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A $(k, n-k)$ Conjugate Boundary Value Problem with Semipositone Nonlinearity

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Abstract: The existence of positive solution is proved for a $(k, n-k)$ conjugate boundary value problem in which the nonlinearity may make negative values and may be singular with respect to the time variable. The main results of Agarwal *et al.* (Agarwal R P, Grace S R, O'Regan D. Semipositive higher-order differential equations. *Appl. Math. Letters*, 2004, **14**: 201–207) are extended. The basic tools are the Hammerstein integral equation and the Krasnosel'skii's cone expansion-compression technique.

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1 Introduction

Let $n \geq 2$, $1 \leq k \leq n-1$ be two positive integers and $\lambda > 0$ be a positive parameter. In this paper, we study the existence of positive solution to the following nonlinear $(k, n-k)$ conjugate boundary value problem:

$$\text{(P)} \qquad \begin{cases} (-1)^{n-p} u^{(n)}(t) = \lambda f(t, u(t)), & 0 < t < 1, \\ u^{(i)}(0) = 0, \quad u^{(j)}(1) = 0, & 0 \leq i \leq k-1, \ 0 \leq j \leq n-k-1. \end{cases}$$

The solution u^* of the problem (P) is called positive if $u^*(t) > 0$ for $0 < t < 1$.

For the function $f(t, x)$, we use the following assumptions:

(A1) $f : (0, 1) \times [0, +\infty) \to (-\infty, +\infty)$ is continuous.

(A2) There exists a nonnegative function $h \in L^1[0, 1] \cap C(0, 1)$ such that

$$f(t, x) + h(t) \geq 0, \qquad (t, x) \in (0, 1) \times [0, +\infty).$$

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(A3) For each $r > 0$, there exists a nonnegative function $j_r \in L^1[0, 1] \cap C(0, 1)$ such that

$$f(t, x) + h(t) \leq j_r(t), \qquad (t, x) \in (0, 1) \times [0, r].$$

The assumptions (A2) and (A3) show that $f(t, x)$ may be singular at $t = 0$ and $t = 1$, and may not have any numerical lower bound. Therefore, the problem (P) is singular and semipositone. The problems of this type arise naturally in chemical reactor theory, see [1].

In applications, one is interested in showing the existence of positive solution for some λ. When $h(t) \equiv M \geq 0$, the problem (P) has been frequently investigated in recent years, for example, see [2–9] and the references therein.

In 2004, Agarwal *et al.*[8] established the following existence theorem of positive solution:

Theorem 1.1 ([8], Theorem 2.3) *Suppose that the following conditions are satisfied*:

(a1) $f : [0, 1] \times [0, +\infty) \to (-\infty, +\infty)$ *is continuous and there exists a constant* $M > 0$ *such that* $f(t, x) + M \geq 0$ *for any* $(t, x) \in [0, 1] \times [0, +\infty)$;

(a2) *There exists a continuous and nondecreasing function* $\zeta : [0, +\infty) \to [0, +\infty)$ *such that*

$$\zeta(x) > 0, \qquad 0 < x < +\infty,$$

and

$$f(t, x) + M \leq \zeta(x), \qquad (t, x) \in [0, 1] \times [0, +\infty);$$

(a3) *There exists a positive number* $r_1 \geq \dfrac{\lambda M}{n!}$ *such that*

$$\lambda \zeta(r_1) \max_{0 \leq t \leq 1} \int_0^1 G(t, s) \mathrm{d}s \leq r_1;$$

(a4) *There exist a* δ *with* $0 < \delta < \dfrac{1}{2}$ *and a continuous and nondecreasing function* $\xi : (0, +\infty) \to (0, +\infty)$ *such that*

$$f(t, x) + M \geq \xi(x), \qquad (t, x) \in [\delta, 1 - \delta] \times (0, +\infty);$$

(a5) *There exists an* ε *with*

$$0 < \varepsilon \leq 1 - \frac{\lambda M}{n! r_2}, \qquad r_2 > r_1$$

such that

$$\lambda \xi(\varepsilon \theta r_2) \max_{0 \leq t \leq 1} \int_\delta^{1-\delta} G(t, s) \mathrm{d}s \geq r_2,$$

where

$$\theta = \begin{cases} \delta^k (1 - \delta)^{n-k}, & n \leq 2k; \\ \delta^{n-k} (1 - \delta)^k, & n \geq 2k. \end{cases}$$

Then the problem (P) *has at least one positive solution* $u^* \in C^{n-1}[0, 1] \cap C^n(0, 1)$.

