed out that negatively associated random variables must be negatively dependent, and therefore, negatively associated random variables are also extended negatively dependent. Some applications for sequences of extended negatively dependent random variables have been found. We refer to Shen[4] for the probability inequalities, Liu[1] for the precise large deviations, and Chen[5] for the strong law of large numbers and applications to risk theory and renewal theory.

The concept of complete convergence was firstly introduced by Hsu and Robbins[6] as follows. A sequence $\{X_n,\, n \geq 1\}$ of random variables is said to converge completely to a constant θ if $\sum\limits_{n=1}^{\infty} P(|X_n - \theta| > \epsilon) < \infty$ for any $\epsilon > 0$. In view of the Borel-Cantelli Lemma, the complete convergence implies almost sure convergence. Therefore the complete convergence is very important tool in establishing almost sure convergence. When $\{X_n,\, n \geq 1\}$ is a sequence of independent and identically distributed random variables, Baum and Katz[7] proved the following remarkable result concerning the convergence rate of the tail probabilities $P(|S_n| > \epsilon n^{1/p})$ for any $\epsilon > 0$, where $S_n = \sum\limits_{i=1}^{n} X_i$.

Theorem 1.1[7] *Let $\{X,\, X_n,\, n \geq 1\}$ be a sequence of independent and identically distributed random variables, $0 < p < 2$ and $r > 1$. Then*

$$\sum_{n=1}^{\infty} n^{r-2} P(|S_n| > \epsilon n^{1/p}) < \infty, \qquad \epsilon > 0,$$

if and only if

$$E|X|^{rp} < \infty,$$

where $EX = 0$ whenever $1 \leq p < 2$.

Since partial sums are particular cases of weighted sums and the weighted sums are often encountered in some actual questions, the complete convergence for the weighted sums seems more important. Li *et al.*[8] investigated the complete convergence for independent weighted sums and obtained the following result. From now on, we denote always $\log x = \ln(\max\{e, x\})$.

Theorem 1.2[8] *Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables. Let $\beta > -1$ and $\{a_{ni} \approx (i/n)^\beta, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of real numbers such that $\sum\limits_{i=1}^{n} a_{ni} = 1$ for all $n \geq 1$. Then the following are equivalent*:

(i)

$$\begin{cases} E|X|^{1/(1+\beta)} < \infty, & -1 < \beta < -1/2, \\ E|X|^2 \log(1+|X|) < \infty, & \beta = -1/2, \\ E|X|^2 < \infty, & \beta > -1/2; \end{cases}$$

(ii)

$$\sum_{n=1}^{\infty} P\Big(\Big|\sum_{i=1}^{n} a_{ni}X_i - nEX\Big| > \epsilon n\Big) < \infty, \qquad \epsilon > 0.$$

Liang[9] extended the conclusions of Li *et al.*[8] from independent and identically distributed random variables to identically distributed negatively associated random variables. The result is as follows:

Theorem 1.3[9] *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed negatively associated random variables and $r > 1$. Assume that $\beta > -1$ and $\{a_{ni} \approx (i/n)^\beta, 1 \leq i \leq n, n \geq 1\}$ is a triangular array of real numbers such that $\sum\limits_{i=1}^{n} a_{ni} = 1$ for all $n \geq 1$. Then the following are equivalent*:

(i)

$$\begin{cases} E|X|^{(r-1)/(1+\beta)} < \infty, & -1 < \beta < -1/r, \\ E|X|^r \log(1+|X|) < \infty, & \beta = -1/r, \\ E|X|^r < \infty, & \beta > -1/r, \\ EX = 0; \end{cases}$$

(ii)

$$\sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1\leq k\leq n} \Big|\sum_{i=1}^{k} a_{ni}X_i\Big| > \epsilon n\Big) < \infty, \qquad \epsilon > 0.$$

The main purpose of this paper is to generalize and improve the result of Theorem 1.3 to the case of sequences of extended negatively dependent random variables. We investigate the complete convergence for weighted sums of sequences of extended negatively dependent random variables under more general conditions. Equivalent conditions of complete convergence for weighted sums of sequences of extended negatively dependent random variables are established. As a result, we not only promote and improve the results of Liang[9] from negatively associated random variables to extended negatively dependent random variables without necessarily imposing any extra conditions, but also relax the range of β. As an application, the Baum-Katz type result for sequences of extended negatively dependent random variables is obtained.

