Communications in Mathematical Research
31(1)(2015), 71–80

Rota-Baxter Operators of Weight 1 on 2×2-matrix Algebras

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Communicated by Du Xian-kun

Abstract: In this paper we determine all Rota-Baxter operators of weight 1 on 2×2-matrix algebras over an algebraically closed field.

Key words: Rota-Baxter operator, weight, associative algebra
2010 MR subject classification: 16A06, 47B99
Document code: A
Article ID: 1674-5647(2015)01-0071-10
DOI: 10.13447/j.1674-5647.2015.01.08

1 Introduction

Rota-Baxter algebras or their corresponding Rota-Baxter relations came from [1], which are on integral equations of fluctuation theory. After that, many mathematicians paid attention to this concept, and especially, Rota[2] fundamental papers around 1970 brought the subject into the areas of combinatorics and algebras. In fact, Rota-Baxter relation may be regarded as one possible generalization of the standard shuffle relation in [2–3]. In the case of Lie algebra, when the weight $\lambda = 0$, the Rota-Baxter relation is just the form of classical Yang-Baxter equation (CYBE) and when the weight $\lambda = 1$, it corresponds to the operator form of the modified classical Yang-Baxter equations. The broad connections of Rota-Baxter algebras with many areas in mathematics and mathematical physics are remarkable. However, the theoretical study of Rota-Baxter algebras is still in its early stage of development. In recent years, some articles have been published about Rota-Baxter algebras in [3–8]. Especially, An and Bai[7] have not only worked over Rota-Baxter operators of weight 0 on pre-Lie algebras, but also computed all Rota-Baxter operators of weight 0 on associative algebras of dimensions ≤ 3; Li and Hou[8] have given all Rota-Baxter operators of weight 1 on associative algebras of dimensions ≤ 3.

In this paper, we study the Rota-Baxter operators of weight 1 on the associative algebra

Received date: Jan. 7, 2013.
Foundation item: The NSF (JC201004) of Heilongjiang Province and the NSF (11171055) of China.
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$M_2(\mathbb{F})$ consisting of 2×2-matrix over an algebraically closed field $\mathbb{F}$. The paper is organized as follows. In Section 2, we give all Rota-Baxter operators of weight 1 on $M_2(\mathbb{F})$. In Section 3, we analyze how to prove the previous theorem and establish the corresponding equations. In Section 4, we give the proof of the main theorem. Throughout this paper, all algebras are of finite dimensions and over an algebraically closed field $\mathbb{F}$.

2 Main Results

We adopt the following definition from [3].

Definition 2.1 *A Rota-Baxter algebra is an associative algebra A over $\mathbb{F}$ with a linear operator $R : A \to A$ satisfying the Rota-Baxter relation*:

$$R(x)R(y) + \lambda R(xy) = R(R(x)y + xR(y)), \qquad x, y \in A, \tag{2.1}$$

where $\lambda \in \mathbb{F}$ is a fixed element, which is called the weight of R.

If R is a Rota-Baxter operator of weight $\lambda \neq 0$, then $\frac{1}{\lambda}R$ is a Rota-Baxter operator of weight 1. Thus, for the Rota-Baxter operators of nonzero weight, it suffices to determine the ones of weight 1. From now on, $RB(A)$ denotes the set of all Rota-Baxter operators on A of weight 1.

Theorem 2.1 *All the Rota-Baxter operators of weight 1 on 2×2-matrix algebra $M_2(\mathbb{F})$ are R_i and $I - R_i \in RB(M_2(\mathbb{F}))$, $i = 1, 2, \cdots, 31$, shown in Table 2.1, where $a, b, c \in \mathbb{F}$ (any denominator is nonzero and non-appeared parameters in matrices are zero).*

Table 2.1 The Rota-Baxter operators set

Operators number	Matrix representations of operators
1	$r_{13} = -r_{21} = a$, $r_{14} = -r_{22} = b$, $r_{33} = -r_{41} = 1 + b$, $r_{34} = -r_{42} = \frac{b+b^2}{a}$
2	$r_{11} = r_{33} = a$, $r_{21} = r_{43} = b$, $r_{22} = r_{44} = 1 - a$, $r_{12} = r_{34} = \frac{a-a^2}{b}$
3	$r_{11} = -r_{41} = a$, $r_{21} = b$, $r_{22} = 1 - a$, $r_{12} = -r_{42} = \frac{a-a^2}{b}$, $r_{31} = -\frac{a^2}{b}$, $r_{32} = \frac{a^3-a^2}{b}$
4	$r_{11} = a$, $r_{21} = b$, $r_{22} = r_{41} = 1 - a$, $r_{12} = r_{31} = -r_{42} = \frac{a-a^2}{b}$, $r_{32} = \frac{a^3-a^2}{b^2}$, $r_{34} = \frac{a}{b}$, $r_{44} = 1$
5	$r_{11} = r_{22} = a$, $r_{13} = r_{24} = b$, $r_{31} = r_{42} = \frac{a-a^2}{b}$, $r_{33} = r_{44} = 1 - a$
6	$r_{14} = -r_{33} = a$, $r_{24} = -r_{43} = b$, $r_{22} = -r_{41} = 1 + a$, $r_{12} = -r_{31} = \frac{a+a^2}{b}$
7	$r_{14} = -r_{44} = a$, $r_{22} = 1 + a$, $r_{24} = b$, $r_{12} = -r_{34} = -r_{42} = \frac{a+a^2}{b}$, $r_{31} = \frac{1+a}{b}$, $r_{33} = 1$, $r_{32} = \frac{-a(1+a)^2}{b^2}$