In Theorem 1.1, $G(t, s)$ is the Green function of the problem (P) with $f(t, x) \equiv 0$. For the expression of $G(t, s)$, see Section 2. The function $h(t) \equiv M$ is a constant and the nonlinearity $f(t, x)$ is continuous on $[0, 1] \times [0, +\infty)$.

The purpose of this paper is to extend Theorem 1.1. In this paper, we study the problem (P) under the assumptions (A1)–(A3). Therefore, we allow $h(t)$ to be an integral function on [0, 1] and $f(t, x)$ to be singular at $t=0$ and $t=1$.

We apply the Anuradha's substitution technique and the Krasnosel'skii's cone expansion-compression method to the problem (P) (see [10–12]). By introducing two height functions and considering the integrals of the height functions, we establish a local existence theorem. Finally, we verify that the theorem extends the Theorem 1.1 and illustrate that our extend is true by an example.

2 Preliminaries

Firstly, we list some symbols used in this paper.

Let α and β be two positive numbers with $0<\alpha<\beta<1$. In real problems, we can choose α and β by the properties of $f(t, x)$, for example $\alpha=\dfrac{1}{4}$, $\beta=\dfrac{3}{4}$.

Let $C[0, 1]$ be the Banach space of all continuous functions on [0, 1] equipped with the norm

$$\|u\|=\max_{0\le t\le 1}|u(t)|.$$

Let the polynomials

$$\begin{aligned} &H(s)=\frac{1}{(k-1)!(n-k-1)!}s^{n-k}(1-s)^k,\\ &p(t)=t^k(1-t)^{n-k},\\ &q(t)=\frac{1}{\min\{k, n-k\}}t^{k-1}(1-t)^{n-k-1}. \end{aligned}$$

Let the sets

$$\begin{aligned} &K=\{u\in C[0, 1]: u(t)\ge\|u\|p(t),\ 0\le t\le 1\},\\ &\Omega(r)=\{u\in K: \|u\|<r\},\\ &\partial\Omega(r)=\{u\in K: \|u\|=r\}. \end{aligned}$$

Then K is a cone of nonnegative functions in $C[0, 1]$.

Let $G(t, s)$ be the Green function of the homogeneous linear $(k, n-k)$ boundary value problem (P) with $f(t, x)\equiv 0$. Then $G(t, s)$ has the exact expression

$$G(t, s)=\begin{cases}\displaystyle\sum_{j=0}^{k-1}\left[\sum_{i=0}^{k-1-j}\binom{n-k+i-1}{i}t^i\right]\frac{t^j(-s)^{n-j-1}}{j!(n-j-1)!}(1-t)^{n-k}, & 0\le s\le t\le 1,\\ \displaystyle-\sum_{j=0}^{n-k-1}\left[\sum_{i=0}^{n-k-1-j}\binom{k+i-1}{i}(1-t)^i\right]\frac{(t-1)^j(1-s)^{n-j-1}}{j!(n-j-1)!}t^k, & 0\le t\le s\le 1.\end{cases}$$

By [2],

$$(-1)^{n-k}G(t, s)>0, \qquad 0<t,\ s<1.$$

Let

$$w(t)=\int_0^1(-1)^{n-k}G(t, s)h(s)\mathrm{d}s.$$

Then

$$\begin{cases} (-1)^{n-p} w^{(n)}(t) = h(t), & 0 < t < 1, \\ w^{(i)}(0) = 0, \quad u^{(j)}(1) = 0, & 0 \le i \le k-1, \ 0 \le j \le n-k-1. \end{cases}$$

Let the constants

$$A = \frac{\min\{k, n-k\}(k-1)!(n-k-1)!(n-2)^{n-2}n^n}{(k-1)^{k-1}k^k(n-k-1)^{n-k-1}(n-k)^{n-k}},$$

$$B = \frac{(n-1)(k-1)!(n-k-1)!}{\alpha^n(1-\beta)^n}.$$

Let $[c]^\flat = \max\{0, c\}$. For $u \in K$, define the operator T as follows:

$$(Tu)(t) = \lambda \int_0^1 (-1)^{n-k} G(t, s)[f(s, [u(s) - \lambda w(s)]^\flat) + h(s)] \mathrm{d}s, \qquad 0 \le t \le 1.$$

If (A1)–(A3) hold, then $T : K \to C[0, 1]$ is well-defined and $Tu \in C[0, 1]$.