Throughout this paper, the symbol C denotes generic positive constants, whose value may vary from one application to another, and $I(A)$ denotes the indicator function of A. Let $a_n \ll b_n$ denotes that there exists a constant $C > 0$ such that $a_n \le Cb_n$ for sufficiently large n, and let $a_n \approx b_n$ mean $a_n \ll b_n$ and $b_n \ll a_n$.

By Definition 1.1, the following properties of extended negatively dependent sequences can be obtained directly.

Lemma 1.1 *Let random variables $\{X_n, n \ge 1\}$ be extended negatively dependent. Let $\{f_n, n \ge 1\}$ be a sequence of Borel functions, all of which are monotonically increasing (or all are monotonically decreasing). Then random variables $\{f_n(X_n), n \ge 1\}$ are still extended negatively dependent.*

The following lemma is the Rosenthal type inequality for extended negatively dependent sequences and is obtained by Shen[4].

Lemma 1.2[4] *Let $\{X_i, i \ge 1\}$ be a sequence of extended negatively dependent random variables with*

$$EX_i = 0, \quad E|X_i|^t < \infty, \qquad i \ge 1,\ t \ge 2.$$

Then there exists a positive constant C depending only on t such that

$$E\Big|\sum_{i=1}^{n} X_i\Big|^t \le C\Big(\sum_{i=1}^{n} E|X_i|^t + \Big(\sum_{i=1}^{n} E|X_i|^2\Big)^{t/2}\Big).$$

By Lemma 1.2 and Theorem 3 in [10], we can obtain the following lemma.

Lemma 1.3 *Let $\{X_i, i \ge 1\}$ be a sequence of extended negatively dependent random variables with*

$$EX_i = 0, \quad E|X_i|^t < \infty, \qquad i \ge 1,\ t \ge 2.$$

Then there exists a positive constant C depending only on t such that

$$E\max_{1\le j\le n}\Big|\sum_{i=1}^{j} X_i\Big|^t \le C\log^t n\Big(\sum_{i=1}^{n} E|X_i|^t + \Big(\sum_{i=1}^{n} E|X_i|^2\Big)^{t/2}\Big).$$

From the proof of Lemma 1.10 in [11] and by Lemma 1.2 we can obtain the following lemma.

Lemma 1.4 *Let $\{X_n, n \ge 1\}$ be a sequence of extended negatively dependent random variables. Then there exists a positive constant C such that*

$$(1 - P(\max_{1\le i\le n} |X_i| > x))^2 \sum_{i=1}^{n} P(|X_i| > x) \le CP(\max_{1\le i\le n} |X_i| > x), \qquad x \ge 0,\ n \ge 1.$$

Using Fubini's theorem, the following lemma can be easily proved.

Lemma 1.5 *Let X be a random variable. Then*

(i)
$$\int_1^\infty u^\beta E|X|^\alpha I(|X| > u^\gamma)\mathrm{d}u \ll E|X|^{(\beta+1)/\gamma+\alpha}, \qquad \alpha \geq 0,\ \gamma > 0,\ \beta > -1;$$

(ii)
$$\begin{aligned}&\int_1^\infty u^\beta \log u E|X|^\alpha I(|X| > u^\gamma)\mathrm{d}u\\ &\ll E|X|^{(\beta+1)/\gamma+\alpha}\log(1+|X|), \qquad \alpha \geq 0,\ \gamma > 0,\ \beta > -1.\end{aligned}$$