(to be continued)

(continued)

Operators number	Matrix representations of operators
8	$r_{14}=-r_{44}=a,\ r_{24}=b,\ r_{12}=-r_{42}=\dfrac{a+a^2}{b},\ r_{22}=1+a,\ r_{32}=-\dfrac{a^2+a^3}{b^2},$ $r_{34}=-\dfrac{a^2}{b}$
9	0
10	$r_{12}=-r_{42}=a,\ r_{22}=1,\ r_{32}=-a^2$
11	$r_{12}=-r_{31}=a,\ r_{22}=-r_{41}=1$
12	$r_{12}=r_{34}=a,\ r_{22}=r_{44}=1$
13	$r_{12}=r_{31}=r_{34}=-r_{42}=-a,\ r_{22}=r_{41}=r_{44}=1,\ r_{32}=-a^2$
14	$-r_{14}=r_{22}=1,\ r_{34}=-r_{42}=a$
15	$-r_{14}=r_{22}=r_{44}=1,\ r_{32}=a,\ r_{34}=b$
16	$r_{11}=r_{22}=1,\ r_{31}=r_{42}=a$
17	$r_{11}=r_{22}=-r_{41}=1,\ r_{31}=a$
18	$r_{11}=r_{22}=r_{44}=1,\ r_{31}=a,\ r_{34}=b$ with $ab=0$
19	$r_{11}=r_{22}=r_{41}=r_{44}=1,\ r_{34}=-r_{42}=a$
20	$r_{11}=r_{14}=r_{22}=1,\ r_{12}=-r_{31}=-r_{34}=-r_{42}=a,\ r_{32}=-a^2$
21	$r_{11}=r_{14}=r_{22}=r_{44}=1,\ r_{12}=-r_{31}=a$
22	$r_{13}=-r_{43}=a,\ r_{14}=-r_{44}=b,\ r_{33}=1+b,\ r_{23}=-\dfrac{a^2}{1+b},\ r_{24}=-\dfrac{ab}{1+b},$ $r_{34}=\dfrac{b+b^2}{a}$ with $a\neq 0, b\neq 0,-1$
23	$r_{33}=r_{44}=-r_{14}=1,\ r_{23}=a,\ r_{24}=b$ with $a\neq 0$
24	$r_{11}=1,\ r_{44}=1+a,\ r_{14}=r_{33}=-a,\ r_{13}=-r_{43}=-r_{24}=b,\ r_{21}=\dfrac{b}{a},\ r_{23}=\dfrac{b^2}{a},$ $r_{34}=\dfrac{a+a^2}{b}$ with $a\neq 0,-1; b\neq 0$
25	$r_{11}=r_{33}=r_{14}=1,\ r_{13}=-r_{21}=-r_{24}=-r_{43}=a,\ r_{23}=-a^2$ with $a\neq 0$
26	$r_{11}=-r_{41}=b,\ r_{13}=-r_{43}=a,\ r_{33}=1-b,\ r_{21}=\dfrac{ab}{b-1},\ r_{23}=\dfrac{a^2}{b-1},\ r_{31}=\dfrac{b-b^2}{a}$
27	$r_{11}=r_{33}=-r_{41}=1,\ r_{21}=a,\ r_{23}=b$ with $b\neq 0$
28	$r_{11}=a,\ r_{44}=1,\ r_{13}=r_{21}=-r_{43}=b,\ r_{33}=r_{41}=1-a,\ r_{23}=\dfrac{b^2}{a-1},\ r_{24}=\dfrac{b}{1-a},$ $r_{31}=\dfrac{a-a^2}{b}$
29	$r_{11}=r_{33}=r_{44}=1,\ r_{21}=a,\ r_{24}=b,\ r_{23}=-ab$ with $ab\neq 0$
30	$r_{12}=-r_{34}=-r_{42}=a,\ r_{13}=r_{21}=-r_{43}=b,\ r_{14}=-r_{44}=c,\ r_{22}=1+c,$ $r_{11}=-r_{41}=\dfrac{ab}{c},\ r_{23}=\dfrac{bc}{a},\ r_{24}=\dfrac{c}{a}+\dfrac{c^2}{a},\ r_{31}=\dfrac{a}{c}-\dfrac{a^2b}{c^2},\ r_{32}=-\dfrac{a^2}{c},$ $r_{33}=1-\dfrac{ab}{c}$ with $abc\neq 0$
31	$r_{12}=-r_{42}=\tau,\ r_{13}=-r_{43}=a,\ r_{14}=-r_{44}=b,\ r_{23}=c,\ r_{11}=-r_{41}=\dfrac{\tau a}{b},$ $r_{21}=\dfrac{\tau c}{b},\ r_{22}=\dfrac{\tau c}{a},r_{24}=\dfrac{bc}{a},\ r_{31}=-\dfrac{\tau a^2}{bc},\ r_{32}=-\dfrac{\tau a}{c},\ r_{33}=-\dfrac{a^2}{c},$ $r_{34}=-\dfrac{ab}{c}$ with $\tau=\dfrac{a^3b+ab^2c+abc}{a^2c+bc^2},\ \tau abc\neq 0$