Secondly, we need the following lemmas in order to prove the main results.

Lemma 2.1 *Assume $u \in C^{n-1}[0, 1] \cap C^n(0, 1)$ such that*

$$\begin{cases} (-1)^{n-k} u^{(n)}(t) \ge 0, & 0 < t < 1, \\ u^{(i)}(0) = 0, \quad u^{(j)}(1) = 0, & 0 \le i \le k-1, \ 0 \le j \le n-k-1. \end{cases}$$

Then

$$u(t) \ge \|u\| p(t), \qquad 0 \le t \le 1.$$

Proof. See Lemma 2.1 in [8].

Lemma 2.2 $\dfrac{1}{n-1} p(t) H(s) \le (-1)^{n-k} G(t, s) \le q(t) H(s)$ *for* $0 \le t, s \le 1$.

Proof. See Lemma 2.1 and Theorem 1 in [7].

Lemma 2.3 $\max\limits_{0 \le t,s \le 1} (-1)^{n-k} G(t, s) \le A^{-1}, \quad \min\limits_{\alpha \le t,s \le \beta} (-1)^{n-k} G(t, s) \ge B^{-1}.$

Proof. It is easy to see that

$$\begin{aligned} \max_{0 \le s \le 1} \{s^{n-k}(1-s)^k\} &= \Big(\frac{n-k}{n}\Big)^{n-k} \Big(1 - \frac{n-k}{n}\Big)^k \\ &= \frac{k^k(n-k)^{n-k}}{n^n}, \\ \max_{0 \le t \le 1} \{t^{k-1}(1-t)^{n-k-1}\} &= 7\Big(\frac{k-1}{n-2}\Big)^{k-1} \Big(1 - \frac{k-1}{n-2}\Big)^{n-k-1} \\ &= \frac{(k-1)^{k-1}(n-k-1)^{n-k-1}}{(n-2)^{n-2}}, \\ \min_{\alpha \le t,s \le \beta} \{t^k(1-t)^{n-k}s^{n-k}(1-s)^k\} &\ge \alpha^n(1-\beta)^n. \end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned} &\max_{0 \le t,s \le 1} (-1)^{n-k} G(t, s) \\ \le\ &\max_{0 \le t \le 1} q(t) \max_{0 \le s \le 1} H(s) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\min\{k, n-k\}(k-1)!(n-k-1)!} \max_{0\le t\le 1}\{t^{k-1}(1-t)^{n-k-1}\} \max_{0\le s\le 1}\{s^{n-k}(1-s)^{k}\} \\
&= \frac{(k-1)^{k-1}k^k(n-k-1)^{n-k-1}(n-k)^{n-k}}{\min\{k, n-k\}(k-1)!(n-k-1)!(n-2)^{n-2}n^n} \\
&= A^{-1},
\end{aligned}
$$

$$
\begin{aligned}
&\min_{\alpha\le t,s\le\beta}(-1)^{n-k}G(t,s) \\
&\ge \frac{1}{n-1}\min_{\alpha\le t,s\le\beta}\{p(t)H(s)\} \\
&= \frac{1}{(n-1)(k-1)!(n-k-1)!}\min_{\alpha\le t,s\le\beta} t^k(1-t)^{n-k}s^{n-k}(1-s)^k \\
&\ge \frac{\alpha^n(1-\beta)^n}{(n-1)(k-1)!(n-k-1)!} \\
&= B^{-1}.
\end{aligned}
$$

Lemma 2.4 *If* (A1)–(A3) *hold, then* $T: K\to K$ *is completely continuous.*

Proof. $T(K)\subset K$ is derived from Lemma 2.1. The remainder is a standard argument, for example, see Step II in the proof of Theorem 2.2 in [12] or Step II in the proof of Theorem 1 in [13].