2 Main Results and Proofs

Theorem 2.1 *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed extended negatively dependent random variables, $r > 1$, $p > 1/2$, $\beta + p > 0$ and suppose that $EX = 0$ for $1/2 < p \leq 1$. Let $\{a_{ni} \approx (i/n)^\beta, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of real numbers. Then the following are equivalent*:

(i)
$$\begin{cases} E|X|^{(r-1)/(p+\beta)} < \infty, & -p < \beta < -p/r,\\ E|X|^{r/p}\log(1+|X|) < \infty, & \beta = -p/r,\\ E|X|^{r/p} < \infty, & \beta > -p/r; \end{cases} \tag{2.1}$$

(ii)
$$\sum_{n=1}^\infty n^{r-2} P\Big(\max_{1\leq k\leq n}\Big|\sum_{i=1}^k a_{ni}X_i\Big| > \epsilon n^p\Big) < \infty, \qquad \epsilon > 0. \tag{2.2}$$

Proof. Firstly, we prove that (2.1) implies (2.2). Note that $a_{ni} = a_{ni}^+ - a_{ni}^-$, where $a_{ni}^+ = \max\{a_{ni}, 0\}$ and $a_{ni}^- = \max\{-a_{ni}, 0\}$. Thus, to prove (2.2), it suffices to show that
$$\sum_{n=1}^\infty n^{r-2} P\Big(\max_{1\leq k\leq n}\Big|\sum_{i=1}^k a_{ni}^+X_i\Big| > \epsilon n^p\Big) < \infty,$$
$$\sum_{n=1}^\infty n^{r-2} P\Big(\max_{1\leq k\leq n}\Big|\sum_{i=1}^k a_{ni}^-X_i\Big| > \epsilon n^p\Big) < \infty.$$
So, without loss of generality, we can assume that $a_{ni} > 0$, $1 \leq i \leq n$, $n \geq 1$. Choose a $\delta > 0$ small enough and an integer K sufficiently large. Set, for any $1 \leq i \leq n$, $n \geq 1$,
$$X_{ni}^{(1)} = -n^{p-\delta}I(a_{ni}X_i < -n^{p-\delta}) + a_{ni}X_iI(|a_{ni}X_i| \leq n^{p-\delta}) + n^{p-\delta}I(a_{ni}X_i > n^{p-\delta}),$$
$$X_{ni}^{(2)} = (a_{ni}X_i - n^{p-\delta})I(n^{p-\delta} < a_{ni}X_i < \epsilon n^p/K),$$
$$X_{ni}^{(3)} = (a_{ni}X_i + n^{p-\delta})I(-\epsilon n^p/K < a_{ni}X_i < -n^{p-\delta});$$
$$X_{ni}^{(4)} = (a_{ni}X_i + n^{p-\delta})I(a_{ni}X_i \leq -\epsilon n^p/K) + (a_{ni}X_i - n^{p-\delta})I(a_{ni}X_i \geq \epsilon n^p/K).$$
It is obvious that
$$\sum_{i=1}^k a_{ni}X_i = \sum_{i=1}^k X_{ni}^{(1)} + \sum_{i=1}^k X_{ni}^{(2)} + \sum_{i=1}^k X_{ni}^{(3)} + \sum_{i=1}^k X_{ni}^{(4)}.$$

Note that

$$\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k a_{ni}X_i\Big|>4\epsilon n^p\Big)\subset\bigcup_{j=1}^4\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k X_{ni}^{(j)}\Big|>\epsilon n^p\Big).$$

Hence, to complete the proof of (2.2), it suffices to show that

$$I_j=:\sum_{n=1}^\infty n^{r-2}P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k X_{ni}^{(j)}\Big|>\epsilon n^p\Big)<\infty,\qquad j=1,2,3,4. \tag{2.3}$$

From the definition of $X_{ni}^{(4)}$, it is easy to see that

$$\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k X_{ni}^{(4)}\Big|>\epsilon n^p\Big)\subset\Big(\max_{1\le i\le n}|a_{ni}X_i|>\epsilon n^p/K\Big).$$