3 Analysis

Lemma 3.1[8] *Let R be a linear operator on the algebra A. Then $R \in RB(A)$ if and only if $I - R \in RB(A)$ (I is the identity mapping of A). In particular, $0, I \in RB(A)$.*

Let A be an associative algebra with a basis $\{e_1, e_2, \cdots, e_n\}$. Suppose that

$$e_i e_j = \sum_{k=1}^{n} c_{ij}^k e_k,$$

where $c_{ij}^k \in \mathbb{F}$ are the structure constants. Then any Rota-Baxter operator R can be presented by a matrix (r_{ij}), where $R(e_i) = \sum_{j=1}^{n} r_{ij} e_j$. Moreover, r_{ij} satisfies the following equations (see [7]):

$$\sum_{k,l,m} [c_{kl}^m r_{ik} r_{jl} - c_{kj}^l r_{ik} r_{lm} - c_{il}^k r_{jl} r_{km}] = 0, \qquad i, j = 1, 2, \cdots, n. \tag{3.1}$$

Let

$$e_1 = e_{11}, \quad e_2 = e_{12}, \quad e_3 = e_{21}, \quad e_4 = e_{22} \tag{3.2}$$

be a basis of $M_2(\mathbb{F})$. Since $e_{ij} e_{kl} = \delta_{jk} e_{il}$,

$$\begin{array}{llll} e_1e_1 = e_1, & e_1e_2 = e_2, & e_2e_3 = e_1, & e_2e_4 = e_2, \\ e_3e_1 = e_3, & e_3e_2 = e_4, & e_4e_3 = e_3, & e_4e_4 = e_4 \end{array} \tag{3.3}$$

are all nonzero products on this basis. Let $R(e_i) = \sum_{j=1}^{4} r_{ij} e_j$, $i = 1, 2, 3, 4$. That is,

$$\begin{pmatrix} R(e_1) \\ R(e_2) \\ R(e_3) \\ R(e_4) \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}. \tag{3.4}$$

We need to compute 4×4-matrix (r_{ij}) to obtain $R \in RB(M_2(\mathbb{F}))$. Because R is a linear operator, we only check that all basis elements satisfy the following identity:

$$R(x)R(y) + R(xy) = R(R(x)y + xR(y)), \quad x, y \in A. \tag{3.5}$$

When x, y are substituted by e_1, e_2, e_3, e_4, respectively, we can obtain the following identities:

$$r_{12}r_{13} + r_{11} = r_{11}^2 + r_{12}r_{21} + r_{13}r_{31}; \tag{3.6}$$

$$r_{12}r_{14} + r_{12} = r_{11}r_{12} + r_{12}r_{22} + r_{13}r_{32}; \tag{3.7}$$

$$r_{13}r_{14} + r_{13} = r_{11}r_{13} + r_{12}r_{23} + r_{13}r_{33}; \tag{3.8}$$

$$r_{13}r_{12} + r_{14}^2 + r_{14} = 2r_{11}r_{14} + r_{12}r_{24} + r_{13}r_{34}; \tag{3.9}$$

$$r_{12}r_{23} + r_{21} = r_{21}r_{11} + r_{21}r_{22} + r_{13}r_{41}; \tag{3.10}$$

$$r_{12}r_{24} + r_{22} = r_{12}r_{21} + r_{22}^2 + r_{13}r_{42}; \tag{3.11}$$

$$r_{14}r_{23} + r_{23} = r_{11}r_{23} + r_{22}r_{23} + r_{13}r_{43}; \tag{3.12}$$

$$r_{13}r_{22} + r_{14}r_{24} + r_{24} = r_{21}r_{14} + r_{11}r_{24} + r_{22}r_{24} + r_{13}r_{44}; \tag{3.13}$$

$$r_{12}r_{33} = r_{12}r_{11} + r_{32}r_{21} + r_{14}r_{31}; \tag{3.14}$$

$$r_{11}r_{32} + r_{12}r_{34} = r_{12}^2 + r_{31}r_{12} + r_{32}r_{22} + r_{14}r_{32}; \tag{3.15}$$

$$r_{12}r_{13} + r_{32}r_{23} = 0; \tag{3.16}$$

$$r_{13}r_{32} = r_{12}r_{14} + r_{31}r_{14} + r_{32}r_{24}; \tag{3.17}$$

$$r_{12}r_{43} = r_{12}r_{21} + r_{42}r_{21} + r_{14}r_{41}; \tag{3.18}$$

$$r_{11}r_{42} + r_{12}r_{44} = r_{41}r_{12} + r_{12}r_{22} + r_{42}r_{22} + r_{14}r_{42}; \tag{3.19}$$