Let $\eta = \sup\limits_{0<t<1}\dfrac{w(t)}{q(t)}$. By Lemma 2.2, we have

$$
\begin{aligned}
\eta &= \sup_{0<t<1}\frac{\int_0^1 G(t,\, s)h(s)\mathrm{d}s}{q(t)} \\
&\le \sup_{0<t<1}\frac{q(t)\int_0^1 H(s)h(s)\mathrm{d}s}{q(t)} \\
&= \int_0^1 H(s)h(s)\mathrm{d}s \\
&< +\infty.
\end{aligned}
$$

Lemma 2.5 *If* $\bar u\in K$ *is a fixed point of the operator* T *and* $\|\bar u\| > \lambda\eta$, *then* $u^*(t)$ *is a positive solution of the problem* (P), *where* $u^* = \bar u - \lambda w$.

Proof. By the definition of η, we have

$$w(t)\le \eta q(t), \qquad 0\le t\le 1.$$

Since $\|\bar u\| > \lambda\eta$, one has

$$\bar u(t) - \lambda w(t) \ge \|\bar u\| q(t) - \lambda\eta q(t)\ge 0, \qquad 0\le t\le 1.$$

It shows that

$$f(t,\, [\bar u(t)-\lambda w(t)]^\flat) = f(t,\, \bar u(t)-\lambda w(t)), \qquad 0\le t\le 1.$$

By the equality and $T\bar u = \bar u$, one has

$$\begin{cases} (-1)^{n-k}\bar{u}^{(n)}(t) = \lambda[f(t,\, \bar{u}(t) - \lambda w(t)) + h(t)], & 0 < t < 1, \\ \bar{u}^{(i)}(0) = 0, \quad \bar{u}^{(j)}(1) = 0, & 0 \le i \le k-1,\ 0 \le j \le n-k-1. \end{cases}$$

Since

$$u^* = \bar{u} - \lambda w,$$

by the properties of $w(t)$, we get

$$\begin{cases} (-1)^{n-k}(u^*)^{(n)}(t) = \lambda f(t,\, u^*(t) - \lambda w(t)), & 0 < t < 1, \\ (u^*)^{(i)}(0) = 0, \quad (u^*)^{(j)}(1) = 0, & 0 \le i \le k-1,\ 0 \le j \le n-k-1. \end{cases}$$

This shows that $u^*(t)$ is a solution of the problem (P). Since

$$u^*(t) = \bar{u}(t) - \lambda w(t) \ge (\|\bar{u}\| - \lambda\eta)q(t) > 0, \qquad 0 < t < 1,$$

the solution $u^*(t)$ is positive.

3 Main Results

We use the following control functions:

$$\varphi(t,\, r) = \max\{f(t,\, [u - \lambda w(t)]^\flat) + h(t) : rp(t) \le u \le r\},$$
$$\psi(t,\, r) = \min\{f(t,\, [u - \lambda w(t)]^\flat) + h(t) : rp(t) \le u \le r\}.$$

In geometry, $\varphi(t,\, r)$ is the maximum height of $f(t,\, [u - \lambda w(t)]^\flat) + h(t)$ on the set $\{t\} \times [rp(t),\, r]$, $\psi(t,\, r)$ is the minimum height of $f(t,\, [u - \lambda w(t)]^\flat) + h(t)$ on the same set. If (A1)–(A3) hold, then $\varphi(t,\, r)$ and $\psi(t,\, r)$ are nonnegative integrable function on $[0,\, 1]$ for any $r > 0$.

We obtain the following local existence results.