Noting that $a_{ni}\approx(i/n)^\beta$, we obtain from Lemma 1.5 that

$$\begin{aligned}
I_4&\le\sum_{n=1}^\infty n^{r-2}\sum_{i=1}^n P\Big(|a_{ni}X_i|>\frac{\epsilon}{K}n^p\Big)\\
&\le\sum_{n=1}^\infty n^{r-2}\sum_{i=1}^n P\Big(|X|>\frac{\epsilon}{CK}n^{p+\beta}i^{-\beta}\Big)\\
&\approx\int_1^\infty x^{r-2}\int_1^x P\Big(|X|>\frac{\epsilon}{CK}x^{p+\beta}y^{-\beta}\Big)\mathrm{d}y\mathrm{d}x\quad(\text{letting } u=x^{p+\beta}y^{-\beta},\ v=y)\\
&=\frac{1}{p+\beta}\int_1^\infty\mathrm{d}u\int_1^{u^{1/p}}u^{(r-1)/(p+\beta)-1}v^{\beta(r-1)/(p+\beta)}P\Big(|X|>\frac{\epsilon}{CK}u\Big)\mathrm{d}v\\
&\approx\begin{cases}\displaystyle\int_1^\infty u^{(r-1)/(p+\beta)-1}P\Big(|X|>\frac{\epsilon}{CK}u\Big)\mathrm{d}u, & -p<\beta<-p/r;\\ \displaystyle\int_1^\infty u^{r/p-1}\log uP\Big(|X|>\frac{\epsilon}{CK}u\Big)\mathrm{d}u, & \beta=-p/r;\\ \displaystyle\int_1^\infty u^{r/p-1}P\Big(|X|>\frac{\epsilon}{CK}u\Big)\mathrm{d}u, & \beta>-p/r\end{cases}\\
&\ll\begin{cases}E|X|^{(r-1)/(p+\beta)}, & -p<\beta<-p/r;\\ E|X|^{r/p}\log(1+|X|), & \beta=-p/r;\\ E|X|^{r/p}, & \beta>-p/r.\end{cases}
\end{aligned}\tag{2.4}$$

Therefore, by (2.1), $I_4<\infty$. From the definition of $X_{ni}^{(2)}$, it is clear that $X_{ni}^{(2)}>0$. Using Definition 1.1, we deduce that

$$\begin{aligned}
&P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k X_{ni}^{(2)}\Big|>\epsilon n^p\Big)\\
&=P\Big(\sum_{i=1}^n X_{ni}^{(2)}>\epsilon n^p\Big)\\
&\le P\quad(\text{there are at least } K \text{ indices } i\in[1,\,n] \text{ such that } a_{ni}X_i>n^{p-\delta})\\
&\ll\sum_{1\le i_1<i_2<\cdots<i_K\le n}\prod_{j=1}^K P(a_{ni_j}X_{i_j}>n^{p-\delta})
\end{aligned}$$

$$\le \Big(\sum_{j=1}^{n} P(a_{nj}X > n^{p-\delta})\Big)^{K}. \tag{2.5}$$

Since (2.1) implies $E|X|^{r/p} < \infty$, one has by Markov's inequality and (2.5) that

$$\begin{aligned}
I_2 &\le \sum_{n=1}^{\infty} n^{r-2}\Big(\sum_{j=1}^{n} P\left(a_{nj}X > n^{p-\delta}\right)\Big)^{K} \\
&\le \sum_{n=1}^{\infty} n^{r-2}\Big(\sum_{j=1}^{n} n^{-r+r\delta/p}|a_{nj}|^{r/p}E|X|^{r/p}\Big)^{K} \\
&\ll \begin{cases} \sum_{n=1}^{\infty} n^{r-2-Kr(p+\beta-\delta)/p}, & -p < \beta < -p/r; \\ \sum_{n=1}^{\infty} n^{r-2-K(r-1-r\delta/p)}\log n, & \beta = -p/r; \\ \sum_{n=1}^{\infty} n^{r-2-K(r-1-r\delta/p)}, & \beta > -p/r. \end{cases}
\end{aligned} \tag{2.6}$$