$$r_{12}r_{23} + r_{42}r_{23} = 0; \tag{3.20}$$

$$r_{13}r_{42} = r_{41}r_{14} + r_{12}r_{24} + r_{42}r_{24}; \tag{3.21}$$

$$r_{22}r_{13} = r_{13}r_{11} + r_{14}r_{21} + r_{23}r_{31}; \tag{3.22}$$

$$r_{13}r_{12} + r_{23}r_{32} = 0; \tag{3.23}$$

$$r_{23}r_{11} + r_{24}r_{13} = r_{21}r_{13} + r_{13}^2 + r_{14}r_{23} + r_{23}r_{33}; \tag{3.24}$$

$$r_{23}r_{12} = r_{21}r_{14} + r_{13}r_{14} + r_{23}r_{34}; \tag{3.25}$$

$$r_{22}r_{23} = r_{11}r_{23} + r_{24}r_{21} + r_{23}r_{41}; \tag{3.26}$$

$$r_{23}r_{12} + r_{23}r_{42} = 0; \tag{3.27}$$

$$r_{23}r_{13} + r_{23}r_{43} = 0; \tag{3.28}$$

$$r_{23}r_{22} = r_{23}r_{14} + r_{21}r_{24} + r_{23}r_{44}; \tag{3.29}$$

$$r_{21}r_{31} + r_{22}r_{33} + r_{11} = r_{22}r_{11} + r_{33}r_{11} + r_{34}r_{21} + r_{24}r_{31}; \tag{3.30}$$

$$r_{21}r_{32} + r_{12} = r_{22}r_{12} + r_{33}r_{12} + r_{24}r_{32}; \tag{3.31}$$

$$r_{23}r_{31} + r_{13} = r_{22}r_{13} + r_{33}r_{13} + r_{34}r_{23}; \tag{3.32}$$

$$r_{23}r_{32} + r_{14} = r_{22}r_{14} + r_{33}r_{14} + r_{34}r_{24}; \tag{3.33}$$

$$r_{21}r_{41} + r_{22}r_{43} + r_{21} = r_{43}r_{11} + r_{22}r_{21} + r_{44}r_{21} + r_{24}r_{41}; \tag{3.34}$$

$$r_{21}r_{32} + r_{22} = r_{22}r_{12} + r_{33}r_{12} + r_{24}r_{32}; \tag{3.35}$$

$$r_{23}r_{41} + r_{23} = r_{43}r_{13} + r_{22}r_{23} + r_{44}r_{23}; \tag{3.36}$$

$$r_{23}r_{42} + r_{24} = r_{43}r_{14} + r_{22}r_{24} + r_{44}r_{24}; \tag{3.37}$$

$$r_{32}r_{13} + r_{31} = r_{11}r_{31} + r_{33}r_{31} + r_{12}r_{41}; \tag{3.38}$$

$$r_{32}r_{14} + r_{32} = r_{11}r_{32} + r_{33}r_{32} + r_{12}r_{42}; \tag{3.39}$$

$$r_{34}r_{13} + r_{33} = r_{31}r_{31} + r_{33}^2 + r_{12}r_{43}; \tag{3.40}$$

$$r_{33}r_{12}+r_{34}r_{14}+r_{34}=r_{31}r_{14}+r_{11}r_{34}+r_{33}r_{34}+r_{12}r_{44}; \tag{3.41}$$

$$r_{32}r_{23}+r_{41}=r_{21}r_{31}+r_{22}r_{41}+r_{33}r_{41}; \tag{3.42}$$

$$r_{32}r_{24}+r_{42}=r_{32}r_{21}+r_{42}r_{22}+r_{33}r_{42}; \tag{3.43}$$

$$r_{34}r_{23}+r_{43}=r_{31}r_{23}+r_{22}r_{43}+r_{33}r_{43}; \tag{3.44}$$

$$r_{33}r_{22}+r_{34}r_{24}+r_{44}=r_{31}r_{24}+r_{21}r_{34}+r_{22}r_{44}+r_{33}r_{44}; \tag{3.45}$$

$$r_{32}r_{33}=r_{32}r_{11}+r_{34}r_{31}+r_{32}r_{41}; \tag{3.46}$$

$$r_{32}r_{12}+r_{32}r_{42}=0; \tag{3.47}$$

$$r_{32}r_{13}+r_{32}r_{43}=0; \tag{3.48}$$

$$r_{33}r_{32}=r_{32}r_{14}+r_{31}r_{34}+r_{32}r_{44}; \tag{3.49}$$

$$r_{32}r_{43}=r_{32}r_{21}+r_{34}r_{41}+r_{42}r_{41}; \tag{3.50}$$

$$r_{31}r_{42}+r_{32}r_{44}=r_{32}r_{22}+r_{41}r_{32}+r_{34}r_{42}+r_{42}^2; \tag{3.51}$$