Theorem 3.1 *Assume that* (A1)–(A3) *hold and there exist two positive numbers* $r_2 > r_1 > \lambda\eta$ *such that one of the following conditions is satisfied*:

(b1)

$$\max_{0\le t\le 1} \int_0^1 (-1)^{n-k} G(t,\, s)\varphi(t,\, r_1)\mathrm{d}t \le \lambda^{-1} r_1,$$
$$\max_{0\le t\le 1} \int_0^1 (-1)^{n-k} G(t,\, s)\psi(t,\, r_2)\mathrm{d}t \ge \lambda^{-1} r_2;$$

(b2)

$$\max_{0\le t\le 1} \int_0^1 (-1)^{n-k} G(t,\, s)\psi(t,\, r_1)\mathrm{d}t \ge \lambda^{-1} r_1,$$
$$\max_{0\le t\le 1} \int_0^1 (-1)^{n-k} G(t,\, s)\varphi(t,\, r_2)\mathrm{d}t \le \lambda^{-1} r_2.$$

Then the problem (P) *has at least one positive solution* u^* *such that*

$$u^* + \lambda w \in K, \qquad r_1 \le \|u^* + \lambda w\| \le r_2.$$

Proof. We only prove the case (b1).

If $u \in \partial\Omega(r_1)$, then

$$r_1 q(t) = \|u\| q(t) \le u(t) \le \|u\| = r_1, \qquad 0 \le t \le 1,$$

By the definition of $\varphi(t, r_1)$, we have

$$f(t, [u(t)-\lambda w(t)]^\flat) + h(t) \le \varphi(t, r_1), \qquad 0 < t < 1.$$

It follows

$$\begin{aligned}\|Tu\| &= \lambda \max_{0\le t\le 1} \int_0^1 (-1)^{n-k} G(t, s)[f(s, [u(s)-\lambda w(s)]^\flat) + h(s)]\mathrm{d}s \\ &\le \lambda \max_{0\le t\le 1} \int_0^1 (-1)^{n-k} G(t, s)\varphi(s, r_1)\mathrm{d}s \\ &\le \lambda\lambda^{-1} r_1 \\ &= \|u\|.\end{aligned}$$

If $u \in \partial\Omega(r_2)$, then

$$r_2 q(t) = \|u\| q(t) \le u(t) \le \|u\| = r_2, \qquad 0 \le t \le 1.$$

By the definition of $\psi(t, r_2)$, one has

$$f(t, [u(t)-\lambda w(t)]^\flat) + h(t) \ge \psi(t, r_2), \qquad 0 < t < 1.$$

It follows

$$\begin{aligned}\|Tu\| &\ge \lambda \max_{0\le t\le 1} \int_0^1 (-1)^{n-k} G(t, s)[f(s, [u(s)-\lambda w(s)]^\flat) + h(s)]\mathrm{d}s \\ &\ge \lambda \max_{0\le t\le 1} \int_0^1 (-1)^{n-k} G(t, s)\psi(s, r_2)\mathrm{d}s \\ &\ge \lambda\lambda^{-1} r_2 \\ &= \|u\|.\end{aligned}$$

By Lemma 2.4, $T : K \to K$ is completely continuous. By the Krasnosel'skii fixed point theorem of cone expansion-compression type, T has a fixed point $\bar{u} \in \overline{K(r_2)}\backslash K(r_1)$. Since

$$\|\bar{u}\| \ge r_1 > \lambda\eta,$$

by Lemma 2.5, we known that $u^* = \bar{u} - \lambda w$ is a positive solution of the problem (P). Further,

$$u^* + \lambda w \in K, \qquad r_1 \le \|u^* + \lambda w\| \le r_2.$$

Corollary 3.1 *Assume that* (A1)–(A3) *hold and there exist two positive numbers* r_1 *and* r_2 *with* $r_2 > r_1 > \lambda\eta$ *such that one of the following conditions is satisfied*:

(c1)

$$\int_0^1 \varphi(t, r_1)\mathrm{d}t \le \lambda^{-1} r_1 A,$$

$$\int_\alpha^\beta \psi(t, r_2)\mathrm{d}t \ge \lambda^{-1} r_2 B;$$

(c2)

$$\int_\alpha^\beta \psi(t, r_1)\mathrm{d}t \ge \lambda^{-1} r_1 B,$$

$$\int_0^1 \varphi(t, r_2)\mathrm{d}t \le \lambda^{-1} r_2 A.$$

Then the problem (P) *has at least one positive solution* u^* *such that*

$$u^* + \lambda w \in K, \qquad r_1 \le \|u^* + \lambda w\| \le r_2.$$

Proof. We only prove the case (c1).