Noting that $r > 1$, $p + \beta > 0$, we can take δ small enough and take integer K sufficiently large so that

$$r - 2 - Kr(p+\beta-\delta)/p < -1,$$
$$r - 2 - K(r - 1 - r\delta/p) < -1.$$

Thus, by (2.6) we get $I_2 < \infty$. Similarly, we can show $I_3 < \infty$. In order to estimate I_1, we first verify that

$$n^{-p}\max_{1\le k\le n}\Big|\sum_{i=1}^{k} EX_{ni}^{(1)}\Big| \to 0 \qquad \text{as } n\to\infty.$$

Note that (2.1) implies $E|X|^{r/p} < \infty$ and $E|X|^{1/p} < \infty$. When $p > 1$, noting that

$$|X_{ni}^{(1)}| \le n^{p-\delta}, \qquad |X_{ni}^{(1)}| \le |a_{ni}X_i|,$$

we obtain by Hölder's inequality that

$$\begin{aligned}
n^{-p}\max_{1\le k\le n}\Big|\sum_{i=1}^{k} EX_{ni}^{(1)}\Big| &\le n^{-p}\sum_{i=1}^{n} E|X_{ni}^{(1)}| \\
&\le n^{-1-\delta(1-1/p)}\sum_{i=1}^{n} E|a_{ni}X_i|^{1/p} \\
&\le n^{-1-\delta(1-1/p)}\sum_{i=1}^{n} |a_{ni}|^{1/p} \\
&\le n^{-1-\delta(1-1/p)}n^{(r-1)/r}\Big(\sum_{i=1}^{n}|a_{ni}|^{r/p}\Big)^{1/r} \\
&\ll n^{-\delta(1-1/p)-1/r}\Big(\sum_{i=1}^{n} n^{-r\beta/p}i^{\beta r/p}\Big)^{1/r}
\end{aligned}$$

$$\approx \begin{cases} n^{-\delta(1-1/p)-1/r-\beta/p}, & -p < \beta < -p/r; \\ n^{-\delta(1-1/p)} \log n, & \beta = -p/r; \\ n^{-\delta(1-1/p)}, & \beta > -p/r \end{cases}$$

$$\to 0 \quad \text{as} \ \ n \to \infty. \tag{2.7}$$

When $1/2 < p \le 1$, noting that $EX = 0$, by choosing a δ small so that

$$-\delta(1 - r/p) + 1 - r < 0,$$

we have

$$\begin{aligned} n^{-p} \max_{1\le k\le n} \Big| \sum_{i=1}^{k} EX_{ni}^{(1)} \Big| &\le 2n^{-p} \sum_{i=1}^{n} E|a_{ni}X_i| I(|a_{ni}X_i| > n^{p-\delta}) \\ &\le 2n^{-r-\delta(1-r/p)} \sum_{i=1}^{n} E|a_{ni}X_i|^{r/p} \\ &\le n^{-r-\delta(1-r/p)} \sum_{i=1}^{n} |a_{ni}|^{r/p} \\ &\ll n^{-r-\delta(1-r/p)} \Big(\sum_{i=1}^{n} n^{-r\beta/p} i^{\beta r/p} \Big) \\ &\approx \begin{cases} n^{-\delta-r(p+\beta-\delta)/p}, & -p < \beta < -p/r; \\ n^{-\delta-r(p+\beta-\delta)/p} \log n, & \beta = -p/r; \\ n^{-\delta(1-r/p)+1-r}, & \beta > -p/r \end{cases} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.8}$$

Therefore, by (2.7) and (2.8), it suffices to prove that

$$I_1^* =: \sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1\le k\le n} \Big| \sum_{i=1}^{k} (X_{ni}^{(1)} - EX_{ni}^{(1)}) \Big| > \epsilon n^p \Big) < \infty. \tag{2.9}$$