$$r_{32}r_{23}+r_{42}r_{43}=0; \tag{3.52}$$

$$r_{33}r_{42}=r_{32}r_{24}+r_{41}r_{34}+r_{42}r_{44}; \tag{3.53}$$

$$r_{42}r_{13}=r_{13}r_{31}+r_{43}r_{31}+r_{14}r_{41}; \tag{3.54}$$

$$r_{13}r_{32}+r_{43}r_{32}=0; \tag{3.55}$$

$$r_{43}r_{11}+r_{44}r_{13}=r_{41}r_{13}+r_{13}r_{33}+r_{43}r_{33}+r_{14}r_{43}; \tag{3.56}$$

$$r_{43}r_{12}=r_{41}r_{14}+r_{13}r_{34}+r_{43}r_{34}; \tag{3.57}$$

$$r_{42}r_{23}=r_{23}r_{31}+r_{43}r_{41}+r_{24}r_{41}; \tag{3.58}$$

$$r_{23}r_{32}+r_{43}r_{42}=0; \tag{3.59}$$

$$r_{43}r_{21}+r_{44}r_{23}=r_{41}r_{23}+r_{23}r_{33}+r_{43}^2+r_{24}r_{43}; \tag{3.60}$$

$$r_{43}r_{22}=r_{41}r_{24}+r_{23}r_{34}+r_{43}r_{44}; \tag{3.61}$$

$$r_{41}r_{31}+r_{42}r_{33}+r_{31}=r_{42}r_{11}+r_{33}r_{31}+r_{44}r_{31}+r_{34}r_{41}; \tag{3.62}$$

$$r_{41}r_{32}+r_{32}=r_{42}r_{12}+r_{33}r_{32}+r_{44}r_{32}; \tag{3.63}$$

$$r_{43}r_{31}+r_{33}=r_{42}r_{13}+r_{33}^2+r_{34}r_{43}; \tag{3.64}$$

$$r_{43}r_{32}+r_{34}=r_{42}r_{14}+r_{33}r_{34}+r_{44}r_{34}; \tag{3.65}$$

$$r_{41}^2+r_{42}r_{43}+r_{41}=r_{42}r_{21}+r_{43}r_{31}+2r_{44}r_{41}; \tag{3.66}$$

$$r_{41}r_{42}+r_{42}=r_{42}r_{22}+r_{43}r_{32}+r_{44}r_{42}; \tag{3.67}$$

$$r_{43}r_{41} + r_{43} = r_{42}r_{23} + r_{43}r_{33} + r_{44}r_{43}; \tag{3.68}$$

$$r_{43}r_{42} + r_{44} = r_{42}r_{24} + r_{43}r_{34} + r_{44}^2. \tag{3.69}$$

4 Proof

Now, we prove Theorem 2.1. When $r_{23} = 0$, it can be obtained that $r_{21}r_{24} = 0$ by (3.26). In order to solve the equations (3.6)–(3.69), it suffices to compute (r_{ij}) in the following five cases:

4.1 Case 1. $r_{23} = r_{24} = 0,\ r_{21} \neq 0.$

Case 1.1. $r_{13} \neq 0$. We have $r_{43} = r_{12} = 0$, $r_{21} = -r_{13}$, $r_{32} = 0$, $r_{22} + r_{33} = 1$, $r_{31} = 0$, $r_{11} = 0$ or 1 by (3.6), (3.7), (3.12), (3.16), (3.24), (3.32) and (3.42).

Case 1.1.1. $r_{11} = 0$. We have

$$r_{22} - r_{41} = 1, \quad r_{44} = 0, \quad r_{33} = 1 + r_{14}, \quad r_{22} + r_{33} = 1, \quad r_{14}^2 + r_{14} = r_{13}r_{34},$$

$$r_{22}^2 + r_{13}r_{42} = r_{22}, \quad r_{34} = -r_{42}, \quad r_{21} = -r_{13} \neq 0,$$

$$r_{11} = r_{12} = r_{23} = r_{24} = r_{31} = r_{32} = r_{43} = r_{44} = 0, \quad -r_{41} = r_{33} = 1 + r_{14} = 1 - r_{22}$$

by (3.8)–(3.11), (3.32) and (3.34). Let $r_{13} = a \neq 0$, $r_{14} = b$. We get $\boldsymbol{R}_1$ in Theorem 2.1, where

$$\boldsymbol{R}_1 = \begin{pmatrix} 0 & 0 & a & b \\ -a & -b & 0 & 0 \\ 0 & 0 & 1+b & \dfrac{b+b^2}{a} \\ -1-b & -\dfrac{b+b^2}{a} & 0 & 0 \end{pmatrix}.$$

Case 1.1.2. $r_{11} = 1$. We conclude that $\boldsymbol{I} - \boldsymbol{R}_1$ is in $RB(M_2(\mathbb{F}))$ by Lemma 3.1.