By Lemma 2.3 and (c1), one has

$$\begin{aligned}
&\max_{0\le t\le 1}\int_0^1 (-1)^{n-k}G(t,\, s)\varphi(t,\, r_1)\mathrm{d}t\\
\le\ &\max_{0\le t,s\le 1}(-1)^{n-k}G(t,\, s)\int_0^1 \varphi(t,\, r_1)\mathrm{d}t\\
\le\ &A^{-1}\lambda^{-1}r_1 A\\
=\ &\lambda^{-1}r_1,\\
&\max_{0\le t\le 1}\int_0^1 (-1)^{n-k}G(t,\, s)\psi(t,\, r_2)\mathrm{d}t\\
\ge\ &\max_{\alpha\le t\le \beta}\int_\alpha^\beta (-1)^{n-k}G(t,\, s)\psi(t,\, r_2)\mathrm{d}t\\
\ge\ &\min_{\alpha\le t,s\le \beta}(-1)^{n-k}G(t,\, s)\int_\alpha^\beta \psi(t,\, r_2)\mathrm{d}t\\
\ge\ &B^{-1}\lambda^{-1}r_2 B\\
=\ &\lambda^{-1}r_2.
\end{aligned}$$

By Theorem 3.1(b1), the proof is completed.

4 A New Extend

In this section, we demonstrate that Theorem 3.1 extends Theorem 1.1.

Proposition 4.1 *Theorem* 1.1 *is a special case of Theorem* 3.1(b1).

Proof. Let r_1, r_2, $\zeta(x)$, $\xi(x)$ be as in Theorem 1.1.

Since $\zeta(x)$ is nondecreasing on $[0,\, +\infty)$, by (a2) and (a3), one has, for $0 < t < 1$,

$$\begin{aligned}
\varphi(t,\, r_1) =\ &\max\{f(t,\, [x-\lambda w(t)]^\flat) + h(t) : r_1 q(t) \le x \le r_1\}\\
\le\ &\max\{f(t,\, x) + h(t) : 0 \le x \le r_1\}\\
\le\ &\max\{\zeta(x) : 0 \le x \le r_1\}\\
=\ &\zeta(r_1)\\
\le\ &\lambda^{-1}\left[\max_{0\le t\le 1}\int_0^1 (-1)^{n-k}G(t,\, s)\mathrm{d}s\right]^{-1} r_1.
\end{aligned}$$

It follows

$$\begin{aligned}
&\max_{0\le t\le 1}\int_0^1 (-1)^{n-k}G(t,\, s)\varphi(s,\, r_1)\mathrm{d}s\\
\le\ &\lambda^{-1}\left[\max_{0\le t\le 1}\int_0^1 (-1)^{n-k}G(t,\, s)\mathrm{d}s\right]^{-1} r_1\cdot \max_{0\le t\le 1}\int_0^1 (-1)^{n-k}G(t,\, s)\mathrm{d}s\\
=\ &\lambda^{-1}r_1.
\end{aligned}$$

Let

$$\alpha = \delta, \qquad \beta = 1-\delta.$$

Then

$$\theta \le \min_{\delta \le t \le 1-\delta} p(t).$$

Since $h(t) \equiv M$, by Lemma 2.2 in [8], one has

$$w(t) \le \frac{1}{n!} M p(t).$$

So

$$\eta \le \frac{1}{n!} M.$$

By (a5), if

$$r_2 p(t) \le x \le r_2, \qquad \delta \le t \le 1-\delta,$$

then

$$\begin{aligned} x - \lambda w(t) &\ge r_2 \Big[1 - \frac{\lambda M}{n! r_2}\Big] p(t) \\ &\ge \varepsilon \theta r_2 \\ &> 0, \qquad \delta \le t \le 1-\delta. \end{aligned}$$

Since $\xi(x)$ is nondecreasing on $(0, +\infty)$, by (a4), one has, for $\delta \le t \le 1-\delta$,

$$\begin{aligned} \psi(t, r_2) =& \min\{f(t, x-\lambda w(t)) + h(t) : r_2 p(t) \le x \le r_2\} \\ \ge& \min\{\xi(x - \lambda w(t)) : r_2 p(t) \le x \le r_2\} \\ \ge& \xi(\varepsilon \theta r_2) \\ \ge& \lambda^{-1} \Big[\max_{0 \le t \le 1} \int_\delta^{1-\delta} G(t, s) \mathrm{d}s\Big]^{-1} r_2. \end{aligned}$$