Note that $\{X_{ni}^{(1)}, 1 \le i \le n, n \ge 1\}$ is still extended negatively dependent by Lemma 1.1. By using Markov's inequality, C_r inequality and Lemma 1.3, we get that for a suitably large t, which will be determined later,

$$\begin{aligned} &P\Big(\max_{1\le k\le n} \Big| \sum_{i=1}^{k} (X_{ni}^{(1)} - EX_{ni}^{(1)}) \Big| > \epsilon n^p \Big) \\ &\ll n^{-pt} (\log n)^t \Big(\sum_{i=1}^{n} E|X_{ni}^{(1)}|^t + \Big(\sum_{i=1}^{n} E(X_{ni}^{(1)})^2 \Big)^{t/2} \Big). \end{aligned} \tag{2.10}$$

Taking a sufficiently large t so that

$$-2 - \delta t + r(\delta - \beta)/p < -1, \qquad -1 - (t - r/p)\delta < -1,$$

we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} n^{-pt} (\log n)^t \sum_{i=1}^{n} E|X_{ni}^{(1)}|^t \mathrm{d}x \\
\le &\sum_{n=1}^{\infty} n^{-2} n^{-\delta(t-r/p)} (\log n)^t \sum_{i=1}^{n} |a_{ni}|^{r/p} \\
\ll &\sum_{n=1}^{\infty} n^{-2-\delta(t-r/p)} (\log n)^t \sum_{i=1}^{n} i^{r\beta/p} n^{-r\beta/p} \\
\ll &\begin{cases}
\sum_{n=1}^{\infty} n^{-2-\delta t + r(\delta-\beta)/p} (\log n)^t, & -p < \beta < -1/r; \\
\sum_{n=1}^{\infty} n^{-2-\delta t + r(\delta-\beta)/p} (\log n)^{t+1}, & \beta = -p/r; \\
\sum_{n=1}^{\infty} n^{-1-(t-r/p)\delta} (\log n)^t, & \beta > -p/r
\end{cases} \\
< &\infty.
\end{aligned} \tag{2.11}$$

When $r/p \ge 2$, (2.1) implies $EX^2 < \infty$. Noting that $p + \beta > 0$, $p > 1/2$, and choosing a sufficiently large t so that

$$r - 2 - t(p + \beta) < -1, \qquad r - 2 - (2p - 1)t/2 < -1,$$

we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} n^{-pt} (\log n)^t \Big(\sum_{i=1}^{n} E(X_{ni}^{(1)})^2 \Big)^{t/2} \\
\le &\sum_{n=1}^{\infty} n^{r-2} n^{-pt} (\log n)^t \Big(\sum_{i=1}^{n} a_{ni}^2 \Big)^{t/2} \\
\ll &\sum_{n=1}^{\infty} n^{r-2-pt} (\log n)^t \Big(\sum_{i=1}^{n} i^{2\beta} n^{-2\beta} \Big)^{t/2} \\
\ll &\begin{cases}
\sum_{n=1}^{\infty} n^{r-2-t(p+\beta)} (\log n)^t, & -p < \beta < -1/2; \\
\sum_{n=1}^{\infty} n^{r-2-(2p-1)t/2} (\log n)^{3t/2}, & \beta = -1/2; \\
\sum_{n=1}^{\infty} n^{r-2-(2p-1)t/2} (\log n)^t, & \beta > -1/2
\end{cases} \\
< &\infty.
\end{aligned} \tag{2.12}$$