Case 1.2. $r_{13} = 0$. We get

$$r_{14} = 0, \quad r_{11} + r_{22} = 1, \quad r_{43}(r_{22} + r_{33} - 1) = 0, \quad r_{43}r_{32} = 0, \quad r_{43}r_{42} = 0, \quad r_{43}r_{31} = 0,$$

$$r_{43}(r_{11} - r_{33}) = 0, \quad r_{43}(r_{12} - r_{34}) = 0, \quad r_{43}r_{41} = 0, \quad r_{43}(r_{43} - r_{21}) = 0,$$

$$r_{43}(r_{22} - r_{44}) = 0$$

by (3.10), (3.13), (3.44), (3.48), (3.52), (3.54), (3.56)–(3.61).

Case 1.2.1. $r_{43} \neq 0$. We get $\boldsymbol{R}_2$ in Theorem 2.1 by (3.44), (3.48), (3.52), (3.54), (3.56)–(3.61) and (3.69).

Case 1.2.2. $r_{43} = 0$, where $r_{13} = r_{14} = r_{23} = r_{24} = r_{43} = 0$, $r_{11} + r_{22} = 1$, $r_{21} \neq 0$. It is not difficult to obtain $r_{33} = 0$ or 1 by (3.40).

(A) $r_{33} = 0$. From (3.18), (3.69), we directly conclude that $r_{12} = -r_{42}$ and $r_{44} = 0$ or 1.

According to $r_{44} = 0$ and $r_{44} = 1$, respectively, we obtain $\boldsymbol{R}_3$ and $\boldsymbol{R}_4$ in Theorem 2.1 by (3.10), (3.30), (3.34), (3.35), (3.45) and (3.63).

(B) $r_{33} = 1$. We conclude that $\boldsymbol{I} - \boldsymbol{R}_3$ and $\boldsymbol{I} - \boldsymbol{R}_4$ are in $RB(M_2(\mathbb{F}))$ by Lemma 3.1.

4.2 Case 2. $r_{21} = r_{23} = 0,\ r_{24} \neq 0$.

Case 2.1. $r_{13} \neq 0$. We obtain

$$r_{13} = r_{24} \neq 0, \quad r_{41} = 0, \quad r_{43} = 0, \quad r_{12} = 0, \quad r_{11} = r_{22}, \quad r_{14} = 0, \quad r_{22} + r_{33} = 1,$$

$$r_{34} = 0, \quad r_{32} = 0, \quad r_{33} = r_{44} = 1 - r_{22} = 1 - r_{11}, \quad r_{31} = r_{42}, \quad r_{22} - r_{22}^2 = r_{31}r_{24}$$

by (3.10), (3.12), (3.16), (3.22), (3.24), (3.25), (3.32), (3.33), (3.43), (3.45), (3.54) and (3.56). Let $r_{11} = a$, $r_{13} = b \neq 0$. We get $\boldsymbol{R}_5$ in Theorem 2.1.

Case 2.2. $r_{13} = 0$. It is not difficult to obtain $r_{11} = 0$ or 1 by (3.6).

Case 2.2.1. $r_{11} = 0$. It is easy to know $r_{22} = 1 + r_{14}$ by (3.13).

(A) $r_{43} \neq 0$. We get $\boldsymbol{R}_6$ in Theorem 2.1 by (3.9), (3.30), (3.37), (3.44), (3.48), (3.52), (3.60), (3.68) and (3.69).

(B) $r_{43} = 0$. We can get $r_{14}^2 + r_{14} = r_{12}r_{24}$, $r_{22} = r_{22}^2 + r_{24}r_{42}$, $r_{12} = -r_{42}$, $r_{41} = 0$, $r_{22} + r_{44} = 1$, $r_{33} = 0$ or 1 by (3.9), (3.35), (3.34), (3.37) and (3.40).

According to $r_{33} = 0$ and $r_{33} = 1$, respectively, we obtain $\boldsymbol{R}_7$ and $\boldsymbol{R}_8$ in Theorem 2.1 by (3.17), (3.30), (3.39) and (3.45).

Case 2.2.2. $r_{11} = 1$. We conclude that $\boldsymbol{I} - \boldsymbol{R}_i$ are in $RB(M_2(\mathbb{F}))$ $(i = 6, 7, 8)$ by Lemma 3.1.

4.3 Case 3. $r_{23} = r_{21} = r_{24} = 0$.

We can obtain $r_{13} = 0$, $r_{43} = 0$ and $r_{33} = 0$ or 1 by (3.24), (3.40) and (3.60).

Case I. $r_{33} = 0$. It is easy to get $r_{11} = 0$ or 1 by (3.6).

(I) $r_{11} = 0$. We can obtain $r_{22} = 0$ or 1 by (3.11).

(I.1) $r_{22} = 0$. It is not difficult to obtain $r_{14} = 0$ or 1 from (3.11).

(I.1.1) $r_{14} = 0$. We obtain $r_{12} = r_{31} = r_{32} = r_{34} = r_{41} = r_{42} = r_{44} = 0$ by (3.31), (3.38), (3.39), (3.41), (3.42), (3.43) and (3.45). So we get $\boldsymbol{R}_9 = \boldsymbol{0}$ in Theorem 2.1.