By (a5), we get

$$\begin{aligned} &\max_{0 \le t \le 1} \int_\delta^{1-\delta} (-1)^{n-k} G(t, s) \psi(s, r_2) \mathrm{d}s \\ \ge& \lambda^{-1} \Big[\max_{0 \le t \le 1} \int_\delta^{1-\delta} (-1)^{n-k} G(t, s) \mathrm{d}s\Big]^{-1} r_2 \cdot \max_{0 \le t \le 1} \int_\delta^{1-\delta} (-1)^{n-k} G(t, s) \mathrm{d}s \\ =& \lambda^{-1} r_2. \end{aligned}$$

Since (A1)–(A3) hold, Theorem 1.1 now can be derived from Theorem 3.1(b1).

Example 4.2 Consider the $(2, 4-2)$ conjugate boundary value problem

$$\begin{cases} u^{(4)}(t) = \lambda[2000000\sqrt{u(t)} - 1], \qquad 0 \le t \le 1, \\ u(0) = u'(0) = u(1) = u'(1) = 0. \end{cases}$$

In this problem,

$$f(t, x) = f(x) = 2000000\sqrt{x} - 1, \qquad h(t) \equiv M = 1.$$

Since

$$f(x) + h(t) = 2000000\sqrt{x}$$

and $2000000\sqrt{x}$ is upward convex, Theorems 1.1 is nonapplicable to the problem.

But the problem has one positive solution for some $\lambda > 0$.

In fact, let
$$\alpha = \frac{1}{4}, \qquad \beta = \frac{3}{4}.$$
Since
$$n = 4, \qquad k = 2,$$
one has
$$\begin{aligned} & q(t) = \frac{1}{2}t(1-t), \\ & A = 128, \\ & B = 196608, \\ & \eta \leq \frac{1}{4!}M = \frac{1}{24}, \\ & \max_{0\leq t\leq 0} q(t) = \frac{1}{8}. \end{aligned}$$
For any $0 \leq t \leq 1$, one has
$$\begin{aligned} \varphi(t,\, 16384\times 10^8) &\leq \max\left\{2000000\sqrt{x} : 0 \leq x \leq 16384\times 10^8\right\} \\ &= 256\times 10^{10}. \end{aligned}$$
Since $w(t) \leq \eta q(t)$, if $\lambda \leq 192$ and $x \geq 30$, then for $0 \leq t \leq 1$,
$$\begin{aligned} x - \lambda w(t) &\geq x - \lambda\eta q(t) \\ &\geq x - 192\cdot\frac{1}{24}\cdot\frac{1}{8} \\ &= x - 1 \\ &\geq 0. \end{aligned}$$
Since
$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} q(t) = \frac{3}{32},$$
one has, for $\frac{1}{4} \leq t \leq \frac{3}{4}$,
$$\begin{aligned} \psi(t,\, 320) &= \min\{2000000\sqrt{[x-\lambda w(t)]^\flat} : 320q(t) \leq x \leq 320\} \\ &\geq \min\left\{2000000\sqrt{x-1} : 30 \leq x \leq 320\right\} \\ &\approx 10770330. \end{aligned}$$
If $11.6830 \leq \lambda \leq 81.92$, then
$$\begin{aligned} & \int_{\frac{1}{4}}^{\frac{3}{4}} \psi(t,\, 320)\mathrm{d}t \geq 5385165 > \lambda^{-1}320B, \\ & \int_0^1 \varphi(t,\, 16384\times 10^8)\mathrm{d}t \leq 256\times 10^{10} \leq \lambda^{-1}16384\times 10^8 A. \end{aligned}$$
By Theorem 3.2(c2), the problem has a positive solution u^* such that
$$u^* + \lambda w \in K, \qquad 320 \leq \|u^* + \lambda w\| \leq 16384\times 10^8$$
for any λ with $11.6830 \leq \lambda \leq 81.92$.

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