When $r/p < 2$, choosing a sufficiently large t so that

$$\begin{aligned}
&r - 2 - [\delta(2 - r/p) + r(p + \beta)/p]t/2 < -1, \\
&r - 2 - [\delta(2 - r/p) + r - 1]t/2 < -1,
\end{aligned}$$

we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} n^{-pt} (\log n)^t \Big(\sum_{i=1}^{n} E(X_{ni}^{(1)})^2 \Big)^{t/2} \\
\ll &\sum_{n=1}^{\infty} n^{r-2} n^{-rt/2-\delta(2-r/p)t/2} (\log n)^t \Big(\sum_{i=1}^{n} a_{ni}^{r/p} \Big)^{t/2} \\
\ll &\sum_{n=1}^{\infty} n^{r-2} n^{-\delta(2-r/p)t/2} (\log n)^t \Big(\sum_{i=1}^{n} i^{r\beta/p} n^{-r\beta/p} \Big)^{t/2} \\
\ll &\begin{cases} \displaystyle\sum_{n=1}^{\infty} n^{r-2-[\delta(2-r/p)+r(p+\beta)/p]t/2} (\log n)^t, & -p<\beta<-1/r; \\ \displaystyle\sum_{n=1}^{\infty} n^{r-2-[\delta(2-r/p)+r-1]t/2} (\log n)^{3t/2}, & \beta=-p/r; \\ \displaystyle\sum_{n=1}^{\infty} n^{r-2-[\delta(2-r/p)+r-1]t/2} (\log n)^t, & \beta>-p/r \end{cases} \\
< &\ \infty. \qquad (2.13)
\end{aligned}
$$

Thus, by (2.9)–(2.13), we have shown that $I_1 < \infty$. Therefore, (2.1) holds by (2.3) and $I_i < \infty$, $i = 1, 2, 3, 4$.

Next, we proceed to prove that (2.2) implies (2.1). Since

$$\max_{1\le k\le n} |a_{nk} X_k| \le 2 \max_{1\le k\le n} \Big| \sum_{i=1}^{k} a_{ni} X_i \Big|,$$

by (2.2), we have

$$\sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1\le k\le n} |a_{nk} X_k| > \epsilon n^p \Big) < \infty, \qquad (2.14)$$

$$P\Big(\max_{1\le k\le n} |a_{nk} X_k| > \epsilon n^p \Big) \to 0 \qquad \text{as } n \to \infty. \qquad (2.15)$$

Therefore, by Lemma 1.4 and (2.15), we obtain

$$\sum_{i=1}^{n} P(|a_{ni} X_i| > \epsilon n^p) \ll P\Big(\max_{1\le k\le n} |a_{nk} X_k| > \epsilon n^p \Big). \qquad (2.16)$$

Combining with (2.14) and (2.16), we deduce that

$$\sum_{n=1}^{\infty} n^{r-2} \sum_{i=1}^{n} P(|a_{ni} X_i| > \epsilon n^p) < \infty. \qquad (2.17)$$

From the proof of (2.4), it is obvious that (2.17) is equivalent to (2.1).

Remark 2.1 It is obvious that Theorem 1.2 of Li *et al.*[8] is a special case of $p = 1$, $r = 2$ of Theorem 2.1 in this paper. Thus, we not only extend the result of Li *et al.*[8] for independent and identically distributed random variables to the case of sequences of extended negatively dependent random variables without necessarily imposing any extra conditions, but also the method used for proving Theorem 2.1 is different from that of Li *et al.*[8].

Remark 2.2 Since negatively associated random variables are a special case of extended negatively dependent random variables, taking $p = 1$ in Theorem 2.1, we obtain Theorem 1.3 of Liang[9]. So Theorem 2.1 generalizes and improves the corresponding result of Liang[9].

Applying Theorem 2.1 with $\beta = 0$ and p replaced by $1/p$, we can get the following Baum-Katz type result for sequences of extended negatively dependent random variables as follows.

Corollary 2.1 *Let* $\{X, X_n, n \geq 1\}$ *be a sequence of identically distributed extended negatively dependent random variables,* $0 < p < 2$ *and* $r > 1$. *Then*

$$\sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1\leq k\leq n}\Big|\sum_{i=1}^{k} X_i\Big| > \epsilon n^{1/p}\Big) < \infty, \qquad \epsilon > 0,$$

if and only if $E|X|^{rp} < \infty$, *where* $EX = 0$ *whenever* $1 \leq p < 2$.

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