(I.1.2) $r_{14} = -1$. We can get $r_{12} = 0$ by (3.31), $r_{14} = 0$ by (3.33), which is inconsistent with $r_{14} = -1$. Hence, the system of the equations (3.6)–(3.69) has no solution in this case.

(I.2) $r_{22} = 1$. By (3.7) and (3.9), we can get $r_{12}r_{14} = 0$ and $r_{14} = 0$ or -1.

(I.2.1) $r_{14} = 0$. We can get $r_{44} = 0$ or 1 from (3.69).

(A) $r_{44} = 0$. It is not difficult to obtain $r_{34} = 0$ from (3.65), $r_{41} = 0$ or -1 from (3.66). By $r_{41} = 0$ and $r_{41} = -1$, respectively, we get $\boldsymbol{R}_{10}$ and $\boldsymbol{R}_{11}$ in Theorem 2.1.

(B) $r_{44} = 1$. Similarly, we obtain $\boldsymbol{R}_{12}$ and $\boldsymbol{R}_{13}$ in Theorem 2.1.

(I.2.2) $r_{14} = -1$. In the same way, we get $\boldsymbol{R}_{14}$ and $\boldsymbol{R}_{15}$ in Theorem 2.1.

(II) $r_{11} = 1$. Here $r_{13} = r_{21} = r_{23} = r_{24} = r_{33} = r_{43} = 0$ and $r_{22} = 0$ or 1.

(II.1) $r_{22} = 0$. We can get $r_{11} = 0$ from (3.30), which is inconsistent with $r_{11} = 1$. Hence, the system of the equations (3.6)–(3.69) has no solution in this case.

(II.2) $r_{22} = 1$. From (3.7) and (3.9), we can conclude $r_{12}r_{14} = r_{12}$ and $r_{14} = 0$ or 1.

(II.2.1) $r_{14} = 0$. We can get $r_{44} = 0$ or 1.

(A) $r_{44} = 0$. We get $r_{41} = 0$ or -1 by (3.66) and we get $\boldsymbol{R}_{16}$ and $\boldsymbol{R}_{17}$ in Theorem 2.1.

(B) $r_{44} = 1$. Similarly, we get $\boldsymbol{R}_{18}$ and $\boldsymbol{R}_{19}$ in Theorem 2.1.

(II.2.2) $r_{14} = 1$. In the same way, we get $\boldsymbol{R}_{20}$ and $\boldsymbol{R}_{21}$ in Theorem 2.1.

Case II. $r_{33} = 1$. We conclude that $\boldsymbol{I} - \boldsymbol{R}_i$ $(i = 9, 10, \cdots, 21)$ are in $RB(M_2(\mathbb{F}))$ by Lemma 3.1.

4.4 Case 4. $r_{23} \neq 0$, $r_{12} = 0$.

We have

$$r_{12} = -r_{42} = 0, \quad r_{13} = -r_{43}, \quad r_{32} = -\frac{r_{12}r_{13}}{r_{23}} = 0, \quad r_{14}r_{41} = -r_{13}r_{12} = 0$$

by (3.20), (3.21), (3.28), (3.52), and by (3.12), (3.36), we get

$$r_{11} + r_{41} = r_{14} + r_{44}. \tag{4.1}$$

It is easy to get $r_{22} = 0$ or 1 by (3.11).

Case 4.1. $r_{22} = 0$.

Case 4.1.1. $r_{14} \neq 0$. We have $r_{31} = 0$, $r_{41} = 0$ and $r_{11} = 0$ or 1 by (3.17), (3.21) and (3.6).

(A) $r_{11} = 0$. It can be obtained that $r_{14}^2 + r_{14} = r_{13}r_{34}$ and $r_{21} = 0$ by (3.9), (3.22).

According to $r_{14}^2 + r_{14} = r_{13}r_{34} \neq 0$ and $r_{14}^2 + r_{14} = r_{13}r_{34} = 0$, respectively. We get $\boldsymbol{R}_{22}$ and $\boldsymbol{R}_{23}$ in Theorem 2.1 by (3.12), (3.13), (3.24), (3.25), (3.45), (3.68) and (4.1).

(B) $r_{11} = 1$. Similarly, we get $\boldsymbol{R}_{24}$ and $\boldsymbol{R}_{25}$ in Theorem 2.1.

Case 4.1.2. $r_{14} = 0$. From (3.25), (3.69), we conclude $r_{34} = 0$ and $r_{44} = 0$ or 1.

(A) $r_{44} = 0$. We can get $r_{11} = -r_{41}$ by (4.1) and $r_{24} = 0$ by (3.37). According to $r_{13} \neq 0$ and $r_{13} = 0$, respectively. We get $\boldsymbol{R}_{26}$ and $\boldsymbol{R}_{27}$ in Theorem 2.1 by (3.6), (3.8), (3.10), (3.12), (3.22), (3.24) and (3.36).

(B) $r_{44} = 1$. Similarly, we get $\boldsymbol{R}_{28}$ and $\boldsymbol{R}_{29}$ in Theorem 2.1.

Case 4.2. $r_{22} = 1$. We conclude that $\boldsymbol{I} - \boldsymbol{R}_i$ $(i = 22, 23, \cdots, 29)$ are in $RB(M_2(\mathbb{F}))$ by Lemma 3.1.

4.5 Case 5. $r_{23} \neq 0$, $r_{12} \neq 0$.

We have

$$r_{12}r_{23} = r_{13}(r_{14} + 1 - r_{11} - r_{33}) \neq 0, \quad r_{42} = -r_{12}, \quad r_{43} = -r_{13}, \quad r_{32} = -\frac{r_{12}r_{13}}{r_{23}},$$

$$r_{14}r_{41} = -r_{12}r_{13}, \quad r_{11} + r_{41} = r_{14} + r_{44}, \quad r_{22} = 1 - r_{11} + r_{14} + \frac{r_{13}^2}{r_{23}}, \quad r_{21} = \frac{r_{12}r_{23}}{r_{14}},$$

$$r_{33} = -\frac{r_{12}r_{13}}{r_{14}} - r_{44} + 1 - \frac{r_{12}r_{23}}{r_{13}}, \quad r_{31} = -\frac{r_{12}^2r_{13}}{r_{14}^2} - \frac{r_{12}^2r_{23}}{r_{13}r_{14}} - \frac{2r_{12}r_{44}}{r_{14}} - r_{12} + \frac{r_{12}}{r_{14}},$$

$$r_{24} = r_{13} - \frac{2r_{23}r_{44}}{r_{13}} - \frac{r_{12}r_{23}}{r_{14}} + \frac{r_{23}}{r_{13}} - \frac{r_{12}r_{23}^2}{r_{13}^2}$$

by (3.7), (3.8), (3.10), (3.12)–(3.14), (3.21), (3.23), (3.27), (3.28), (3.36) and (3.68). $r_{14} + r_{44} = 0$ or 1 from (3.6).

Case 5.1. $r_{14} + r_{44} = 0$. We have $(r_{13}r_{14} - r_{12}r_{23})(r_{14}r_{23} - r_{13}r_{24}) = 0$ by (3.37).

Case 5.1.1. $r_{13}r_{14} = r_{12}r_{23}$. Let $r_{12} = a$, $r_{13} = b$, $r_{14} = c$. We get $\boldsymbol{R}_{30}$ in Theorem 2.1 by (3.50).

Case 5.1.2. $r_{24} = \dfrac{r_{14}r_{23}}{r_{13}} \neq 0$. We get

$$r_{13}^3 r_{14} + r_{13}r_{14}r_{23} + r_{13}r_{14}^2 r_{23} - r_{12}r_{13}^2 r_{23} - r_{12}r_{14}r_{23}^2 = 0. \tag{4.2}$$

Let $r_{12} = \tau$, $r_{13} = a$, $r_{14} = b$, $r_{23} = c$. We get $\boldsymbol{R}_{31}$ in Theorem 2.1 by (3.10), (3.21), (3.23), (3.25), (3.29), (3.32), (3.62) and (3.68).

Here $\tau abc \neq 0$, $a^3b + ab^2c + abc - \tau a^2c - \tau bc^2 = 0$ by (4.2), that is,

$$\tau = \frac{a^3b + ab^2c + abc}{a^2c + bc^2} \neq 0,$$

where $abc \neq 0$. In fact, we have $a^3b + ab^2c + abc \neq 0$ and $a^2c + bc^2 \neq 0$ by (3.12) and (3.69).

Case 5.2. $r_{14} + r_{44} = 1$. We conclude that $\boldsymbol{I} - \boldsymbol{R}_{30}$ and $\boldsymbol{I} - \boldsymbol{R}_{31}$ are in $RB(M_2(\mathbb{F}))$ by Lemma 3.1.

References

[1] Baxter G. An analytic problem whose solution follows from a simple algebraic identity. *Pacific J. Math.*, 1960, **10**: 731–742.

[2] Rota G C. Rota-Baxter algebras and combinatorial identities I. *Bull. Amer. Math. Soc.*, 1969, **75**: 325–329.

[3] Guo L. What is a Rota-Baxter algebra? *Notice Amer. Math. Soc.*, 2009, **56**: 1436–1437.

[4] Guo L, Keigher W. Free commutative Rota-Baxter algebras and shuffle products. *Adv. Math.*, 2000, **150**: 117–149.

[5] Guo L, Ebrahimi-Fard K. Free noncommutative Rota-Baxter algebras and rooted trees. *J. Algebra Appl.*, 2008, **7**: 167–194.

[6] Miller J B. Baxter operators and endomorphisms on Banach algebras. *J. Math. Anal. Appl.*, 1969, **25**: 503–520.

[7] An H H, Bai C M. From Rota-Baxter algebras to pre-Lie algebras. *J. Phys. A*, 2008, **41**: 015201–015219.

[8] Li X X, Hou D P, Bai C M. Rota-Baxter operators on pre-Lie algebras. *J. Nonlinear Math. Phys.*, 2007, **14**: 269–289.